

FIXED POINTS OF KANNAN INTERPOLATIVE, RIECH INTERPOLATIVE, AND RATIONAL CONTRACTIONS IN A-METRIC SPACES

ABSTRACT: (λ, α) -interpolative Kannan contraction, (λ, α, β) -interpolative Kannan contraction, $(\lambda, \alpha, \beta, \gamma)$ -interpolative Riech contraction and (λ, α, β) -interpolative Dass-Gupta rational contraction are presented in this study. Furthermore, we prove a few fixed-point theorems for interpolative contractions in complete A-metric spaces. These theorems also extend and apply to an A-metric setting several interesting results from metric fixed-point theory.

Keywords: fixed-point; interpolative contraction; A-metric spaces.

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1. Introduction and Preliminaries

Fixed point theory is a fascinating field of research in analysis and topology. In 1922, Banach [12] proposed an important result that became known as the Banach contraction principle. Its relevance to metric fixed-point theory was investigated. Let (D, d) be a full metric space and Y a self-map on a nonempty set D . If there exists a constant $c \in [0, 1)$ such that

$$d(Y\sigma, Y\zeta) \leq c d(\sigma, \zeta), \text{ for all } \sigma, \zeta \in D, (1)$$

then it possesses a unique fixed point in D . The Banach contraction principle was then widely generalized in the literature (see [13–14]). Both pure and applied mathematics make extensive use of it. Kannan [2] defined a new variation of this theory in 1968 and eliminated the continuity condition from it.

Theorem 1.1 (see [2]). *Let (D, d) be a complete metric space and a self-map $Y: D \rightarrow D$ be a Kannan contraction mapping, i.e.,*

$$d(Y\sigma, Y\zeta) \leq k[d(\sigma, Y\sigma) + d(\zeta, Y\zeta)], (2)$$

for all $\sigma, \zeta \in D$, where $k \in [0, 1/2)$. Then, Y admits a unique fixed point in D .

The idea of b-metric space, which is a generalization of the well-known Banach contraction mapping principle, was first presented by Bakhtin [9] in 1989. In 1993, Czerwik [10,11] expanded on the idea of b-metric space. “Kannan fixed-point theorem is the first significant variant of the outstanding result of Banach on the metric fixed-point theory” [12]. The concept of A-metric space was first developed by Abbas et al. [1] in 2015.

Definition 1.2(see [1]) Let D be a nonempty set. A mapping $A: D^n \rightarrow [0, +\infty)$ is called an A-metric on D if and only if for all $\sigma_i, a \in D, i = 1, 2, 3, \dots, n$: the following conditions hold:

$$(A1). A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_{n-1}, \sigma_n) \geq 0,$$

$$(A2). A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_{n-1}, \sigma_n) = 0 \text{ if and only if } \sigma_1 = \sigma_2 = \dots = \sigma_{n-1} = \sigma_n,$$

$$(A3). A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_{n-1}, \sigma_n) \leq A(\sigma_1, \sigma_1, \sigma_1, \dots, (\sigma_1)_{n-1}, a) \\ + A(\sigma_2, \sigma_2, \sigma_2, \dots, (\sigma_2)_{n-1}, a) \\ + A(\sigma_3, \sigma_3, \sigma_3, \dots, (\sigma_3)_{n-1}, a) + \dots \\ + A(\sigma_{n-1}, \sigma_{n-1}, \sigma_{n-1}, \dots, (\sigma_{n-1})_{n-1}, a) \\ + A(\sigma_n, \sigma_n, \sigma_n, \dots, (\sigma_n)_{n-1}, a)]$$

The pair (D, A) is called an A-metric space.

The following is the intuitive geometric example for A-metric spaces.

Example 1.3(see [1]) Let $D = [1, +\infty)$. Define $A: D^n \rightarrow [0, +\infty)$ by

$$A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_{n-1}, \sigma_n) = \sum_{i=1}^n \sum_{i < j} |\sigma_i - \sigma_j|$$

for all $\sigma_i \in D, i = 1, 2, \dots, n$.

Example 1.4(see [1]) Let $D = \mathbb{R}$. Define $A: D^n \rightarrow [0, +\infty)$ by

$$A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_{n-1}, \sigma_n) = \left| \sum_{i=n}^2 \sigma_i - (n-1)\sigma_1 \right| \\ + \left| \sum_{i=n}^3 \sigma_i - (n-2)\sigma_2 \right| + \dots$$

$$\begin{aligned}
& + \left| \sum_{i=n}^{n-3} \sigma_i - 3\sigma_{n-3} \right| \\
& + \left| \sum_{i=n}^{n-2} \sigma_i - 2\sigma_{n-2} \right| \\
& + |\sigma_n - \sigma_{n-1}|
\end{aligned}$$

for all $\sigma_i \in D, i = 1, 2, \dots, n$.

Lemma 1.5(see [1]) Let (D, A) be an A -metric space. Then for all $\sigma, \varsigma \in D$,

$$A(\sigma, \sigma, \sigma, \sigma, \dots, (\sigma)_{n-1}, \varsigma) = A(\varsigma, \varsigma, \varsigma, \varsigma, \dots, (\varsigma)_{n-1}, \sigma)$$

Lemma 1.6(see [1]) Let (D, A) be an A -metric space. Then for all $\sigma, \varsigma, z \in D$,

$$\begin{aligned}
A(\sigma, \sigma, \sigma, \sigma, \dots, (\sigma)_{n-1}, z) & \leq (n-1)A(\sigma, \sigma, \sigma, \sigma, \dots, (\sigma)_{n-1}, \varsigma) \\
& + A(z, z, z, z, \dots, (z)_{n-1}, \varsigma)
\end{aligned}$$

and

$$\begin{aligned}
A(\sigma, \sigma, \sigma, \sigma, \dots, (\sigma)_{n-1}, z) & \leq (n-1)A(\sigma, \sigma, \sigma, \sigma, \dots, (\sigma)_{n-1}, \varsigma) \\
& + A(\varsigma, \varsigma, \varsigma, \varsigma, \dots, (\varsigma)_{n-1}, z)
\end{aligned}$$

Lemma 1.7(see [1]) Let (D, A) be an A -metric space. Then $(D \times D, D_A)$ is an A -metric space on $D \times D$, where D_A is given by for all $\sigma_i, \varsigma_j \in D, i, j = 1, 2, \dots, n$:

$$\begin{aligned}
& D_A((\sigma_1, \varsigma_1), (\sigma_2, \varsigma_2), (\sigma_3, \varsigma_3), \dots, (\sigma_n, \varsigma_n)) \\
& = A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n) + A(\varsigma_1, \varsigma_2, \varsigma_3, \dots, \varsigma_n).
\end{aligned}$$

Definition 1.7 (see [1]) Let (D, A) be an A -metric space. Then

1. A sequence $\{\sigma_k\}$ is called convergent to σ in (\mathfrak{D}, A) if

$$\lim_{k \rightarrow +\infty} A(\sigma_k, \sigma_k, \sigma_k, \sigma_k, \dots, (\sigma_k)_{n-1}, \sigma) = 0.$$

That is, for each $\epsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k \geq n_0$, we have

$$A(\sigma_k, \sigma_k, \sigma_k, \sigma_k, \dots, (\sigma_k)_{n-1}, \sigma) \leq \epsilon$$

and we write $\lim_{k \rightarrow +\infty} \sigma_k = \sigma$.

2. A sequence $\{\sigma_k\}$ is called Cauchy in (D, A) if

$$\lim_{k, m \rightarrow +\infty} A(\sigma_k, \sigma_k, \sigma_k, \sigma_k, \dots, (\sigma_k)_{n-1}, \sigma_m) = 0.$$

That is, for each $\epsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k, m \geq n_0$, we have

$$A(\sigma_k, \sigma_k, \sigma_k, \sigma_k, \dots, (\sigma_k)_{n-1}, \sigma_m) \leq \epsilon.$$

3. (D, A) is said to be complete if every Cauchy sequence in (D, A) is a convergent.

Lemma 1.8 (see [1]) Let (D, A) be an A -metric space. If the sequence $\{\sigma_k\}$ in D converges to σ , then σ is unique.

Lemma 1.9 (see [1]) Every convergent sequence in A -metric space (D, A) is a Cauchy sequence.

This study defines and discusses interpolative contraction of the Kannan, Riech, and rational types within the context of A -metric space. Furthermore, the concept of interpolation is used to establish a few popular fixed-point results. These theorems also extend and apply to an A -metric setting a number of interesting results from metric fixed-point theory.

2. Main Result

We begin by defining the terms below.

Definition 2.1 Let (D, A) be an A -metric space. A self-map $Y: D \rightarrow D$ is called a (λ, α) -interpolative Kannan contraction, if there exist $\lambda \in [0, 1)$, $\alpha \in (0, 1)$ such that

$$A\left(\underbrace{Y\sigma, Y\sigma, \dots, Y\sigma}_{(n-1) \text{ times}}, Y\zeta\right) \leq \lambda \left(A\left(\underbrace{\sigma, \sigma, \dots, \sigma}_{(n-1) \text{ times}}, Y\sigma\right) \right)^\alpha \left(A\left(\underbrace{\zeta, \zeta, \dots, \zeta}_{(n-1) \text{ times}}, Y\zeta\right) \right)^{1-\alpha} \quad (3)$$

for all $\sigma, \zeta \in D$, with $\sigma \neq \zeta$.

Definition 2.2 Let (D, A) be an A- metric space. A self-map $Y: D \rightarrow Dis$ (λ, α, β) - interpolative Kannan contraction, if there exist $\lambda \in [0, 1), \alpha, \beta \in (0, 1), \alpha + \beta < 1$ such that

$$A\left(\underbrace{Y\sigma, Y\sigma, \dots, Y\sigma}_{(n-1) \text{ times}}, Y\zeta\right) \leq \lambda \left(A\left(\underbrace{\sigma, \sigma, \dots, \sigma}_{(n-1) \text{ times}}, Y\sigma\right) \right)^\alpha \left(A\left(\underbrace{\zeta, \zeta, \dots, \zeta}_{(n-1) \text{ times}}, Y\zeta\right) \right)^\beta \quad (4)$$

for all $\sigma, \zeta \in D$, with $\sigma \neq \zeta$.

Definition 2.3 Let (D, A) be an A- metric space. A self-map $Y: D \rightarrow Dis$ $(\lambda, \alpha, \beta, \gamma)$ - interpolative Reich contraction, if there exist $\lambda \in [0, 1), \alpha, \beta, \gamma \in (0, 1), \alpha + \beta + \gamma < 1$ such that

$$A\left(\underbrace{Y\sigma, Y\sigma, \dots, Y\sigma}_{(n-1) \text{ times}}, Y\zeta\right) \leq \lambda \left(A\left(\underbrace{\sigma, \sigma, \dots, \sigma}_{(n-1) \text{ times}}, \zeta\right) \right)^\alpha \left(A\left(\underbrace{\sigma, \sigma, \dots, \sigma}_{(n-1) \text{ times}}, Y\sigma\right) \right)^\beta \left(A\left(\underbrace{\zeta, \zeta, \dots, \zeta}_{(n-1) \text{ times}}, Y\zeta\right) \right)^\gamma \quad (5)$$

for all $\sigma, \zeta \in D$, with $\sigma \neq \zeta$.

Definition 2.4 Let (D, A) be an A- metric space. A self-map $Y: D \rightarrow Dis$ (λ, α, β) - interpolative Dass-Gupta rational contraction, if there exist $\lambda \in [0, 1), \alpha, \beta \in (0, 1), \alpha + \beta < 1$ such that

$$A\left(\underbrace{Y\sigma, Y\sigma, \dots, Y\sigma}_{(n-1) \text{ times}}, Y\zeta\right) \leq \lambda \left(A\left(\underbrace{\sigma, \sigma, \dots, \sigma}_{(n-1) \text{ times}}, \zeta\right) \right)^\alpha \left(\frac{\left[1 + A\left(\underbrace{\sigma, \sigma, \dots, \sigma}_{(n-1) \text{ times}}, Y\sigma\right) \right] A\left(\underbrace{\zeta, \zeta, \dots, \zeta}_{(n-1) \text{ times}}, Y\zeta\right)}{1 + A\left(\underbrace{\sigma, \sigma, \dots, \sigma}_{(n-1) \text{ times}}, \zeta\right)} \right)^\beta \quad (6)$$

for all $\sigma, \zeta \in D$, with $\sigma \neq \zeta$.

Our first main result as follows.

Theorem 2.5 Let (D, A) be a complete A-metric space. Let $Y: D \rightarrow D$ be a (λ, α) - interpolative Kannan contraction. Then Y has a unique fixed point.

Proof. Let $\sigma_0 \in D$ be initial point. Define $s\sigma_{n+1} = Y\sigma_n, \forall n \in \mathbb{N}$. Obviously, if $\exists n_0 \in \mathbb{N}$ for which $\sigma_{n_0+1} = \sigma_{n_0}$, then $Y\sigma_{n_0} = \sigma_{n_0}$, and the proof is finished. Thus, we suppose that $\sigma_{n+1} \neq \sigma_n$ for each $n \in \mathbb{N}$. Thus, by (3), we have

$$\begin{aligned}
A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1}\right) &= A\left(\underbrace{Y\sigma_{n-1}, Y\sigma_{n-1}, \dots, Y\sigma_{n-1}}_{(n-1) \text{ times}}, Y\sigma_n\right) \\
&\leq \lambda \left(A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, Y\sigma_{n-1}\right) \right)^\alpha \left(A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, Y\sigma_n\right) \right)^{1-\alpha} \\
&= \lambda \left(A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n\right) \right)^\alpha \left(A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1}\right) \right)^{1-\alpha}
\end{aligned}$$

The last inequality gives

$$\left(A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1}\right) \right)^\alpha \leq \lambda \left(A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n\right) \right)^\alpha \quad (7)$$

Since $\alpha < 1$, we have

$$\begin{aligned}
A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1}\right) &\leq \lambda^{\frac{1}{\alpha}} A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n\right) \\
&\leq \lambda A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n\right)
\end{aligned}$$

and then

$$\begin{aligned}
A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1}\right) &\leq \lambda A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n\right) \\
&\leq \lambda^2 A\left(\underbrace{\sigma_{n-2}, \sigma_{n-2}, \dots, \sigma_{n-2}}_{(n-1) \text{ times}}, \sigma_{n-1}\right) \\
&\leq \dots \leq \lambda^n A\left(\underbrace{\sigma_0, \sigma_0, \dots, \sigma_0}_{(n-1) \text{ times}}, \sigma_1\right) \quad (8)
\end{aligned}$$

For all $n, m \in \mathbb{N}$ and $n < m$, we have

$$\begin{aligned}
& A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_m\right) \leq (n-1)A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1}\right) \\
& \quad + A\left(\underbrace{\sigma_m, \sigma_m, \dots, \sigma_m}_{(n-1) \text{ times}}, \sigma_{n+1}\right) \\
& = (n-1)A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1}\right) \\
& \quad + A\left(\underbrace{\sigma_{n+1}, \sigma_{n+1}, \dots, \sigma_{n+1}}_{(n-1) \text{ times}}, \sigma_m\right) \\
& \leq (n-1)A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1}\right) \\
& \quad + (n-1)A\left(\underbrace{\sigma_{n+1}, \sigma_{n+1}, \dots, \sigma_{n+1}}_{(n-1) \text{ times}}, \sigma_{n+2}\right) \\
& \quad + A\left(\underbrace{\sigma_m, \sigma_m, \dots, \sigma_m}_{(n-1) \text{ times}}, \sigma_{n+2}\right) \\
& \leq (n-1)A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1}\right) \\
& \quad + (n-1)A\left(\underbrace{\sigma_{n+1}, \sigma_{n+1}, \dots, \sigma_{n+1}}_{(n-1) \text{ times}}, \sigma_{n+2}\right) \\
& \quad + A\left(\underbrace{\sigma_{n+2}, \sigma_{n+2}, \dots, \sigma_{n+2}}_{(n-1) \text{ times}}, \sigma_m\right) \\
& \leq (n-1)A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1}\right) \\
& \quad + (n-1)A\left(\underbrace{\sigma_{n+1}, \sigma_{n+1}, \dots, \sigma_{n+1}}_{(n-1) \text{ times}}, \sigma_{n+2}\right) + \dots
\end{aligned}$$

$$\begin{aligned}
& +(n-1)A\left(\underbrace{\sigma_{m-2}, \sigma_{m-2}, \dots, \sigma_{m-2}}_{(n-1) \text{ times}}, \sigma_{m-1}\right) \\
& +A\left(\underbrace{\sigma_{m-1}, \sigma_{m-1}, \dots, \sigma_{m-1}}_{(n-1) \text{ times}}, \sigma_m\right) \\
& \leq (n-1)[\lambda^n + \lambda^{n+1} + \dots + \lambda^{m-2}]A\left(\underbrace{\sigma_0, \sigma_0, \dots, \sigma_0}_{(n-1) \text{ times}}, \sigma_1\right) \\
& \quad +\lambda^{m-2}A\left(\underbrace{\sigma_0, \sigma_0, \dots, \sigma_0}_{(n-1) \text{ times}}, \sigma_1\right) \\
& \leq (n-1)[\lambda^n + \lambda^{n+1} + \dots + \lambda^{m-2} + \dots]A\left(\underbrace{\sigma_0, \sigma_0, \dots, \sigma_0}_{(n-1) \text{ times}}, \sigma_1\right) \\
& \leq (n-1)[\lambda^n + \lambda^{n+1} + \dots + \lambda^{m-2} + \dots]A\left(\underbrace{\sigma_0, \sigma_0, \dots, \sigma_0}_{(n-1) \text{ times}}, \sigma_1\right) \\
& \leq (n-1)\frac{\lambda^n}{1-\lambda}A\left(\underbrace{\sigma_0, \sigma_0, \dots, \sigma_0}_{(n-1) \text{ times}}, \sigma_1\right)
\end{aligned}$$

Letting $n, m \rightarrow \infty$, we obtain

$$\lim_{n, m \rightarrow \infty} A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_m\right) = 0 \quad (9)$$

Thus, the sequence $\{\sigma_n\}$ is Cauchy in the complete A-metric space (D, A) . So, there is some $\sigma^* \in D$. So that

$$\lim_{n \rightarrow \infty} A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma^*\right) = 0; \quad (10)$$

that is, $\sigma_n \rightarrow \sigma^*$ as $n \rightarrow \infty$. Now, we will prove that σ^* is a fixed point of F. By (3) and condition (A3), we get

$$\begin{aligned}
A\left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1) \text{ times}}, Y\sigma^*\right) &\leq (n-1)A\left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1) \text{ times}}, \sigma_{n+1}\right) + A\left(\underbrace{Y\sigma^*, Y\sigma^*, \dots, Y\sigma^*}_{(n-1) \text{ times}}, \sigma_{n+1}\right) \\
&= (n-1)A\left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1) \text{ times}}, \sigma_{n+1}\right) + A\left(\underbrace{Y\sigma^*, Y\sigma^*, \dots, Y\sigma^*}_{(n-1) \text{ times}}, Y\sigma_n\right) \\
&\leq (n-1)A\left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1) \text{ times}}, \sigma_{n+1}\right) + \lambda \left(\left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1) \text{ times}}, Y\sigma^* \right) \right)^\alpha \left(\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, Y\sigma_n \right) \right)^{1-\alpha} \\
&\leq (n-1)A\left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1) \text{ times}}, \sigma_{n+1}\right) + \lambda \left(\left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1) \text{ times}}, Y\sigma^* \right) \right)^\alpha \left(\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1} \right) \right)^{1-\alpha}
\end{aligned}
\tag{11}$$

Taking the limit as $n \rightarrow \infty$, we obtain that

$$A\left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1) \text{ times}}, Y\sigma^*\right) = 0 \tag{12}$$

This yields that $\sigma^* = Y\sigma^*$. Now, we prove the uniqueness of σ^* . Let ζ^* be another fixed point of Y in D , then $Y\zeta^* = \zeta^*$. Now, by (3), we have

$$\begin{aligned}
A\left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1) \text{ times}}, \zeta^*\right) &= A\left(\underbrace{Y\sigma^*, Y\sigma^*, \dots, Y\sigma^*}_{(n-1) \text{ times}}, Y\zeta^*\right) \\
&\leq \lambda \left(A\left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1) \text{ times}}, Y\sigma^*\right) \right)^\alpha \left(A\left(\underbrace{\zeta^*, \zeta^*, \dots, \zeta^*}_{(n-1) \text{ times}}, Y\zeta^*\right) \right)^{1-\alpha} = 0
\end{aligned}
\tag{13}$$

This yields that $\sigma^* = \zeta^*$. It completes the proof.

Theorem 2.6 Let (D, A) be a complete A-metric space. Let $Y: D \rightarrow D$ be a (λ, α, β) -interpolative Kannan contraction. Then Y has a unique fixed point.

Proof Following the steps of proof of Theorem 2.5, we construct the sequence $\{\sigma_n\}$ by iterating

$$\sigma_{n+1} = Y\sigma_n, \forall n \in \mathbb{N},$$

where $\sigma_0 \in Dis$ arbitrary starting point. Then, by (4), we have

$$\begin{aligned} A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1}\right) &= A\left(\underbrace{Y\sigma_{n-1}, Y\sigma_{n-1}, \dots, Y\sigma_{n-1}}_{(n-1) \text{ times}}, Y\sigma_n\right) \\ &\leq \lambda \left(A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, Y\sigma_{n-1}\right) \right)^\alpha \left(A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, Y\sigma_n\right) \right)^\beta \\ &= \lambda \left(A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n\right) \right)^\alpha \left(A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1}\right) \right)^\beta \end{aligned}$$

Since $\alpha < 1 - \beta$, the last inequality gives

$$\begin{aligned} \left(A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1}\right) \right)^{1-\beta} &\leq \lambda \left(A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n\right) \right)^\alpha \\ &\leq \lambda \left(A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n\right) \right)^{1-\beta} \quad (14) \end{aligned}$$

$$\begin{aligned} A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1}\right) &\leq \lambda^{\frac{1}{1-\beta}} A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n\right) \\ &\leq \lambda A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n\right) \end{aligned}$$

and then

$$A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1}\right) \leq \lambda A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n\right)$$

$$\begin{aligned}
&\leq \lambda^2 A \left(\underbrace{\sigma_{n-2}, \sigma_{n-2}, \dots, \sigma_{n-2}}_{(n-1) \text{ times}}, \sigma_{n-1} \right) \\
&\leq \dots \leq \lambda^n A \left(\underbrace{\sigma_0, \sigma_0, \dots, \sigma_0}_{(n-1) \text{ times}}, \sigma_1 \right) \quad (15)
\end{aligned}$$

A fixed-point $\sigma^* \in D$ is produced by the classical process, as was previously shown in the proof of Theorem 2.5. Now, we prove the uniqueness of σ^* . Let ζ^* be another fixed point of Y in D , then $Y\zeta^* = \zeta^*$. Now, by (4), we have

$$\begin{aligned}
A \left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1) \text{ times}}, \zeta^* \right) &= A \left(\underbrace{Y\sigma^*, Y\sigma^*, \dots, Y\sigma^*}_{(n-1) \text{ times}}, Y\zeta^* \right) \\
&\leq \lambda \left(A \left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1) \text{ times}}, Y\sigma^* \right) \right)^\alpha \left(A \left(\underbrace{\zeta^*, \zeta^*, \dots, \zeta^*}_{(n-1) \text{ times}}, Y\zeta^* \right) \right)^\beta = 0
\end{aligned} \quad (16)$$

This yields that $\sigma^* = \zeta^*$. This completes the proof.

Theorem 2.7 Let (D, A) be a complete A -metric space. Let $Y: D \rightarrow D$ be a $(\lambda, \alpha, \beta, \gamma)$ -interpolative Reich contraction. Then Y has a unique fixed point.

Proof Following the steps of proof of Theorem 2.5, we construct the sequence $\{\sigma_n\}$ by iterating

$$\sigma_{n+1} = Y\sigma_n, \forall n \in \mathbb{N},$$

where $\sigma_0 \in D$ is arbitrary starting point. Then, by (5), we have

$$\begin{aligned}
A \left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1} \right) &= A \left(\underbrace{Y\sigma_{n-1}, Y\sigma_{n-1}, \dots, Y\sigma_{n-1}}_{(n-1) \text{ times}}, Y\sigma_n \right) \\
&\leq \lambda \left(A \left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n \right) \right)^\alpha \left(A \left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, Y\sigma_{n-1} \right) \right)^\beta \left(A \left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, Y\sigma_n \right) \right)^\gamma
\end{aligned}$$

$$\leq \lambda \left(A \left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n \right) \right)^\alpha \left(A \left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n \right) \right)^\beta \left(A \left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1} \right) \right)^\gamma$$

Since $\alpha + \beta < 1 - \gamma$, the last inequality gives

$$\begin{aligned} \left(A \left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1} \right) \right)^{1-\gamma} &\leq \lambda \left(A \left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n \right) \right)^{\alpha+\beta} \\ &\leq \lambda \left(A \left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n \right) \right)^{1-\gamma} \end{aligned} \quad (18)$$

$$\begin{aligned} A \left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1} \right) &\leq \lambda^{\frac{1}{1-\gamma}} A \left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n \right) \\ &\leq \lambda A \left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n \right) \end{aligned}$$

and then

$$\begin{aligned} A \left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1} \right) &\leq \lambda A \left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n \right) \\ &\leq \lambda^2 A \left(\underbrace{\sigma_{n-2}, \sigma_{n-2}, \dots, \sigma_{n-2}}_{(n-1) \text{ times}}, \sigma_{n-1} \right) \\ &\leq \dots \leq \lambda^n A \left(\underbrace{\sigma_0, \sigma_0, \dots, \sigma_0}_{(n-1) \text{ times}}, \sigma_1 \right) \end{aligned} \quad (19)$$

A fixed-point $\sigma^* \in D$ is produced by the classical process, as was previously explained in the proof of Theorem 2.5. We now demonstrate σ^* 's uniqueness. If ζ^* be another fixed point of Y in D , then $Y\zeta^* = \zeta^*$. As of (5), we now have

$$\begin{aligned}
A\left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1) \text{ times}}, \zeta^*\right) &= A\left(\underbrace{Y\sigma^*, Y\sigma^*, \dots, Y\sigma^*}_{(n-1) \text{ times}}, Y\zeta^*\right) \\
&\leq \lambda \left(A\left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1) \text{ times}}, \zeta^*\right) \right)^\alpha \left(A\left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1) \text{ times}}, Y\sigma^*\right) \right)^\beta \left(A\left(\underbrace{\zeta^*, \zeta^*, \dots, \zeta^*}_{(n-1) \text{ times}}, Y\zeta^*\right) \right)^\gamma = 0
\end{aligned} \tag{20}$$

This yields that $\sigma^* = \zeta^*$. This completes the proof.

Theorem 2.8 Let (D, A) be a complete A-metric space. Let $Y: D \rightarrow D$ be a (λ, α, β) -interpolative Dass-Gupta rational contraction. Then Y has a unique fixed point.

Proof Following the steps of proof of Theorem 2.5, we construct the sequence $\{\sigma_n\}$ by iterating

$$\sigma_{n+1} = Y\sigma_n, \forall n \in \mathbb{N},$$

where $\sigma_0 \in D$ is arbitrary starting point. Then, by (6), we have

$$\begin{aligned}
A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1}\right) &= A\left(\underbrace{Y\sigma_{n-1}, Y\sigma_{n-1}, \dots, Y\sigma_{n-1}}_{(n-1) \text{ times}}, Y\sigma_n\right) \\
&\leq \lambda \left(A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n\right) \right)^\alpha \left(\frac{\left[1 + A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, Y\sigma_{n-1}\right) \right] A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, Y\sigma_n\right)}{1 + A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n\right)} \right)^\beta \\
&\leq \lambda \left(A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n\right) \right)^\alpha \left(\frac{\left[1 + A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n\right) \right] A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1}\right)}{1 + A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n\right)} \right)^\beta \\
&= \lambda \left(A\left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n\right) \right)^\alpha \left(A\left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1}\right) \right)^\beta
\end{aligned}$$

Since $\alpha + \beta < 1$, the last inequality gives

$$\begin{aligned}
\left(A \left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1} \right) \right)^{1-\beta} &\leq \lambda \left(A \left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n \right) \right)^\alpha \\
&\leq \lambda \left(A \left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n \right) \right)^{1-\beta} \quad (21) \\
A \left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1} \right) &\leq \lambda^{\frac{1}{1-\beta}} A \left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n \right) \\
&\leq \lambda A \left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n \right)
\end{aligned}$$

and then

$$\begin{aligned}
A \left(\underbrace{\sigma_n, \sigma_n, \dots, \sigma_n}_{(n-1) \text{ times}}, \sigma_{n+1} \right) &\leq \lambda A \left(\underbrace{\sigma_{n-1}, \sigma_{n-1}, \dots, \sigma_{n-1}}_{(n-1) \text{ times}}, \sigma_n \right) \\
&\leq \lambda^2 A \left(\underbrace{\sigma_{n-2}, \sigma_{n-2}, \dots, \sigma_{n-2}}_{(n-1) \text{ times}}, \sigma_{n-1} \right) \\
&\leq \dots \leq \lambda^n A \left(\underbrace{\sigma_0, \sigma_0, \dots, \sigma_0}_{(n-1) \text{ times}}, \sigma_1 \right) \quad (22)
\end{aligned}$$

A fixed-point $\sigma^* \in D$ is produced by the classical process, as was previously shown in the proof of Theorem 2.5. Now, we prove the uniqueness of σ^* . Let ζ^* be another fixed point of Y in D , then $Y\zeta^* = \zeta^*$. Now, by (6), we have

$$A \left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1) \text{ times}}, \zeta^* \right) = A \left(\underbrace{Y\sigma^*, Y\sigma^*, \dots, Y\sigma^*}_{(n-1) \text{ times}}, Y\zeta^* \right)$$

$$\leq \lambda \left(A \left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1) \text{ times}}, \zeta^* \right) \right)^\alpha \left(\frac{\left[1 + A \left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1) \text{ times}}, \gamma \sigma^* \right) \right] A \left(\underbrace{\zeta^*, \zeta^*, \dots, \zeta^*}_{(n-1) \text{ times}}, \gamma \zeta^* \right)}{1 + A \left(\underbrace{\sigma^*, \sigma^*, \dots, \sigma^*}_{(n-1) \text{ times}}, \zeta^* \right)} \right)^\beta$$

= 0(23)

This yields

that $\sigma^* = \zeta^*$. This completes the proof.

4. Conclusion

In this work, we presented the notion of (λ, α) -interpolative Kannan contraction, (λ, α, β) -interpolative Kannan contraction and $(\lambda, \alpha, \beta, \gamma)$ -interpolative Riech contraction and (λ, α, β) -interpolative Dass-Gupta rational contraction. We also demonstrated the existence of fixed points for self-mapping. All of these concepts were introduced using the new framework of A-metric spaces.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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