Original Research Article

FIXED POINTS OF KANNAN INTERPOLATIVE, RIECH INTERPOLATIVE, AND RATIONAL CONTRACTIONS IN A-METRIC SPACES

ABSTRACT: In this paper, we introduce (λ, α)-interpolative Kannan contraction, $(\lambda\ \alpha, \beta)$-interpolative Kannan contraction, $(\lambda, \alpha, \beta, \gamma)$-interpolative Riech contraction and $(\lambda\ \alpha, \beta)$-interpolative Dass-Gupta rational contraction. Also, we establish some fixed-point theorems in complete A-metric spaces for interpolative contractions. Additionally, these theorems expand and apply several intriguing findings from metric fixed-point theory to an A-metric setting.

Keywords: fixed-point; iterative methods; interpolative; contraction; A-metric spaces.

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1. Introduction and Preliminaries

Research on fixed point theory is an exciting area of study in both topology and analysis. A significant conclusion known as the Banach contraction principle was presented by Banach [12] in 1922, and its significance in metric fixed-point theory was examined. Let T be a self-map on a nonempty set X and (X, d) be a complete metric space. If there exists a constant $c \in [0, 1)$ such that

$$d(T\omega, T\mu) \leq c\, d(\omega, \mu), \text{ for all } \omega, \mu \in X, \tag{1}$$

then it possesses a unique fixed point in X. The Banach contraction principle was then widely generalized in the literature (see [13–14]). Both pure and applied mathematics make extensive use of it. Kannan [2] defined a new variation of this theory in 1968 and eliminated the continuity condition from it.

Theorem 1.1 (see [2]). *Let (X, d) be a complete metric space and a self-map $T: X \longrightarrow X$ be a Kannan contraction mapping, i.e.,*

$$d(T\omega, T\mu) \leq k[d(\omega, T\omega) + d(\mu, T\mu)], \tag{2}$$

for all $\omega, \mu \in X$, where $k \in [0, 1/2)$. Then, T admits a unique fixed point in X.

In 1989, Bakhtin [9] introduced the concept of b-metric space which is generalization of renowned Banach contraction mapping principle. Czerwik [10,11] extended the concept of b-metric space in 1993. Kannan fixed-point theorem is the first significant variant of the outstanding result of Banach on the metric fixed-point theory [12]. In 2015, Abbas et al. [1] introduced the notion of A-metric space.

Definition 1.2 (see [1]) Let X be a nonempty set. A mapping $A: X^n \to [0,+\infty)$ is called an A-metric on X if and only if for all $\omega_i, a \in X, i = 1,2,3, .. n$: the following conditions hold:

(A1). $A(\omega_1, \omega_2, \omega_3, \dots, \omega_{n-1}, \omega_n) \geq 0,$

(A2). $A(\omega_1, \omega_2, \omega_3, \dots, \omega_{n-1}, \omega_n) = 0$ if and only if $\omega_1 = \omega_2 = \cdots = \omega_{n-1} = \omega_n,$

(A3). $A(\omega_1, \omega_2, \omega_3, \dots, \omega_{n-1}, \omega_n) \leq A(\omega_1, \omega_1, \omega_1, \dots, (\omega_1)_{n-1}, a)$

$$+A(\omega_2, \omega_2, \omega_2, \dots, (\omega_2)_{n-1}, a)$$
$$+A(\omega_3, \omega_3, \omega_3, \dots, (\omega_3)_{n-1}, a) + \cdots$$
$$+A(\omega_{n-1}, \omega_{n-1}, \omega_{n-1}, \dots, (\omega_{n-1})_{n-1}, a)$$
$$+A(\omega_n, \omega_n, \omega_n, \dots, (\omega_n)_{n-1}, a)]$$

The pair (X, A) is called an A-metric space.

The following is the intuitive geometric example for A-metric spaces.

Example 1.3 (see [1]) Let $X = [1,+\infty)$. Define $A: X^n \to [0,+\infty)$ by

$$A(\omega_1,\omega_2,\omega_3,\dots.,\omega_{n-1},\omega_n)=\sum_{i=1}^{n}\sum_{i<j}\left|\omega_i-\omega_j\right|$$

for all $\omega_i \in X, i = 1,2, \dots n$.

Example 1.4 (see [1]) Let $X = \mathbb{R}$. Define $A: X^n \to [0,+\infty)$ by

$$A(\omega_1,\omega_2,\omega_3,\dots.,\omega_{n-1},\omega_n)=\left|\sum_{i=n}^{2}\omega_i-(n-1)\omega_1\right|$$

$$+\left|\sum_{i=n}^{3}\omega_i-(n-2)\omega_2\right|+\cdots$$

$$+\left|\sum_{i=n}^{n-3}\omega_i-3\omega_{n-3}\right|$$

$$+\left|\sum_{i=n}^{n-2}\omega_i-2\omega_{n-2}\right|$$

$$+\left|\omega_n-\omega_{n-1}\right|$$

for all $\omega_i \in X, i = 1,2, \dots n$.

Lemma 1.5 (see [1]) Let (X, A) be an A-metric space. Then for all $\omega, \mu \in X$,

$$A(\omega,\omega,\omega,\omega,\dots,(\omega)_{n-1},\mu)=A(\mu,\mu,\mu,\mu\dots.,(\mu)_{n-1},\omega)$$

Lemma 1.6 (see [1]) Let (X, A) be an A-metric space. Then for all $\omega, \mu, z \in X$,

$$A(\omega,\omega,\omega,\omega,\dots,(\omega)_{n-1},z)\le(n-1)A(\omega,\omega,\omega,\omega,\dots,(\omega)_{n-1},\mu)$$

$$+A(z,z,z,z,\dots,(z)_{n-1},\mu)$$

and

$$A(\omega,\omega,\omega,\omega,\dots,(\omega)_{n-1},z)\le(n-1)A(\omega,\omega,\omega,\omega,\dots,(\omega)_{n-1},\mu)$$

$$+A(\mu,\mu,\mu,\mu,\dots,(\mu)_{n-1},z)$$

Lemma 1.7 (see [1]) Let (X, A) be an A-metric space. Then $(X \times X, D_A)$ is an A-metric space on $X \times X$, where D_A is given by for all $\omega_i, \mu_j \in X, i, j = 1,2, \dots, n$:

$$D_A\big((\omega_1,\mu_1),(\omega_2,\mu_2),(\omega_3,\mu_3),\dots,(\omega_n,\mu_n)\big)$$

$$= A(\omega_1,\omega_2,\omega_3,\dots.,\omega_n) + A(\mu_1,\mu_2,\mu_3,\dots.,\mu_n).$$

Definition 1.8 (see [1]) Let (X,A) be an A-metric space. Then

1. A sequence $\{\omega_k\}$ is called convergent to ω in (X,A) if

$$\lim_{k\to+\infty} A(\omega_k,\omega_k,\omega_k,\omega_k,..(\omega_k)_{n-1},\omega) = 0.$$

That is, for each $\epsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k \geq n_0$, we have

$$A(\omega_k,\omega_k,\omega_k,\omega_k,..(\omega_k)_{n-1},\omega) \leq \epsilon$$

and we write $\lim_{k\to+\infty} \omega_k = \omega$.

2. A sequence $\{\omega_k\}$ is called Cauchy in (X,A) if

$$\lim_{k,m\to+\infty} A(\omega_k,\omega_k,\omega_k,\omega_k,..(\omega_k)_{n-1},\omega_m) = 0.$$

That is, for each $\epsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k,m \geq n_0$, we have

$$A(\omega_k,\omega_k,\omega_k,\omega_k,..(\omega_k)_{n-1},\omega_m) \leq \epsilon.$$

3. (X,A) is said to be complete if every Cauchy sequence in (X,A) is a convergent.

Lemma 1.9 (see [1]) Let (X,A) be an A-metric space. If the sequence $\{\omega_k\}$ in X converges to ω, then ω is unique.

Lemma 1.10 (see [1]) Every convergent sequence in A-metric space (X,A) is a Cauchy sequence.

In the present research paper, Kannan type, Riech type and rational type interpolative contraction is defined and discussed in the framework of A-metric space. Further, some common fixed-point results are proved using the notion of interpolation. Additionally, these theorems expand and apply several intriguing findings from metric fixed-point theory to an A-metric setting.

2. **Main Result**

We begin by defining the terms below.

Definition 2.1 Let (X, A) be an A- metric space. Let $T: X \longrightarrow X$ be a self-map. We shall call T a (λ, α)-interpolative Kannan contraction, if there exist $\lambda \in [0,1), \alpha \in (0,1)$ such that

$$A\left(\underbrace{T\omega, T\omega, \dots, T\omega}_{(n-1)\,times}, T\mu\right) \leq \lambda\left(A\left(\underbrace{\omega, \omega, \dots, \omega}_{(n-1)\,times}, T\omega\right)\right)^{\alpha}\left(A\left(\underbrace{\mu, \mu, \dots, \mu}_{(n-1)\,times}, T\mu\right)\right)^{1-\alpha} \qquad (3)$$

for all $\omega, \mu \in X$, with $\omega \neq \mu$.

Definition 2.2 Let (X, A) be an A- metric space. Let $T: X \longrightarrow X$ be a self-map. We shall call T a (λ, α, β)-interpolative Kannan contraction, if there exist $\lambda \in [0,1), \alpha, \beta \in (0,1)$, $\alpha + \beta < 1$ such that

$$A\left(\underbrace{T\omega, T\omega, \dots, T\omega}_{(n-1)\,times}, T\mu\right) \leq \lambda\left(A\left(\underbrace{\omega, \omega, \dots, \omega}_{(n-1)\,times}, T\omega\right)\right)^{\alpha}\left(A\left(\underbrace{\mu, \mu, \dots, \mu}_{(n-1)\,times}, T\mu\right)\right)^{\beta} \qquad (4)$$

for all $\omega, \mu \in X$, with $\omega \neq \mu$.

Definition 2.3 Let (X, A) be an A- metric space. Let $T: X \longrightarrow X$ be a self-map. We shall call T a $(\lambda, \alpha, \beta, \gamma)$-interpolative Reich contraction, if there exist $\lambda \in [0,1), \alpha, \beta, \gamma \in (0,1)$, $\alpha + \beta + \gamma < 1$ such that

$$A\left(\underbrace{T\omega, T\omega, \dots, T\omega}_{(n-1)\,times}, T\mu\right) \leq$$

$$\lambda\left(A\left(\underbrace{\omega, \omega, \dots, \omega}_{(n-1)\,times}, \mu\right)\right)^{\alpha}\left(A\left(\underbrace{\omega, \omega, \dots, \omega}_{(n-1)\,times}, T\omega\right)\right)^{\beta}\left(A\left(\underbrace{\mu, \mu, \dots, \mu}_{(n-1)\,times}, T\mu\right)\right)^{\gamma} \qquad (5)$$

for all $\omega, \mu \in X$, with $\omega \neq \mu$.

Definition 2.4 Let (X, A) be an A- metric space. Then a self-map $T: X \longrightarrow X$ is (λ, α, β)-interpolative Dass-Gupta rational contraction, if there exist $\lambda \in [0,1), \alpha, \beta \in (0,1)$, $\alpha + \beta < 1$ such that

$$A\left(\underbrace{T\omega, T\omega, \dots, T\omega}_{(n-1)\,times}, T\mu\right) \leq \lambda\left(A\left(\underbrace{\omega, \omega, \dots, \omega}_{(n-1)\,times}, \mu\right)\right)^{\alpha}\left(\frac{\left[1 + A\left(\underbrace{\omega, \omega, \dots, \omega}_{(n-1)\,times}, T\omega\right)\right] A\left(\underbrace{\mu, \mu, \dots, \mu}_{(n-1)\,times}, T\mu\right)}{1 + A\left(\underbrace{\omega, \omega, \dots, \omega}_{(n-1)\,times}, \mu\right)}\right)^{\beta} \qquad (6)$$

for all $\omega, \mu \in X$, with $\omega \neq \mu$.

Our first main result as follows.

Theorem 2.5 Let (X, A) be a complete A-metric space. Let $T: X \longrightarrow X$ be a (λ, α)-interpolative Kannan contraction. Then T has a unique fixed point.

Proof. Let $\omega_0 \in X$ be initial point. Define a sequence $\{\omega_n\}$ as $\omega_{n+1} = T\omega_n, \forall\, n \in \mathbb{N}$. Obviously, if $\exists\, n_0 \in \mathbb{N}$ for which $\omega_{n_0+1} = \omega_{n_0}$, then $T\omega_{n_0} = \omega_{n_0}$, and the proof is finished. Thus, we suppose that $\omega_{n+1} \neq \omega_n$ for each $n \in \mathbb{N}$. Thus, by (3), we have

$$A\Big(\underbrace{\omega_n, \omega_n, \dots, \omega_n}_{(n-1)\ times}, \omega_{n+1}\Big) = A\Big(\underbrace{T\omega_{n-1}, T\omega_{n-1}, \dots, T\omega_{n-1}}_{(n-1)\ times}, T\omega_n\Big)$$

$$\leq \lambda\left(A\Big(\underbrace{\omega_{n-1}, \omega_{n-1}, \dots, \omega_{n-1}}_{(n-1)\ times}, T\omega_{n-1}\Big)\right)^{\alpha}\left(A\Big(\underbrace{\omega_n, \omega_n, \dots, \omega_n}_{(n-1)\ times}, T\omega_n\Big)\right)^{1-\alpha}$$

$$= \lambda\left(A\Big(\underbrace{\omega_{n-1}, \omega_{n-1}, \dots, \omega_{n-1}}_{(n-1)\ times}, \omega_n\Big)\right)^{\alpha}\left(A\Big(\underbrace{\omega_n, \omega_n, \dots, \omega_n}_{(n-1)\ times}, \omega_{n+1}\Big)\right)^{1-\alpha}$$

The last inequality gives

$$\left(A\Big(\underbrace{\omega_n, \omega_n, \dots, \omega_n}_{(n-1)\ times}, \omega_{n+1}\Big)\right)^{\alpha} \leq \lambda\left(A\Big(\underbrace{\omega_{n-1}, \omega_{n-1}, \dots, \omega_{n-1}}_{(n-1)\ times}, \omega_n\Big)\right)^{\alpha} \tag{7}$$

Since $\alpha < 1$, we have

$$A\Big(\underbrace{\omega_n, \omega_n, \dots, \omega_n}_{(n-1)\ times}, \omega_{n+1}\Big) \leq \lambda^{\frac{1}{\alpha}} A\Big(\underbrace{\omega_{n-1}, \omega_{n-1}, \dots, \omega_{n-1}}_{(n-1)\ times}, \omega_n\Big)$$

$$\leq \lambda A\Big(\underbrace{\omega_{n-1}, \omega_{n-1}, \dots, \omega_{n-1}}_{(n-1)\ times}, \omega_n\Big)$$

and then

$$A\left(\underbrace{\omega_n,\omega_n,\dots,\omega_n}_{(n-1)\ times},\omega_{n+1}\right)\leq\lambda A\left(\underbrace{\omega_{n-1},\omega_{n-1},\dots,\omega_{n-1}}_{(n-1)\ times},\omega_n\right)$$

$$\leq\lambda^2 A\left(\underbrace{\omega_{n-2},\omega_{n-2},\dots,\omega_{n-2}}_{(n-1)\ times},\omega_{n-1}\right)$$

$$\leq\cdots\leq\lambda^n A\left(\underbrace{\omega_0,\omega_0,\dots,\omega_0}_{(n-1)\ times},\omega_1\right)\tag{8}$$

For all $n,m\in\mathbb{N}$ and $n<m$, we have

$$A\left(\underbrace{\omega_n,\omega_n,\dots,\omega_n}_{(n-1)\ times},\omega_m\right)\leq(n-1)A\left(\underbrace{\omega_n,\omega_n,\dots,\omega_n}_{(n-1)\ times},\omega_{n+1}\right)$$

$$+A\left(\underbrace{\omega_m,\omega_m,\dots,\omega_m}_{(n-1)\ times},\omega_{n+1}\right)$$

$$=(n-1)A\left(\underbrace{\omega_n,\omega_n,\dots,\omega_n}_{(n-1)\ times},\omega_{n+1}\right)$$

$$+A\left(\underbrace{\omega_{n+1},\omega_{n+1},\dots,\omega_{n+1}}_{(n-1)\ times},\omega_m\right)$$

$$\leq(n-1)A\left(\underbrace{\omega_n,\omega_n,\dots,\omega_n}_{(n-1)\ times},\omega_{n+1}\right)$$

$$+(n-1)A\left(\underbrace{\omega_{n+1},\omega_{n+1},\dots,\omega_{n+1}}_{(n-1)\ times},\omega_{n+2}\right)$$

$$+A\left(\underbrace{\omega_m,\omega_m,\dots,\omega_m}_{(n-1)\ times},\omega_{n+2}\right)$$

$$\leq(n-1)A\left(\underbrace{\omega_n,\omega_n,\dots,\omega_n}_{(n-1)\ times},\omega_{n+1}\right)$$

$$+(n-1)A\left(\underbrace{\omega_{n+1},\omega_{n+1},\dots,\omega_{n+1}}_{(n-1)\ times},\omega_{n+2}\right)$$

$$+A\left(\underbrace{\omega_{n+2},\omega_{n+2},\dots,\omega_{n+2}}_{(n-1)\ times},\omega_{m}\right)$$

$$\leq (n-1)A\left(\underbrace{\omega_{n},\omega_{n},\dots,\omega_{n}}_{(n-1)\ times},\omega_{n+1}\right)$$

$$+(n-1)A\left(\underbrace{\omega_{n+1},\omega_{n+1},\dots,\omega_{n+1}}_{(n-1)\ times},\omega_{n+2}\right)+\cdots$$

$$+(n-1)A\left(\underbrace{\omega_{m-2},\omega_{m-2},\dots,\omega_{m-2}}_{(n-1)\ times},\omega_{m-1}\right)$$

$$+A\left(\underbrace{\omega_{m-1},\omega_{m-1},\dots,\omega_{m-1}}_{(n-1)\ times},\omega_{m}\right)$$

$$\leq (n-1)[\lambda^{n}+\lambda^{n+1}+\cdots+\lambda^{m-2}]A\left(\underbrace{\omega_{0},\omega_{0},\dots,\omega_{0}}_{(n-1)\ times},\omega_{1}\right)$$

$$+\lambda^{m-2}A\left(\underbrace{\omega_{0},\omega_{0},\dots,\omega_{0}}_{(n-1)\ times},\omega_{1}\right)$$

$$\leq (n-1)[\lambda^{n}+\lambda^{n+1}+\cdots+\lambda^{m-2}+\cdots]A\left(\underbrace{\omega_{0},\omega_{0},\dots,\omega_{0}}_{(n-1)\ times},\omega_{1}\right)$$

$$\leq (n-1)[\lambda^{n}+\lambda^{n+1}+\cdots+\lambda^{m-2}+\cdots]A\left(\underbrace{\omega_{0},\omega_{0},\dots,\omega_{0}}_{(n-1)\ times},\omega_{1}\right)$$

$$\leq (n-1)\frac{\lambda^{n}}{1-\lambda}A\left(\underbrace{\omega_{0},\omega_{0},\dots,\omega_{0}}_{(n-1)\ times},\omega_{1}\right)$$

Letting $n,m\longrightarrow\infty$, we obtain

$$\lim_{n,m\to\infty} A\left(\underbrace{\omega_n,\omega_n,\dots,\omega_n}_{(n-1)\ times},\omega_m\right)=0 \tag{9}$$

Thus, the sequence $\{\omega_n\}$ is Cauchy in the complete A-metric space (X, d_c). So, there is some $\omega^* \in X$. So that

$$\lim_{n\to\infty} A\left(\underbrace{\omega_n,\omega_n,\dots,\omega_n}_{(n-1)\ times},\omega^*\right)=0; \tag{10}$$

that is, $\omega_n \longrightarrow \omega^*$ as $n \longrightarrow \infty$. Now, we will prove that ω^* is a fixed point of F. By (3) and condition (A3), we get

$$A\left(\underbrace{\omega^*,\omega^*,\dots,\omega^*}_{(n-1)\ times},T\omega^*\right)\le(n-1)A\left(\underbrace{\omega^*,\omega^*,\dots,\omega^*}_{(n-1)\ times},\omega_{n+1}\right)+A\left(\underbrace{T\omega^*,T\omega^*,\dots,T\omega^*}_{(n-1)\ times},\omega_{n+1}\right)$$

$$=(n-1)A\left(\underbrace{\omega^*,\omega^*,\dots,\omega^*}_{(n-1)\ times},\omega_{n+1}\right)+A\left(\underbrace{T\omega^*,T\omega^*,\dots,T\omega^*}_{(n-1)\ times},T\omega_n\right)$$

$$\le(n-1)A\left(\underbrace{\omega^*,\omega^*,\dots,\omega^*}_{(n-1)\ times},\omega_{n+1}\right)$$

$$+\lambda\left(\left(\underbrace{\omega^*,\omega^*,\dots,\omega^*}_{(n-1)\ times},T\omega^*\right)\right)^{\alpha}\left(\left(\underbrace{\omega_n,\omega_n,\dots,\omega_n}_{(n-1)\ times},T\omega_n\right)\right)^{1-\alpha}$$

$$\le(n-1)A\left(\underbrace{\omega^*,\omega^*,\dots,\omega^*}_{(n-1)\ times},\omega_{n+1}\right)$$

$$+\lambda\left(\left(\underbrace{\omega^*,\omega^*,\dots,\omega^*}_{(n-1)\ times},T\omega^*\right)\right)^{\alpha}\left(\left(\underbrace{\omega_n,\omega_n,\dots,\omega_n}_{(n-1)\ times},\omega_{n+1}\right)\right)^{1-\alpha} \tag{11}$$

Taking the limit as $n \longrightarrow \infty$, we obtain that

$$A\left(\underbrace{\omega^*,\omega^*,\dots,\omega^*}_{(n-1)\ times},T\omega^*\right)=0 \tag{12}$$

This yields that $\omega^* = T\omega^*$. Now, we prove the uniqueness of ω^*. Let μ^* be another fixed point of T in X, then $T\mu^* = \mu^*$. Now, by (3), we have

$$A\Big(\underbrace{\omega^*, \omega^*, , \dots, \omega^*}_{(n-1)\ times}, \mu^*\Big) = A\Big(\underbrace{T\omega^*, T\omega^*, \dots, T\omega^*}_{(n-1)\ times}, T\mu^*\Big)$$
$$\leq \lambda\left(A\Big(\underbrace{\omega^*, \omega^*, \dots, \omega^*}_{(n-1)\ times}, T\omega^*\Big)\right)^{\alpha}\left(A\Big(\underbrace{\mu^*, \mu^*, \dots, \mu^*}_{(n-1)\ times}, T\mu^*\Big)\right)^{1-\alpha} = 0 \tag{13}$$

This yields that $\omega^* = \mu^*$. It completes the proof.

Theorem 2.6 Let (X, A) be a complete A-metric space. Let $T: X \longrightarrow X$ be a (λ, α, β)-interpolative Kannan contraction. Then T has a unique fixed point.

Proof Following the steps of proof of Theorem 2.5, we construct the sequence $\{\omega_n\}$ by iterating

$$\omega_{n+1} = T\omega_n, \forall\, n \in \mathbb{N},$$

where $\omega_0 \in X$ is arbitrary starting point. Then, by (4), we have

$$A\Big(\underbrace{\omega_n, \omega_n, \dots, \omega_n}_{(n-1)\ times}, \omega_{n+1}\Big) = A\Big(\underbrace{T\omega_{n-1}, T\omega_{n-1}, \dots, T\omega_{n-1}}_{(n-1)\ times}, T\omega_n\Big)$$

$$\leq \lambda\left(A\Big(\underbrace{\omega_{n-1}, \omega_{n-1}, \dots, \omega_{n-1}}_{(n-1)\ times}, T\omega_{n-1}\Big)\right)^{\alpha}\left(A\Big(\underbrace{\omega_n, \omega_n, \dots, \omega_n}_{(n-1)\ times}, T\omega_n\Big)\right)^{\beta}$$

$$= \lambda\left(A\Big(\underbrace{\omega_{n-1}, \omega_{n-1}, \dots, \omega_{n-1}}_{(n-1)\ times}, \omega_n\Big)\right)^{\alpha}\left(A\Big(\underbrace{\omega_n, \omega_n, \dots, \omega_n}_{(n-1)\ times}, \omega_{n+1}\Big)\right)^{\beta}$$

Since $\alpha < 1 - \beta$, the last inequality gives

$$\left(A\Big(\underbrace{\omega_n, \omega_n, \dots, \omega_n}_{(n-1)\ times}, \omega_{n+1}\Big)\right)^{1-\beta} \leq \lambda\left(A\Big(\underbrace{\omega_{n-1}, \omega_{n-1}, \dots, \omega_{n-1}}_{(n-1)\ times}, \omega_n\Big)\right)^{\alpha}$$

$$\leq \lambda\left(A\left(\underbrace{\omega_{n-1},\omega_{n-1},\dots,\omega_{n-1}}_{(n-1)\,times},\omega_n\right)\right)^{1-\beta} \tag{14}$$

$$A\left(\underbrace{\omega_{n},\omega_{n},\dots,\omega_{n}}_{(n-1)\,times},\omega_{n+1}\right) \leq \lambda^{\frac{1}{1-\beta}} A\left(\underbrace{\omega_{n-1},\omega_{n-1},\dots,\omega_{n-1}}_{(n-1)\,times},\omega_n\right)$$

$$\leq \lambda A\left(\underbrace{\omega_{n-1},\omega_{n-1},\dots,\omega_{n-1}}_{(n-1)\,times},\omega_n\right)$$

and then

$$A\left(\underbrace{\omega_{n},\omega_{n},\dots,\omega_{n}}_{(n-1)\,times},\omega_{n+1}\right) \leq \lambda A\left(\underbrace{\omega_{n-1},\omega_{n-1},\dots,\omega_{n-1}}_{(n-1)\,times},\omega_n\right)$$

$$\leq \lambda^2 A\left(\underbrace{\omega_{n-2},\omega_{n-2},\dots,\omega_{n-2}}_{(n-1)\,times},\omega_{n-1}\right)$$

$$\leq \cdots \leq \lambda^n A\left(\underbrace{\omega_{0},\omega_{0},\dots,\omega_{0}}_{(n-1)\,times},\omega_1\right) \tag{15}$$

As already elaborated in the proof of Theorem 2.5, the classical procedure leads to the existence of a fixed-point $\omega^* \in X$. Now, we prove the uniqueness of ω^*. Let μ^* be another fixed point of T in X, then $T\mu^* = \mu^*$. Now, by (4), we have

$$A\left(\underbrace{\omega^*,\omega^*,,\dots,\omega^*}_{(n-1)\,times},\mu^*\right) = A\left(\underbrace{T\omega^*,T\omega^*,\dots,T\omega^*}_{(n-1)\,times},T\mu^*\right)$$

$$\leq \lambda\left(A\left(\underbrace{\omega^*,\omega^*,\dots,\omega^*}_{(n-1)\,times},T\omega^*\right)\right)^{\alpha}\left(A\left(\underbrace{\mu^*,\mu^*,\dots,\mu^*}_{(n-1)\,times},T\mu^*\right)\right)^{\beta} = 0 \tag{16}$$

This yields that $\omega^* = \mu^*$. This completes the proof.

Theorem 2.7 Let (X, A) be a complete A-metric space. Let $T: X \longrightarrow X$ be a $(\lambda, \alpha, \beta, \gamma)$-interpolative Reich contraction. Then T has a unique fixed point.

Proof Following the steps of proof of Theorem 2.5, we construct the sequence $\{\omega_n\}$ by iterating

$$\omega_{n+1} = T\omega_n, \forall\, n \in \mathbb{N},$$

where $\omega_0 \in X$ is arbitrary starting point. Then, by (5), we have

$$A\left(\underbrace{\omega_n, \omega_n, \dots, \omega_n}_{(n-1)\ times}, \omega_{n+1}\right) = A\left(\underbrace{T\omega_{n-1}, T\omega_{n-1}, \dots, T\omega_{n-1}}_{(n-1)\ times}, T\omega_n\right)$$

$$\leq \lambda\left(A\left(\underbrace{\omega_{n-1}, \omega_{n-1}, \dots, \omega_{n-1}}_{(n-1)\ times}, \omega_n\right)\right)^{\alpha}\left(A\left(\underbrace{\omega_{n-1}, \omega_{n-1}, \dots, \omega_{n-1}}_{(n-1)\ times}, T\omega_{n-1}\right)\right)^{\beta}\left(A\left(\underbrace{\omega_n, \omega_n, \dots, \omega_n}_{(n-1)\ times}, T\omega_n\right)\right)^{\gamma}$$

$$\leq \lambda\left(A\left(\underbrace{\omega_{n-1}, \omega_{n-1}, \dots, \omega_{n-1}}_{(n-1)\ times}, \omega_n\right)\right)^{\alpha}\left(A\left(\underbrace{\omega_{n-1}, \omega_{n-1}, \dots, \omega_{n-1}}_{(n-1)\ times}, \omega_n\right)\right)^{\beta}\left(A\left(\underbrace{\omega_n, \omega_n, \dots, \omega_n}_{(n-1)\ times}, \omega_{n+1}\right)\right)^{\gamma}$$

Since $\alpha + \beta < 1 - \gamma$, the last inequality gives

$$\left(A\left(\underbrace{\omega_n, \omega_n, \dots, \omega_n}_{(n-1)\ times}, \omega_{n+1}\right)\right)^{1-\gamma} \leq \lambda\left(A\left(\underbrace{\omega_{n-1}, \omega_{n-1}, \dots, \omega_{n-1}}_{(n-1)\ times}, \omega_n\right)\right)^{\alpha+\beta}$$

$$\leq \lambda\left(A\left(\underbrace{\omega_{n-1}, \omega_{n-1}, \dots, \omega_{n-1}}_{(n-1)\ times}, \omega_n\right)\right)^{1-\gamma} \qquad (18)$$

$$A\left(\underbrace{\omega_n, \omega_n, \dots, \omega_n}_{(n-1)\ times}, \omega_{n+1}\right) \leq \lambda^{\frac{1}{1-\gamma}} A\left(\underbrace{\omega_{n-1}, \omega_{n-1}, \dots, \omega_{n-1}}_{(n-1)\ times}, \omega_n\right)$$

$$\leq \lambda A\left(\underbrace{\omega_{n-1}, \omega_{n-1}, \dots, \omega_{n-1}}_{(n-1)\ times}, \omega_n\right)$$

and then

$$A\Big(\underbrace{\omega_n,\omega_n,\dots,\omega_n}_{(n-1)\,times},\omega_{n+1}\Big)\leq \lambda A\Big(\underbrace{\omega_{n-1},\omega_{n-1},\dots,\omega_{n-1}}_{(n-1)\,times},\omega_n\Big)$$

$$\leq \lambda^2 A\Big(\underbrace{\omega_{n-2},\omega_{n-2},\dots,\omega_{n-2}}_{(n-1)\,times},\omega_{n-1}\Big)$$

$$\leq \cdots \leq \lambda^n A\Big(\underbrace{\omega_0,\omega_0,\dots,\omega_0}_{(n-1)\,times},\omega_1\Big) \tag{19}$$

A fixed-point $\omega^* \in X$ is produced by the classical process, as was previously explained in the proof of Theorem 2.5. We now demonstrate ω^*'s uniqueness. If μ^* be another fixed point of T in X, then $T\mu^* = \mu^*$. As of (5), we now have

$$A\Big(\underbrace{\omega^*,\omega^*,,\dots,\omega^*}_{(n-1)\,times},\mu^*\Big) = A\Big(\underbrace{T\omega^*,T\omega^*,\dots,T\omega^*}_{(n-1)\,times},T\mu^*\Big)$$

$$\leq \lambda\Bigg(A\Big(\underbrace{\omega^*,\omega^*,,\dots,\omega^*}_{(n-1)\,times},\mu^*\Big)\Bigg)^{\alpha}\Bigg(A\Big(\underbrace{\omega^*,\omega^*,\dots,\omega^*}_{(n-1)\,times},T\omega^*\Big)\Bigg)^{\beta}\Bigg(A\Big(\underbrace{\mu^*,\mu^*,\dots,\mu^*}_{(n-1)\,times},T\mu^*\Big)\Bigg)^{\gamma} = 0 \tag{20}$$

This yields that $\omega^* = \mu^*$. This completes the proof.

Theorem 2.8 Let (X, A) be a complete A-metric space. Let $T: X \longrightarrow X$ be a (λ,α,β)-interpolative Dass-Gupta rational contraction. Then T has a unique fixed point.

Proof Following the steps of proof of Theorem 2.5, we construct the sequence $\{\omega_n\}$ by iterating

$$\omega_{n+1} = T\omega_n, \forall\, n \in \mathbb{N},$$

where $\omega_0 \in X$ is arbitrary starting point. Then, by (6), we have

$$A\Big(\underbrace{\omega_n,\omega_n,\dots,\omega_n}_{(n-1)\,times},\omega_{n+1}\Big)=A\Big(\underbrace{T\omega_{n-1},T\omega_{n-1},\dots,T\omega_{n-1}}_{(n-1)\,times},T\omega_n\Big)$$

$$\leq\lambda\left(A\Big(\underbrace{\omega_{n-1},\omega_{n-1},\dots,\omega_{n-1}}_{(n-1)\,times},\omega_n\Big)\right)^{\alpha}\left(\frac{\left[1+A\Big(\underbrace{\omega_{n-1},\omega_{n-1},\dots,\omega_{n-1}}_{(n-1)\,times},T\omega_{n-1}\Big)\right]A\Big(\underbrace{\omega_n,\omega_n,\dots,\omega_n}_{(n-1)\,times},T\omega_n\Big)}{1+A\Big(\underbrace{\omega_{n-1},\omega_{n-1},\dots,\omega_{n-1}}_{(n-1)\,times},\omega_n\Big)}\right)^{\beta}$$

$$\leq\lambda\left(A\Big(\underbrace{\omega_{n-1},\omega_{n-1},\dots,\omega_{n-1}}_{(n-1)\,times},\omega_n\Big)\right)^{\alpha}\left(\frac{\left[1+A\Big(\underbrace{\omega_{n-1},\omega_{n-1},\dots,\omega_{n-1}}_{(n-1)\,times},\omega_n\Big)\right]A\Big(\underbrace{\omega_n,\omega_n,\dots,\omega_n}_{(n-1)\,times},\omega_{n+1}\Big)}{1+A\Big(\underbrace{\omega_{n-1},\omega_{n-1},\dots,\omega_{n-1}}_{(n-1)\,times},\omega_n\Big)}\right)^{\beta}$$

$$=\lambda\left(A\Big(\underbrace{\omega_{n-1},\omega_{n-1},\dots,\omega_{n-1}}_{(n-1)\,times},\omega_n\Big)\right)^{\alpha}\left(A\Big(\underbrace{\omega_n,\omega_n,\dots,\omega_n}_{(n-1)\,times},\omega_{n+1}\Big)\right)^{\beta}$$

Since $\alpha+\beta<1$, the last inequality gives

$$\left(A\Big(\underbrace{\omega_n,\omega_n,\dots,\omega_n}_{(n-1)\,times},\omega_{n+1}\Big)\right)^{1-\beta}\leq\lambda\left(A\Big(\underbrace{\omega_{n-1},\omega_{n-1},\dots,\omega_{n-1}}_{(n-1)\,times},\omega_n\Big)\right)^{\alpha}$$

$$\leq\lambda\left(A\Big(\underbrace{\omega_{n-1},\omega_{n-1},\dots,\omega_{n-1}}_{(n-1)\,times},\omega_n\Big)\right)^{1-\beta}\qquad(21)$$

$$A\Big(\underbrace{\omega_n,\omega_n,\dots,\omega_n}_{(n-1)\,times},\omega_{n+1}\Big)\leq\lambda^{\frac{1}{1-\beta}}A\Big(\underbrace{\omega_{n-1},\omega_{n-1},\dots,\omega_{n-1}}_{(n-1)\,times},\omega_n\Big)$$

$$\leq\lambda A\Big(\underbrace{\omega_{n-1},\omega_{n-1},\dots,\omega_{n-1}}_{(n-1)\,times},\omega_n\Big)$$

and then

$$A\Big(\underbrace{\omega_n,\omega_n,\dots,\omega_n}_{(n-1)\,times},\omega_{n+1}\Big)\leq\lambda A\Big(\underbrace{\omega_{n-1},\omega_{n-1},\dots,\omega_{n-1}}_{(n-1)\,times},\omega_n\Big)$$

$$\leq \lambda^2 A\left(\underbrace{\omega_{n-2}, \omega_{n-2}, \dots, \omega_{n-2}}_{(n-1)\ times}, \omega_{n-1}\right)$$

$$\leq \cdots \leq \lambda^n A\left(\underbrace{\omega_0, \omega_0, \dots, \omega_0}_{(n-1)\ times}, \omega_1\right) \tag{22}$$

As already elaborated in the proof of Theorem 2.5, the classical procedure leads to the existence of a fixed-point $\omega^* \in X$. Now, we prove the uniqueness of ω^*. Let μ^* be another fixed point of T in X, then $T\mu^* = \mu^*$. Now, by (6), we have

$$A\left(\underbrace{\omega^*, \omega^*,, \dots, \omega^*}_{(n-1)\ times}, \mu^*\right) = A\left(\underbrace{T\omega^*, T\omega^*, \dots, T\omega^*}_{(n-1)\ times}, T\mu^*\right)$$

$$\leq \lambda\left(A\left(\underbrace{\omega^*, \omega^*,, \dots, \omega^*}_{(n-1)\ times}, \mu^*\right)\right)^{\alpha}\left(\frac{\left[1+A\left(\underbrace{\omega^*, \omega^*, \dots, \omega^*}_{(n-1)\ times}, T\omega^*\right)\right]A\left(\underbrace{\mu^*, \mu^*, \dots, \mu^*}_{(n-1)\ times}, T\mu^*\right)}{1+A\left(\underbrace{\omega^*, \omega^*,, \dots, \omega^*}_{(n-1)\ times}, \mu^*\right)}\right)^{\beta}$$

$$= 0 \tag{23}$$

This yields that $\omega^* = \mu^*$. This completes the proof.

4. Conclusion

In this paper, using the new framework of A-metric spaces, we introduced the concept of (λ, α)-interpolative Kannan contraction, $(\lambda\ \alpha, \beta)$-interpolative Kannan contraction and $(\lambda, \alpha, \beta, \gamma)$-interpolative Riech contraction and $(\lambda\ \alpha, \beta)$-interpolative Dass-Gupta rational contraction and proved the existence of fixed points for self-mapping.

References

[1] Mujahid Abbas, Bashir Ali, Yusuf I Suleiman: Generalized coupled common fixed-point results in partially ordered *A*-metric spaces, *Fixed-point Theory and Applications*, (2015) 2015:64, doi: 10.1186/s13663-015-0309-2.

[2] R. Kannan: Some results on fixed-point s. Bull. Calcutta Math. Soc. **60**, 71–76 (1968).

[3] S. Reich: Kannan's fixed-point theorem. Boll. Un. Mat. Ital. (4) **4**, 1–11 (1971).

[4] Yae Ulrich Gaba, E. Karapinar, A new approach to the interpolative contractions, *Axioms*, 8 (2019), 1-4.

[5] E. Karapinar, Revisiting the Kannan Type Contractions via Interpolation, *Advances in theory of nonlinear analysis and its applications,* 2, 2(2018), 8587.

[6] B. K. Dass and S. Gupta, "An extension of Banach contraction principle through rational expressions," Indian J. Pure Appl. Math. vol. 6, 1455-1458, 1975.

[7] M. Nazam, H. Aydi and M. Arshad, "A real generalization of the Dass-Gupta fixed point theorem," TWMS J. Pure Appl. Math. 11(1) (2020), 109-118.

[8] E. Karapinar, Revisiting the Kannan Type Contractions via Interpolation, *Advances in theory of nonlinear analysis and its applications,* 2, 2(2018), 8587.

[9] I. A. Bakhtin, "The contraction mapping principle in almost metric space," *Functional Analysis and its Applications*, vol. 30, pp. 26–37, 1989.

[10] S. Czerwik, "Contraction mapping in b-metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, no. 1, pp. 5–11, 1993.

[11] S. Czerwik, "Nonlinear set-valued contraction mappings in b-metric spaces," *Atti del Seminario Matematico e Fisico dell'Università di Modena*, vol. 46, no. 2, pp. 263–276, 1998.

[12] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[13] I. A. Bakhtin, "The contraction principle in quasi-metric spaces," *Journal of Functional Analysis*, vol. 30, pp. 26–37, 1989.

[14] S. G. Matthews, “Partial metric topology,” *Annals of the New York Academy of Sciences*, vol. 728, no. 1, pp. 183–197, 1994.