

**A NEW LIFE-TIME MODEL WITH BATHTUB AND INVERTED BATHTUB  
FAILURE RATE FUNCTION.**

**ABSTRACT**

In this paper, a three parameter model called Zubair- Kumaraswamy (Z-Kum) distribution is proposed. The extension was done using Zubair G-Family (2018) of continuous probability distribution to extend the well known Kumaraswamy distribution to make it more flexible in modeling and predicting real world phenomenon. Some basic structural properties of the new distributions like cdf, pdf, quantile functions, moments, moment generating functions, characteristics functions and order statistics was obtained. Survival function, hazard function, reversed hazard rate function and a cumulative hazard rate function was also obtained. Behavior of the hazard rate plot exhibit increase, decrease, Bathtub and inverted Bathtub shape. Maximum likelihood estimate was used to estimate the Z-Kum distribution parameters. Monte Carlo simulation was carried out to evaluate the performance of MLE method adopted. Result of the simulation studies indicates that MLE is good for the estimation of our distribution parameters. To compare the proposed model with the other fitted existing models, analytical measure of goodness of fit of some information criterion was considered using three real life data sets. From the results obtained, it is evident that our proposed model gives better fit than the other competing models and therefore, our proposed model provide greater flexibility in modeling real world phenomenon.

**Key Words:** Kumaraswamy, Bathtub, Quantile Function, Moments, Hazard rate

**1. Introduction.**

The quality of the procedures used in statistical analysis depends heavily on the assumed probability model or distribution that the random variable follows. Many lifetime data used for statistical analysis follow a particular probability distribution and therefore knowledge of the

appropriate distribution that any phenomenon follows greatly improves the sensitivity, reliability and efficiency of the statistical analysis associated with it. Furthermore, it is true that several probability distributions exist for modeling lifetime data; however, some of these lifetime data do not follow any of the existing and well known standard probability distributions (models) or at least are inappropriately described by them. Due to change in the world population and rapid global development in science and technology, developing new distributions which could better describe some of these phenomena and provide greater flexibility and wider acceptability in the modeling of lifetime data become an inevitable.

Recently, attempts have been made to define new models that extend well known distributions and provide a greater flexibility in modeling real data and to improve the goodness-of-fit of the generated family. For instance, Eugene *et al.* (2002) introduced a new class of distributions generated from the beta distribution. Zografos and Balakrishnan (2009) proposed the gamma generated family. Bourguignon *et al.* (2014) presented the Weibull-generated (*W-G*) family of distributions with two extra parameters, Kummer beta generalized family of distributions by Pescim *et al.* (2012), geometric exponential-Poisson family of distributions by Nadarajah *et al.* (2013), exponentiated T-X family of distributions by Alzaghal *et al.* (2013), weibull generalized family of distributins by Bourguignon *et al.* (2014); modified beta generalized family distributions by Nadarajah *et al.* (2014), kumaraswamy Weibul by Cordeiro *et al.* (2010) kumaraswamy gumbel by Cordeiro *et al.* (2012), Kumaraswamy Birnbaum-Saunders by Saulo and Bourguibnon (2012), Kumaraswamy Pareto by Bourguibnon *et al.* (2013), Kumaraswamy generalized Rayleigh by Gomes *et al.* (2014). Kumaraswamy inverse by Rayleigh Roges *et al.* (2014), Kumaraswamy modified inverse Weibull by Aryal and Elbatal (2015), Kumaraswamy Laplace by Aryal (2015), Kumaraswamy exponential-Weibull by Cordeiro *et al.* (2016), Kumaraswamy exponentiated inverse Rayleigh by Haq (2016); Olalekan *et al.* (2021); Ayed *et al.* (2022)

## 2.0 Some Existing Probability Distributions

### 2.1 Kumarasawamy Distribution

The pdf and cdf of Kumarasawamy distribution are as given in (1) and (2) respectively.

$$G(x; a, b) = 1 - (1 - x^a)^b, \quad 0 < x < 1, a > 0, b > 0. \quad (1)$$

$$g(x; a, b) = abx^{a-1} (1 - x^a)^{b-1}, \quad 0 < x < 1, a > 0, b > 0. \quad (2)$$

Where a and b are two shape parameters.

### 2.2 Zubair G Family of distribution

A family of life distributions, called the Zubair-G family was introduced by Zubair (2018). The benefit of using this family is that its cdf has a closed form solution and capable of data modeling with monotonic and non-monotonic failure rates. The CDF and PDF of the new family defined by Zubair (2018) for random Variable  $X$  is given in (3) and (4) respectively.

$$F(x, \alpha, \varphi) = \frac{\exp\{\alpha G(x; \varphi)^2\} - 1}{\exp(\alpha) - 1}, \quad \alpha > 0, x \in \mathbb{R}. \quad (3)$$

$$f(x, \alpha, \varphi) = \frac{2\alpha g(x; \varphi) G(x; \varphi) \exp\{\alpha G(x; \varphi)^2\}}{\exp(\alpha) - 1}, \quad \alpha > 0, x \in \mathbb{R}, \quad (4)$$

Where  $\varphi$  is vector of the baseline distribution parameter,  $\alpha$  is the parameter of Zubair G-family  $g(x; \varphi)$  and  $G(x; \varphi)$  are pdf and cdf of the baseline distribution respectively.

### 3.0 Proposed Distribution.

#### 3.1 Zubair-Kumarasawamy (Z-Kum) Distribution

In this paper, Zubair-Kumarasawamy distribution (Z-Kum) was proposed as version of Kumarasawamy distribution (Kum distribution) is obtained by putting (1) into (3). Then

$$F(x; \alpha, a, b)_{Z-Kum} = \frac{\exp\{\alpha(1 - (1 - x^a)^b)^2\} - 1}{\exp(\alpha) - 1} \quad 0 < x < 1, a > 0, b > 0. \quad (4)$$

The corresponding probability density function (pdf) of Zubair-Kumarasawamy (Z-Kum) distribution denoted by  $f(x; \alpha, a, b)_{Z-Kw}$  is obtained by differentiating (4) with respect to  $x$ .

$$f(x; \alpha, a, b) = \frac{d}{dx} \left[ \frac{\exp\{\alpha(1 - (1 - x^a)^b)^2\} - 1}{\exp(\alpha) - 1} \right] = \frac{1}{\exp(\alpha) - 1} \frac{d}{dx} \left[ \exp\{\alpha(1 - (1 - x^a)^b)^2\} - 1 \right] \quad (5)$$

put  $U = (1 - (1 - x^a)^b)^2$  and  $\frac{du}{dx} = abx^{a-1}(1 - x^a)^{b-1}$  in eq (5); we obtained

$$f(x; \alpha, a, b)_{Z-Kum} = \frac{2\alpha abx^{a-1}\{(1 - x^a)^{b-1}(1 - (1 - x^a)^b)\} \exp\{\alpha(1 - (1 - x^a)^b)^2\}}{\exp(\alpha) - 1} \quad (6)$$

$$0 < x < 1, a > 0, b > 0.$$

**Theorem 3.1:** To verify the proposed  $f(x; \alpha, a, b)_{Z-Kum}$  is true probability density function. Then

$$\int_{x=0}^1 f(x; \alpha, a, b)_{Z-Kum} dx = 1 \quad (7)$$

**Proof:** Let  $T_2 = \int_{x=0}^1 f(x; \alpha, a, b)_{Z-Kum} dx$

$$T_2 = \int_{x=0}^1 \frac{2\alpha ab x^{\alpha-1} \{(1-x^a)^{b-1} (1 - (1-x^a)^b)\} \exp\{\alpha(1 - (1-x^a)^b)^2\}}{\exp(\alpha) - 1} dx \quad (8)$$

Put  $u_2 = 1 - x^a$ ,  $\frac{du_2}{dx} = -ax^{\alpha-1} \Rightarrow dx = -\frac{du_2}{ax^{\alpha-1}}$  and  $0 < u_2 < 1$  in (8), we obtained

$$T_2 = \frac{2\alpha ab}{\exp(\alpha) - 1} \int_{u=0}^1 -\frac{(u_2^{b-1} - u_2^b) \exp[\alpha(1 - u_2^b)^2]}{a} du \quad (9)$$

by putting  $v = 1 - u_2^b$ ,  $\frac{dv}{du_2} = -bu_2^{b-1} \Rightarrow du_2 = -\frac{dv}{bu_2^{b-1}}$  and  $0 < v < 1$  in (9), we get

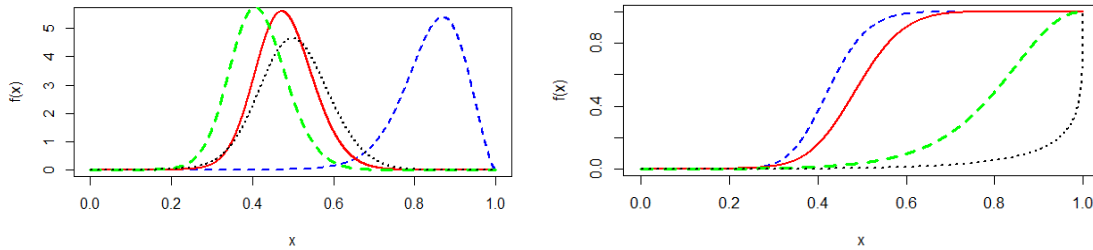
$$T_2 = \frac{2\alpha ab}{\exp(\alpha) - 1} \cdot \frac{1}{a} \left( \int_{v=0}^1 -\frac{\exp(\alpha v^2)(v)}{b} dv \right) \quad (10)$$

Let  $w = \alpha v^2$ ,  $\frac{dw}{dv} = 2\alpha v \Rightarrow dv = \frac{dw}{2ab}$  and  $0 < w < \alpha$  in (10), we get

$$T_2 = \frac{2\alpha ab}{\exp(\alpha) - 1} \cdot \frac{1}{ab} \left[ \frac{\exp(w)}{2\alpha} \right]_0^\alpha = \frac{2\alpha ab}{\exp(\alpha) - 1} \cdot \frac{\exp(\alpha) - 1}{2\alpha ab} = 1 \quad (11)$$

Hence, the proposed model of  $(f(x; \alpha, a, b)_{Z-Kum})$  is true probability density function.

Figures 1 and 2 below displayed the plots of the *pdf* and *cdf* of the Z-Bx distribution for some selected parameter values respectively.



**Figure 1: Plot of Z-Kw PDF**

**Figure 2: Plot of Z-Kw CDF**

**4.0 Properties of the Proposed model.**

This section studies the statistical properties Z-Kum distribution such as the quintile function, order statistics and moments. Survival functions, Hazard rate function, Reversed Hazard rate function, Cumulative hazard rate functions are also discussed in details.

**4.1 Quantile Function of Proposed Distribution.**

The quantile function of Z-Kum distribution is obtained by inverting (4) as given in (12)

$$Q(U) = \left[ 1 - \left\{ 1 - \left( \frac{1}{\alpha} \ln(u(\exp(\alpha) - 1) + 1) \right)^{\frac{1}{2}} \right\}^{\frac{1}{b}} \right]^{\frac{1}{a}} \tag{12}$$

where,  $u \sim U(0,1)$

If we put  $u = 0.25, 0.5$  and  $0.75$  in (12) then first quartile, median and third quartile will be obtained.

**4.2 Order Statistics**

The order statistics and their moments have great importance in many statistical problems and applications in reliability analysis and life testing. Suppose  $X_1, X_2, \dots, X_n$  is a random sample from a distribution with pdf,  $f(x)$ , and let  $X_{1n}, X_{2n}, \dots, X_{in}$  denote the corresponding order statistic obtained from this sample. The  $i^{th}$  order statistic of the proposed distributions can be using (13)

$$f_{in}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) F(x)^{i-1} (1-F(x))^{n-i} \tag{13}$$

Using binomial expansion,

$$(1-F(x))^{n-i} = \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} F(x)^k \tag{14}$$

$$f_{in}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) F(x)^{i-1} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} F(x)^k \tag{15}$$

Inserting (3) and (6) in (15), the pdf of the  $i^{th}$  order statistics can be given as in (16)

$$f_{in}(x) = \frac{n!}{(i-1)!(n-i)!} \left( \frac{2\alpha abx^{a-1} \left( (1-x^a)^{b-1} - (1-x^a)^{2b-1} \right) \exp(\alpha(1-(1-x^a)^b)^2)}{\exp(\alpha) - 1} \right)$$

$$\times \left( \frac{\exp(\alpha(1-(1-x^a)^b)^2) - 1}{\exp(\alpha) - 1} \right)^{i-1} \sum_{k=0}^{n-i} (-1)^k \binom{n-1}{k} \left( \frac{\exp(\alpha(1-(1-x^a)^b)^2) - 1}{\exp(\alpha) - 1} \right)^k \quad (16)$$

### 4.3 Moments.

The moments of a random variable are important in statistical inference. They are used to investigate important characteristics of a distribution such as the measures of central tendency, measures of dispersion and measures of shapes. In this subsection, the  $r^{\text{th}}$  non-central moment of the Z-Kum random variable is derived.

$$\begin{aligned} \mu'_r &= E(X^r) = \int_{-\infty}^{\infty} X^r f(x, \phi) dx \\ &= \int_{-\infty}^{\infty} X^r \frac{2aabx^{a-1} \{(1-x^a)^{b-1} (1-(1-x^a)^b)\} \exp\{\alpha(1-(1-x^a)^b)^2\}}{\exp(\alpha) - 1} dx \end{aligned} \quad (17)$$

putting  $y = (1-x)^b$ ,  $\frac{dy}{dx} = -b(1-x^a)^{b-1} \cdot ax^{a-1} \Rightarrow dx = -\frac{dy}{b(1-x^a)^{b-1} ax^{a-1}}$  in (17)

$$\mu'_r = -\frac{2\alpha}{\exp(\alpha) - 1} \int_0^1 \left[ \left( y^{\frac{1}{b}} + 1 \right)^a \right]^r \exp[\alpha(1-y)^2] dy \quad (18)$$

put  $\left[ \left( y^{\frac{1}{b}} + 1 \right)^a \right]^r = \sum_{j=0}^r \binom{r}{j} \left( y^{\frac{1}{b}} \right)^j (1)^{r-j}$  and  $\exp[\alpha(1-y)^2] = \sum_{j=0}^{\infty} \frac{[\alpha(1-y)^2]^j}{j!}$  in (18)

$$\mu'_r = -\frac{2\alpha}{\exp(\alpha) - 1} \sum_{j=0}^r \sum_{j=0}^{\infty} \int_0^1 \left( y^{\left(\frac{1}{b}+1\right)-1} \right) (1-y)^{(2j+1)-1} dy \quad (19)$$

Hence,

$$\mu'_r = -\frac{2\alpha \alpha^j}{j! \exp(\alpha) - 1} \sum_{j=0}^r \sum_{j=0}^{\infty} B\left(\left(\frac{j}{b} + 1\right), 2(j+1)\right) \quad (20)$$

where  $B\left(\left(\frac{j}{b} + 1\right), 2(j+1)\right) = \int_0^1 \left( y^{\left(\frac{1}{b}+1\right)-1} \right) (1-y)^{(2j+1)-1} dy$

Where Mean =  $(x) = \mu'_1$ ,  $V(x) = \mu'_2 - \mu_1^2$ , Skewness =  $\frac{\mu'_3 - 3\mu'_2\mu'_1 + 4\mu_1^3}{(\mu'_2 - \mu_1^2)^{\frac{3}{2}}}$ ,

Kurtosis =  $\frac{\mu'_4 - 4\mu'_3\mu'_1 + 6\mu_1^2\mu'_2 - 3\mu_1^4}{(\mu'_3 - \mu_1^3)^2}$

#### 4.4 Moment generating function of proposed distribution.

**Theorem 4.1** Let X has a proposed distribution. Then the moment generating function is given by

$$M_X(t) = -\frac{2\alpha a^j}{j!\exp(\alpha)-1} \sum_{j=0}^r \sum_{r=0}^{\infty} \left(\frac{t^r}{r!}\right) B\left(\left(\frac{j}{b}+1\right), 2(j+1)\right) \quad (21)$$

**Proof:**

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x) dx \quad (22)$$

Using Taylor series expansion, the moment generating function can be given as

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} u_r \quad (23)$$

where  $u_r$  is the  $r^{\text{th}}$  non-central moment. Substituting the  $r^{\text{th}}$  non-central moment as in (20) gives the moment generating function of Z-kum distribution as

$$M_X(t) = -\frac{2\alpha a^j}{j!\exp(\alpha)-1} \sum_{j=0}^r \sum_{r=0}^{\infty} \left(\frac{t^r}{r!}\right) B\left(\left(\frac{j}{b}+1\right), 2(j+1)\right) \quad (24)$$

Hence the proof.

#### 4.5 Characteristic Function of proposed distribution

**Theorem 4.2** Let X has a Z-kum distribution. Then the characteristic function is given by

$$\phi_X(t) = -\frac{2\alpha a^j}{j!\exp(\alpha)-1} \sum_{j=0}^r \sum_{r=0}^{\infty} \left(\frac{(it)^r}{r!}\right) B\left(\left(\frac{j}{b}+1\right), 2(j+1)\right) \quad (25)$$

**Proof:**

$$\phi_X(t) = E(e^{itx}) = \int_0^{\infty} e^{itx} f(x) dx \quad (26)$$

Using Taylor series expansion, the moment generating function can be given as

$$\phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} u_r \quad (27)$$

where  $u_r$  is the  $r^{\text{th}}$  non-central moment. Substituting the  $r^{\text{th}}$  non-central moment as in (20) gives the characteristics function of Z-kum distribution as

$$\varphi_x(t) = -\frac{2\alpha a^j}{j! \exp(\alpha) - 1} \sum_{j=0}^r \sum_{i=0}^{\infty} \left( \frac{(it)^r}{r!} \right) B\left( \left( \frac{j}{b} + 1 \right), 2(j+1) \right) \quad (28)$$

Hence the proof.

#### 4.6 Survival Function of proposed distribution.

Using (39) the survival function of Z-Kum distribution is obtain as in (30)

$$S(x) = 1 - F(x) \quad (29)$$

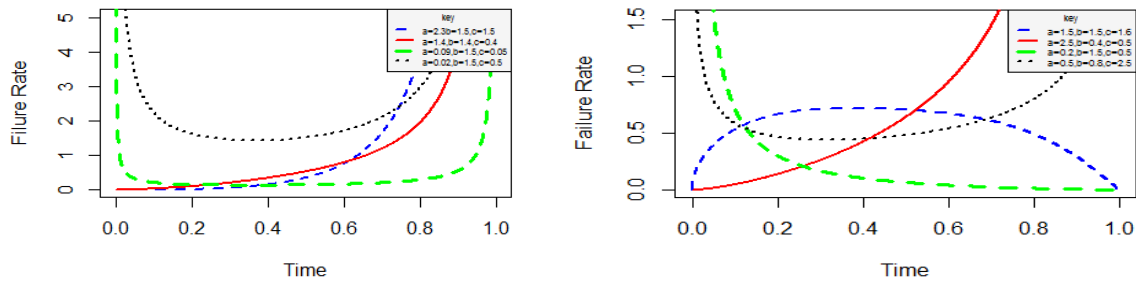
$$S(x) = \frac{\exp(\alpha) - \exp\left(\alpha + \alpha(-x^a + 1)^{2b} - 2\alpha(-x^a + 1)^b\right)}{\exp(\alpha) - 1} \quad (30)$$

#### 4.7 Hazard Rate Function of Z-Kum distribution.

The hazard rate functions of Z-Kum distribution is obtain using (31) as given in (32)

$$h(x) = \frac{f(x)}{S(x)} \quad (31)$$

$$h(x) = \frac{2\alpha abx^{a-1} \left( (1-x^a)^{b-1} (1 - (1-x^a)^b) \right) \exp\left(\alpha(1 - (1-x^a)^b)^2\right)}{\exp(\alpha) - \exp\left(\alpha + \alpha(-x^a + 1)^{2b} - 2\alpha(-x^a + 1)^b\right)} \quad (33)$$



**Figure 3: Plot of of Z-kum hr.**

Plot of the hazard rate function of Z-Kum distribution for specific sets of parameter values is of Bath-Tub shape, inverted Bath-Tub shape, increasing and decreasing shape.

#### 4.8 Reverse Hazard Rate Function of Z-Kum distribution.

The Reverses hazard rate functions of Z-Kum distribution is obtain using (33) as in (34)

$$r(x) = \frac{f(x)}{F(x)} \quad (33)$$

$$r(x) = \frac{2\alpha abx^{a-1} \left( (1-x^a)^{b-1} - (1-x^a)^{2b-1} \right) \exp\left(\alpha(1-(1-x^a)^b)^2\right)}{\exp\left(\alpha(1-(1-x^a)^b)^2\right) - 1} \quad (34)$$

#### 4.9 Cumulative Hazard Rate Function of Z-Kum distribution.

The cumulative hazard rate function of Z-Kum distribution is obtain using (35) as given in (36)

$$H(x) = -\log(1-F(x)) \quad (35)$$

$$H(x) = -\ln\left(\exp(\alpha) - \exp\left(\alpha + \alpha(-x^a + 1)^{2b} - 2\alpha(-x^a + 1)^b\right)\right) - \ln(\exp(\alpha) - 1) \quad (36)$$

### 5.0 Parameters Estimation and Simulation Studies of Z-Burr Distribution.

#### 5.1 Parameter Estimation

To illustrate the applications of the developed distributions with regards to modeling real data sets, it is vital to develop estimators for estimating the parameters of the distributions. In this section, estimators are developed for estimating the parameters of the new distributions using the well known method of maximum likelihood estimate (MLE).

Let  $X_1, X_2, \dots, X_n$  be a random sample from Z-Kum distribution with unknown parameter vector  $\phi = (\alpha, a, b)^T$ , the likelihood function of the distribution is obtain using (37)

$$L(\phi) = \prod_{i=1}^n f(x_i, \phi) = \prod_{i=1}^n \left[ \frac{2\alpha abx_i^{a-1} \{ (1-x_i^a)^{b-1} (1 - (1-x_i^a)^b) \} \exp\left\{ \alpha(1 - (1-x_i^a)^b)^2 \right\}}{\exp(\alpha) - 1} \right] \quad (39)$$

$$L(\phi) = \left[ \frac{2\alpha ab}{\exp(\alpha) - 1} \right]^n \left[ \left( \prod_{i=1}^n x_i^{a-1} \right) \left( \prod_{i=1}^n (1-x_i^a)^{b-1} \right) \prod_{i=1}^n (1-x_i^a)^{2b-1} \exp\left\{ \alpha^n \sum_{i=1}^n (1 - (1-x_i^a)^b)^2 \right\} \right]$$

Taking natural log likelihood is thus obtained as

$$\begin{aligned} \ell(\phi) &= \log\{L(\phi)\} \\ &= \left[ \begin{aligned} &n(\ln 2 + \ln \alpha + \ln b) + (a-1) \sum_{i=1}^n \ln x_i + (b-1) \left( \sum_{i=1}^n \ln(1-x_i^a) \right) \\ &+ (2b-1) \left( \sum_{i=1}^n \ln(1-x_i^a) \right) + \alpha^n \sum_{i=1}^n (1 - (1-x_i^a)^b)^2 - n \ln(\exp(\alpha) - 1) \end{aligned} \right] \dots (40) \end{aligned}$$

After differentiating (40) with respect to parameters  $\alpha$ ,  $a$ , and  $b$ , we get

$$\begin{aligned} \frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} + n\alpha^{n-1} \sum_{i=1}^n (1 - (1-x_i^a)^b)^2 - \frac{n \exp(\alpha)}{\exp(\alpha) - 1} \\ &= \frac{n}{\alpha} \left[ 1 + \alpha^n \sum_{i=1}^n (1 - (1-x_i^a)^b)^2 \right] - \frac{n \exp(\alpha)}{\exp(\alpha) - 1} = 0 \end{aligned} \quad (41)$$

$$\begin{aligned} \frac{\partial \ell}{\partial a} &= \left[ \begin{aligned} &\frac{n}{a} + \sum_{i=1}^n \ln x_i - (b-1) \sum_{i=1}^n \frac{x_i^a \ln a}{1-x_i^a} - (2b-1) \sum_{i=1}^n \frac{\ln a}{1-x_i^a} \\ &+ 2\alpha^n b \sum_{i=1}^n [1 - (1-x_i^a)^b] (1-x_i^a)^{b-1} x_i^a \ln a \end{aligned} \right] \\ &= \left[ \begin{aligned} &\frac{n}{a} + \sum_{i=1}^n \ln x_i - (b-1) \sum_{i=1}^n \frac{x_i^a \ln a}{1-x_i^a} - (2b-1) \sum_{i=1}^n \frac{\ln a}{1-x_i^a} \\ &+ 2\alpha^n b \sum_{i=1}^n \left[ \frac{(1-x_i^a)^b - (1-x_i^a)^{2b}}{1-x_i^a} \right] x_i^a \ln a \end{aligned} \right] = 0 \end{aligned} \quad (42)$$

$$\begin{aligned} \frac{\partial \ell}{\partial b} &= \frac{n}{b} + \sum_{i=1}^n \ln(1-x_i^a) + 2 \sum_{i=1}^n \ln(1-x_i^a) - 2\alpha^n \sum_{i=1}^n (1 - (1-x_i^a)^b) (1-x_i^a)^b \ln(1-x_i^a) \\ &= \frac{n}{b} + 3 \sum_{i=1}^n \ln(1-x_i^a) - 2\alpha^n \sum_{i=1}^n [(1-x_i^a)^b - (1-x_i^a)^{2b}] \ln(1-x_i^a) = 0 \end{aligned} \quad \dots (43)$$

Equations (41), (42) and (43) cannot be solved analytically, statistical software like R can be used to simultaneously solve them numerically using iterative methods. Solutions of these equations provides the maximum likelihood estimate  $\hat{\phi} = (\hat{\alpha}, \hat{a}, \hat{b})$  of  $\phi = (\alpha, a, b)^T$

## 5.2. Simulation Studies

The performance of the maximum likelihood estimates for the Z-Kum distribution parameters was evaluated using Monte Carlo simulation for a three parameter combinations. Different sample sizes ( $n = 50, 75$  and  $100$ ) and some selected parameter values ( $\alpha = 0.09, a = 0.08, b = 0.09$ ) were used to perform the simulation. Result of the simulation is presented in the table below.

**Table 1. Average MLEs, Variance and MSE of the MLEs of parameters of Z-Kum distribution with actual parameter values ( $\alpha = 0.09, a = 0.08, b = 0.09$ )**

N	Estimates			Variance			MSE		
	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$
<b>50</b>	0.0724	0.0880	0.0980	0.0012	0.0003	0.0003	0.0015	0.5071	0.0003
<b>75</b>	0.0735	0.0874	0.0975	0.0011	0.0002	0.0002	0.0014	0.5069	0.0003
<b>100</b>	0.0767	0.0860	0.0964	0.0009	0.0002	0.0002	0.0011	0.5061	0.0002

## 5.3 Model Comparison and Selection Criteria

To show how applicable and flexible our proposed model is, its performance is compared with other established models with reference to information lost. So, we tend to use information criteria techniques and goodness-of-fit statistics that correct model for complexity, to constrain the model from over fitting to assess the most effective model from a range different models which can have different number of parameters.

In this case, we will consider the generally well known criteria such as Akaike Information Critareion (AIC), the Bayesian Information Criterion (BIC), the Consistant Akaike Information Cretarion (CAIC) and Hannan-Quinn Information Criterion (HQIC) and illustrate the flexibility and applicability Z-Kum distribution, using three (3) real life data set.

### Data set 1

This data was used and analyze by Musa et al. (2021)

0.68879, 0.50813, 0.66621, 0.74526, 0.86947, 0.88076, 0.84688, 0.91463, 0.75655, 0.55329, 0.79042, 0.82429, 0.92593, 0.80172, 0.79042, 0.83559, 0.68879, 0.74526, 0.80172, 0.93722,

0.85818, 0.98238, 0.29359, 0.99368, 0.67751, 0.80172, 0.93722, 0.63234, 0.64363, 0.73397,  
0.89205, 0.64363, 0.77913, 0.41779, 0.58717, 0.88076, 0.91463, 0.80172, 0.68879, 0.72267,  
0.90334, 0.76784, 0.93722, 0.21454, 0.38392

## Data set 2

This data was used and analyze by Musa et al (2021)

0.42909, 0.83559, 0.85818, 0.79042, 0.67751, 0.99368, 0.88076, 0.88076, 0.93722, 0.74526,  
0.76784, 0.82429, 0.77913, 0.68879, 0.98238, 0.71138, 0.76784, 0.51942, 0.77913, 0.70009,  
0.54200, 0.75655, 0.86947, 0.99368, 0.76784, 0.92593, 0.80172, 0.46296, 0.76784, 0.76784,  
0.48555, 0.89205, 0.36134, 0.65492, 0.79042, 0.84688, 0.80172, 0.64363, 0.42909, 0.74526,  
0.80172, 0.48555, 0.67751, 0.75655, 0.47425, 0.94851, 0.92593, 0.63234, 0.93722, 0.73397,  
0.71138, 0.90334, 0.72267, 0.99368, 0.63234, 0.45167, 0.65492, 0.92593, 0.41779, 0.72267,  
0.75655, 0.47425, 0.94851, 0.48555, 0.63234, 0.54201, 0.89205, 0.80172, 0.65492, 0.46296,  
0.75655, 0.84688, 0.47425, 0.65492, 0.51942, 0.39521, 0.91463, 0.37263, 0.66621, 0.49684,  
0.86947, 0.82429, 0.63234, 0.41779, 0.74526, 0.80172, 0.12421, 0.16938, 0.15808, 0.09033,  
0.88076, 0.37263, 0.66621, 0.18067, 0.85818, 0.83559, 0.64363, 0.49684, 0.76784, 0.77913,  
0.89205, 0.35005, 0.99368, 0.60976, 0.75655, 0.77913, 0.65492, 0.39521, 0.74526, 0.82429,  
0.92593, 0.97109, 0.68879, 0.94851, 0.7904, 0.99368, 0.71138, 0.49684, 0.06775, 0.91463,  
0.97109, 0.91463, 0.86947, 0.76784, 0.86947, 0.79042, 0.79042, 0.41779, 0.77913, 0.99368,  
0.51942, 0.67751, 0.84688, 0.80172, 0.90334, 0.80172, 0.90334, 0.71138, 0.63234, 0.74526,  
0.54201, 0.39295, 0.76784, 0.71138, 0.67751, 0.63234, 0.77913, 0.85818, 0.63234, 0.99368,  
0.55329, 0.75655, 0.82429, 0.37263, 0.56459, 0.15808, 0.45167, 0.64363, 0.67751, 0.99368,  
0.92593, 0.67751, 0.84689, 0.68879, 0.76784, 0.50813, 0.68879, 0.82429, 0.67751, 0.28229,  
0.49684, 0.62105, 0.66621, 0.62105, 0.86947, 0.89205, 0.68879, 0.50813, 0.66621, 0.74526,  
0.86947, 0.88076, 0.84688, 0.91463, 0.75655, 0.55329, 0.79042, 0.82429, 0.92593, 0.80172,  
0.79042, 0.83559, 0.68879, 0.74526, 0.80172, 0.93722, 0.85818, 0.98238, 0.29359, 0.99368,  
0.67751, 0.80172, 0.93722, 0.63234, 0.64363, 0.73397, 0.89205, 0.64363, 0.77913, 0.41779,  
0.58717, 0.88076, 0.91463, 0.80172, 0.68879, 0.72267, 0.90334, 0.76784, 0.93722, 0.21454,  
0.38392

## Data set 3

The following data was used and analyze by Saboor et al (2021). It consist 48 rock samples from petroleum reservoir obtained from the measurements on petroleum rock samples

0.0903296, 0.2036540, 0.2043140, 0.2808870, 0.1976530, 0.3286410, 0.1486220, 0.1623940, 0.2627270, 0.1794550, 0.3266350, 0.2300810, 0.1833120, 0.1509440, 0.2000710, 0.1918020, 0.1541920, 0.4641250, 0.1170630, 0.1481410, 0.1448100, 0.1330830, 0.2760160, 0.4204770, 0.1224170, 0.2285950, 0.1138520, 0.2252140, 0.1769690, 0.2007440, 0.1670450, 0.2316230, 0.2910290, 0.3412730, 0.4387120, 0.2626510, 0.1896510, 0.1725670, 0.2400770, 0.3116460, 0.1635860, 0.1824530, 0.1641270, 0.1534810, 0.1618650, 0.2760160, 0.2538320, 0.2004470

**Table 2.** Performance of Z-Kum distribution’s goodness of fit using data set 1

	MODEL		
	Z-KUM	LIB-KUM	KUM
<i>AIC</i>	-101.1618	-97.37334	-79.19838
<i>CAIC</i>	-99.82845	-96.04001	-77.86504
<i>BIC</i>	-97.88865	-94.10021	-75.92525
HQIC	-100.3907	-96.60229	-78.42733
Rank	<b>1</b>	<b>2</b>	<b>3</b>

**Table 3.** Performance of Z-Kum distribution’s goodness of fit using data set 2

	MODEL		
	Z-KUM	LIB-KKUM	KUM
<i>AIC</i>	-977.89	-670.72	-816.81
<i>CAIC</i>	-977.78	-670.61	-816.70
<i>BIC</i>	-967.70	-660.52	-806.61
HQIC	-973.78	-666.61	-812.69
Rank	<b>1</b>	<b>2</b>	<b>3</b>

**Table 4.** Performance of Z-Kum distribution's goodness of fit using data set 3

	MODEL		
	Z-KUM	LIB-KUM	KUM
<i>AIC</i>	-134.4426	-126.1913	-118.3165
<i>CAIC</i>	-134.3220	-126.0807	-118.2060
<i>BIC</i>	-124.2481	-115.9968	-108.1221
HQIC	-130.3263	-122.0749	-114.2002
Rank	<b>1</b>	<b>2</b>	<b>3</b>

It can be seen from Table 2, 3 and 4 that based on the values of the information criterion from the three different real life data sets, Z-Kum distribution having the less values performed better than the other two distributions in term of fitting/modeling real life data.

## 6. Discussion and Conclusion

In this paper, we developed new three parameter model called Zubair- Kumaraswamy (Z-Kum) distribution. The extension was done using Zubair G-Family (2018) of continuous probability distribution was used to extend well known Kumaraswamy distribution to make it more flexible in modeling and predicting real world phenomenon. Some basic structural properties of the new distributions like Quantile functions, moments, moment generating functions, characteristics functions and order statistics were obtained. Survival function, hazard function, reversed hazard rate function and a cumulative hazard rate function was also obtained. Behaviour of the hazard rate plot exhibit increase, decrease, Bathtub and inverted Bathtub shape. Maximum likelihood estimate was used to estimate the Z-Kum distribution parameters, Monte Carlo simulation also was carried out to evaluate the performance of MLE in estimating our distribution parameters. Result of the simulation studies revealed that as the sample size increases, the estimate values approaches actual parameter values, while the values of mean square errors approaches zero, this indicates that MLE is good for the estimation of our distribution parameters. To show how flexible and more efficient our proposed model is over some existing distributions, we compare the model with the other fitted existing models. Analytical measure of goodness of fit of some information criterion such as Akaike information criterion (AIC), consistent Akaike information

criterion (CAIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC) was considered using three real life data sets. From the results obtained, it is evident that our proposed model give better fit than the other competing models and is therefore, more flexible in modeling and predicting real world phenomenon.

## REFERENCES

- Aryal, G., & Elbatal, I. (2015). Kumaraswamy modified inverse Weibull distribution: theory and application, *Application of Mathematical Information Science*, 9,651-660.
- Ayed, R. A., Alanzi, M., Qaisar, R., Tahir, M. H., Waqas, S., and Farrukh, J. (2022) A New Modified Kumaraswamy Distribution: Acturial Measures and Applications. *Journal of Mathematics. Volum 2022* <https://doi.org/10.1155/2022/4288286>
- Bourguignon, M., Silva, R. B., L., Zea, M., & Cordeiro, G., M. (2013) The Kumaraswamy Pareto distribution, *Journal of Statistical Theory and Applications*, 12, 129-144.
- Bourguignon, M., Silva, R.B., & Cordeiro, G., M.( 2014) The Weibull Generalized family of probability distributions. *Journal of Data Science*, 12,53–68.
- Cordeiro, G., M., Saboor, A., Khan, M., N., Ozel, G., & Pascoa, M., A. (2016). The Kumaraswamy Exponential-Weibull Distribution: Theory and Applications, *Hacetatepe journal of mathematics and statistics*, 45, 1203-1229.
- Eugene, N., Lee, C. and Famoye, F. (2002). Beta-normal distribution and its applications. *Communications in Statistics-Theory and methods*, 31, 497-512.
- Haq, M., A. (2016). Kumaraswamy Exponentiated Inverse Rayleigh Distribution, *Mathematical Theory and Modeling*, 6, 93-104.
- Laba H., & Subrata, C. (2016) The Marshall-Olkin-Kumaraswamy-G family of distributions
- Lishamol, T., & Jiju, G. (2018) The marshall-olkin ikum distribution *Biometrics & Biostatistics International Journa*,.7, (1) ,10-16
- Musa, Y., Muhammad, A.B., Usman, U.,and Zakari, Y.(2021). On the properties of Burr X-Topp Leone Distribution and its application. *Lapai Journal of Applied and Natural Sciences*, 6(1),114-120
- Nadarajah, S.,Teimouri, M., & Shih, S. H. (2014). Modified beta distributions. *Sankhya*. 76, 19–48.
- Olalekan, A. B., Sani I. D., Abukakar Y. and Haruna M. J. (2021) A Type II Half Logistic Exponentiated-G Family of Distributions with Applications to Survival Analysis. *FUDMA Journal of Sciences (FJS) Vol. 5 No. 3, September, 2021, pp 177 - 190*
- Saboor, A., Iqbal, Z., Hanif, M., & Ahmad, M. (2021). Libby-Novick kumaraswamy distribution with its properties and applications. *International Journal of Analysis and Applications*,19(3), 405-439.
- Zubair, A. (2018). The Zubair-G family of distributions: Properties and applications. *Annals of Data Science*.