The Minimum Superior Dominating Energy of Graphs

ABSTRACT

Kathiresan and Marimuthu were the pioneers of superior distance in graphs. The same authors put forth the concept of superior domination in 2008. Superior distance is the shortest walk between any two vertices including their respective neighbours. The minimum superior dominating energy $\mathbb{E}_{sd}(G)$ is defined by the sum of the eigenvalues and it is obtained from the minimum superior dominating $n \times n$ matrix $\mathbb{A}_{sd}(G) = (e_{ij})$. The minimum superior dominating energy for star and crown graphs are computed. Properties of eigenvalues of minimum superior dominating matrix for star, cocktail party, complete and crown graphs are discussed. Results related to upper and lower bounds of minimum superior dominating energy for star, cocktail party, complete and crown graphs are stated and proved.

Keywords: Minimum superior dominating set, Minimum superior dominating energy, Superior distance, Superior neighbour, Superior dominating eigenvalues.

1. INTRODUCTION

Kathiresan and Marimuthu[1] were the pioneers of superior distance in graphs. Let $D_{uv} = N[u] \cup N[v]$. A D_{uv} walk is defined as a $u - v$ walk in Gthat contains every vertex of D_{uv}. The superior distance $d_s(u,v)$ from u to v is the length of theshortest D_{uv} walk.The superior neighbour of a vertex u is given by $e_{sn}(u) = min\{d_s(u,v) : v \in V(G) - \{u\}\}$. A vertex $v(\neq u)$ is called a superior neighbourhood vertex of u if $d_s(u,v) = e_{sn}(u)$. The same authors[2]"put forth the concept of superior domination in 2008. Superior distance is the shortest walk between any two vertices including their respective neighbours". Different types of domination were also developed. A subset D of V is said to be a dominating set, if every vertex not in D isadjacent to at least one vertex in D. IvanGutman[3] in 1978 introduced "energy of a graph. Inspired by Gutman many authors have explored different types of energy in graph theory". Kanna et al[4] found "the minimum dominating energy of a graph". Inspired by Kanna et al[4]"minimum superior dominating energy $\mathbb{E}_{sd}(G)$ of graphs is introduced in this paper, also $\mathbb{E}_{sd}(G)$ of standard graphs, properties of $\mathbb{E}_{sd}(G)$ and bounds of $\mathbb{E}_{sd}(G)$ are studied".

2. THE MINIMUM SUPERIOR DOMINATING ENERGY

In this section, minimum superior dominating matrix $\mathbb{A}_{sd}(G)$ and minimum superior dominating energy $\mathbb{E}_{sd}(G)$ are defined. The $\mathbb{E}_{sd}(G)$ of some standard graphs are obtained.

Definition 2.1: For $G = (V,E)$ the superior neighbourhood vertex set of a vertex u is given by $E_{sn}(u) = \{v \,/\, d_s(u,v) = e_{sn}(u) \;\forall\, u,v \in V(G)\}$. Let D be a minimum superior dominating set of G then the minimum superior dominating matrix of G is a $n \times n$ defined by $\mathbb{A}_{sd}(G) = (e_{ij})$, where

$$(e_{ij}) = \begin{cases} 1, & if\ v_i \in E_{sn}(v_j)\ or\ v_j \in E_{sn}(v_i), \\ 1, & if\ i = j\ and\ v_i \in D, \\ 0, & otherwise \end{cases}$$

Definition 2.2: The characteristic polynomial of $\mathbb{A}_{sd}(G)$ is $\mathcal{L}_n(G,\alpha) = \det(\mathbb{A}_{sd} - \alpha I)$, where I is the identity matrix.

Definition 2.3: The minimum superior dominating eigenvalues of G are the eigenvalues of $\mathbb{A}_{sd}(G)$. Since $\mathbb{A}_{sd}(G)$ is symmetric and real, the eigenvalues are real. The eigenvalues are arranged in non-increasing order $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$.

Definition 2.4: The minimum superior dominating energy $\mathbb{E}_{sd}(G)$ of G is defined by $\mathbb{E}_{sd}(G) = \sum_{i=1}^{n} |\alpha_i|$.

Remark 2.1: The trace of $\mathbb{A}_{sd}(G)$= Superior Domination number.

Example 2.1: The R graph consists of 6 vertices and 6 edges given in Fig.1

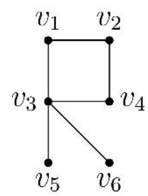

Fig. 1.R Graph

Table 1. Superior neighbour distance and superior neighbour vertex set of R graph

Vertex	Superior neighbour distance $d_s(v) = \min\{d_s(v,u) \forall u \in V(G)\}$	Superior neighbour vertex set $E_{sn}(v) = \{u/d_s(v,u) = e_{sn}(v) \forall u \in V(G)\}$
v_1	3	$\{v_2\}$
v_2	3	$\{v_1, v_4\}$
v_3	7	$\{v_1, v_4, v_5, v_6\}$
v_4	3	$\{v_2\}$
v_5	2	$\{v_6\}$
v_6	2	$\{v_5\}$

The minimum superior dominating sets of an R graph are $D_1 = \{v_1, v_3\}$, $D_2 = \{v_2, v_3\}$ and $D_3 = \{v_3, v_4\}$.

1. $D_1 = \{v_1, v_3\}$,

$$\mathbb{A}_{sd}(G) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial $\mathcal{L}_n(G,\alpha) = \alpha^6 - 2\alpha^5 - 6\alpha^4 + 6\alpha^3 + 8\alpha^2 - 2\alpha - 1$.
Minimum superior dominating eigenvalues are $\alpha_1 \approx 3.0675, \alpha_2 \approx 1.5033, \alpha_3 \approx 0.4512, \alpha_4 \approx -0.2751, \alpha_5 \approx -1, \alpha_6 \approx -1.7469$.
Minimum superior dominating energy $\mathbb{E}_{sd}(G) \approx 8.044$.

2. $D_2 = \{v_2, v_3\}$,

$$\mathbb{A}_{sd}(G)=\begin{pmatrix}0&1&1&0&0&0\\1&1&0&1&0&0\\1&0&1&1&1&1\\0&1&1&0&0&0\\0&0&1&0&0&1\\0&0&1&0&1&0\end{pmatrix}$$

The characteristic polynomial $\mathcal{L}_n(G,\alpha)=\alpha^6-2\alpha^5-6\alpha^4+6\alpha^3+9\alpha^2$.
Minimum superior dominating eigenvalues are $\alpha_1\approx 3, \alpha_2\approx 1.7320, \alpha_3\approx 0, \alpha_4\approx 0, \alpha_5\approx -1, \alpha_6\approx -1.7320$.
Minimum superior dominating energy $\mathbb{E}_{sd}(G)\approx 7.464$.

3. $D_3=\{v_3,v_4\}$,

$$\mathbb{A}_{sd}(G)=\begin{pmatrix}0&1&1&0&0&0\\1&0&0&1&0&0\\1&0&1&1&1&1\\0&1&1&1&0&0\\0&0&1&0&0&1\\0&0&1&0&1&0\end{pmatrix}$$

The characteristic polynomial $\mathcal{L}_n(G,\alpha)=\alpha^6-2\alpha^5-6\alpha^4+6\alpha^3+8\alpha^2-2\alpha-1$.
Minimum superior dominating eigenvalues are $\alpha_1\approx 3.0675, \alpha_2\approx 1.5033, \alpha_3\approx 0.4512, \alpha_4\approx -0.2751, \alpha_5\approx -1, \alpha_6\approx -1.7469$.
Minimum superior dominating energy $\mathbb{E}_{sd}(G)\approx 8.044$.

$\mathbb{E}_{sd}(G)$ of D_1 and D_3 is 8.044, but $\mathbb{E}_{sd}(G)$ of D_2 is 7.464. Therefore $\mathbb{E}_{sd}(G)$ varies based on the superior dominating set.

Remark 2.2: The minimum superior dominating energy $\mathbb{E}_{sd}(G)$ depends on the superior dominating set.

Theorem 2.1: For a star graph S_n where $n\geq 2$ the minimum superior dominating energy of star $\mathbb{E}_{sd}(S_n)\ =\ |-1|(n-2)+\left|\frac{(n-1)+\sqrt{(n-1)^2+4}}{2}\right|+\left|\frac{(n-1)-\sqrt{(n-1)^2+4}}{2}\right|$.

Proof: Consider a star graph S_n with the vertex set $V=\{v_1,v_2,\dots,v_n\}$. The minimum superior dominating set is $D=\{v_1\}$ where $\deg(v_1)=\Delta(S_n)$ then

$$\mathbb{A}_{sd}(S_n)=\begin{pmatrix}1&1&1&&1&1&1\\1&0&1&\cdots&1&1&1\\1&1&0&&1&1&1\\&\vdots&&\ddots&&\vdots&\\1&1&1&&0&1&1\\1&1&1&\cdots&1&0&1\\1&1&1&&1&1&0\end{pmatrix}_{n\times n}$$

Characteristic polynomial is $\mathcal{L}_n(S_n,\alpha)=\det(\mathbb{A}_{sd}(S_n)-\alpha I)$.

$$=\begin{vmatrix}1-\alpha&1&1&&1&1&1\\1&-\alpha&1&\cdots&1&1&1\\1&1&-\alpha&&1&1&1\\&\vdots&&\ddots&&\vdots&\\1&1&1&&-\alpha&1&1\\1&1&1&\cdots&1&-\alpha&1\\1&1&1&&1&1&-\alpha\end{vmatrix}$$

The characteristic polynomial $\mathcal{L}_n(S_n,\alpha)=(\alpha+1)^{n-2}(\alpha^2-(n-1)\alpha-1)$
The minimum superior dominating eigenvalues are

$$\alpha=-1\ ((n-2)\ time),$$

$$\alpha=\frac{(n-1)+\sqrt{(n-1)^2+4}}{2},$$

$$\alpha = \frac{(n-1)-\sqrt{(n-1)^2+4}}{2}.$$

The minimum superior dominating energy of the star S_n is given by

$$\mathbb{E}_{sd}(S_n) = |-1|(n-2) + \left|\frac{(n-1)+\sqrt{(n-1)^2+4}}{2}\right| + \left|\frac{(n-1)-\sqrt{(n-1)^2+4}}{2}\right|.$$

Remark 2.3: The energy of cocktail party and complete graph is same as star graph.

Theorem 2.2: For a crown graph H_n where $n \geq 8$ the minimum superior dominating energy of crown $\mathbb{E}_{sd}(H_n) = \left|\frac{\left(\frac{n}{2}+1\right)+\sqrt{\left(\frac{n}{2}-1\right)^2+4}}{2}\right| + \left|\frac{\left(\frac{n}{2}+1\right)-\sqrt{\left(\frac{n}{2}-1\right)^2+4}}{2}\right| + \left|\frac{\left(1-\frac{n}{2}\right)+\sqrt{\left(\frac{n}{2}+1\right)^2-4}}{2}\right| + \left|\frac{\left(1-\frac{n}{2}\right)-\sqrt{\left(\frac{n}{2}+1\right)^2-4}}{2}\right|.$

Proof: Let H_n be a crown graph with the vertex set $V = \{v_1, v_2, \dots, v_n\}$. The minimum superior dominating set is $D = \{v_1, v_{\frac{n}{2}+1}\}$ then

$$\mathbb{A}_{sd}(H_n) = \begin{pmatrix} 1 & 0 & 0 & & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & & 1 & 1 & 1 \\ & \vdots & & \ddots & & \vdots & \\ 1 & 1 & 1 & & 1 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & & 0 & 0 & 0 \end{pmatrix}_{n \times n}$$

Characteristic polynomial is $\mathcal{L}_n(H_n, \alpha) = \det(\mathbb{A}_{sd}(H_n) - \alpha I)$.

$$= \begin{vmatrix} 1-\alpha & 0 & 0 & & 1 & 1 & 1 \\ 0 & -\alpha & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & -\alpha & & 1 & 1 & 1 \\ & \vdots & & \ddots & & \vdots & \\ 1 & 1 & 1 & & 1-\alpha & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & -\alpha & 0 \\ 1 & 1 & 1 & & 0 & 0 & -\alpha \end{vmatrix}$$

The characteristic polynomial

$$\mathcal{L}_n(H_n, \alpha) = \alpha^{n-4}\left[\alpha^2 - \left(\frac{n}{2}+1\right)\alpha + \left(\frac{n}{2}-1\right)\right]\left[\alpha^2 + \left(\frac{n}{2}-1\right)\alpha - \left(\frac{n}{2}-1\right)\right]$$

The minimum superior dominating eigenvalues are

$$\alpha = 0 \ ((n-4)\ time),$$

$$\alpha = \frac{\left(\frac{n}{2}+1\right)+\sqrt{\left(\frac{n}{2}-1\right)^2+4}}{2},$$

$$\alpha = \frac{\left(\frac{n}{2}+1\right)-\sqrt{\left(\frac{n}{2}-1\right)^2+4}}{2},$$

$$\alpha = \frac{\left(1-\frac{n}{2}\right)+\sqrt{\left(\frac{n}{2}+1\right)^2-4}}{2},$$

$$\alpha = \frac{\left(1-\frac{n}{2}\right)-\sqrt{\left(\frac{n}{2}+1\right)^2-4}}{2}.$$

The minimum superior dominating energy of the crown H_n is given by

$$\mathbb{E}_{sd}(H_n) = \left|\frac{\left(\frac{n}{2}+1\right)+\sqrt{\left(\frac{n}{2}-1\right)^2+4}}{2}\right| + \left|\frac{\left(\frac{n}{2}+1\right)-\sqrt{\left(\frac{n}{2}-1\right)^2+4}}{2}\right| + \left|\frac{\left(1-\frac{n}{2}\right)+\sqrt{\left(\frac{n}{2}+1\right)^2-4}}{2}\right| + \left|\frac{\left(1-\frac{n}{2}\right)-\sqrt{\left(\frac{n}{2}+1\right)^2-4}}{2}\right|.$$

3. PROPERTIES OF MINIMUM SUPERIOR DOMINATING EIGENVALUES

In this section properties of eigenvalues of $\mathbb{A}_{sd}(G)$ for star, cocktail party, complete and crown graphs are discussed. Bounds for minimum superior dominating energy $\mathbb{E}_{sd}(G)$ of some standard graphs are obtained.

Theorem 3.1: If D is the minimum superior dominating set and $\alpha_1, \alpha_2, \dots, \alpha_n$ are the eigenvalues of minimum superior dominating matrix $\mathbb{A}_{sd}(G)$ then

1. For any graph G, $\sum_{i=1}^{n} \alpha_i = |D|$,
2. For a star graph S_n, cocktail party and complete graph, $\sum_{i=1}^{n} \alpha_i^2 = |D| + \sum_{i=1}^{n} |E_{sn}(v_i)| + (n-1)$.
3. For a crown graph H_n, $\sum_{i=1}^{n} \alpha_i^2 = |D| + \sum_{i=1}^{n} |E_{sn}(v_i)|$.

Proof:

1. The trace of $\mathbb{A}_{sd}(G)$ is the sum of eigenvalues of $\mathbb{A}_{sd}(G)$.
$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} e_{ii} = |D|$.
2. For a star graph S_n, cocktail party and complete graph, the trace of $[\mathbb{A}_{sd}(G)]^2$ is equal to the summation of the squares of eigenvalues of $\mathbb{A}_{sd}(G)$.
$$\sum_{i=1}^{n} \alpha_i^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} e_{ij} e_{ij} = \sum_{i=1}^{n} (e_{ii})^2 + \sum_{i \neq j} e_{ij} e_{ij} = \sum_{i=1}^{n} (e_{ii})^2 + 2 \sum_{i<j} (e_{ij})^2$$
$$\sum_{i=1}^{n} \alpha_i^2 = |D| + \sum_{i=1}^{n} |E_{sn}(v_i)| + (n-1) \quad \left[Since, 2 \sum_{i<j} (e_{ij})^2 = \sum_{i=1}^{n} |E_{sn}(v_i)| + (n-1)\right]$$
3. For a crown graph H_n, the trace of $[\mathbb{A}_{sd}(H_n)]^2$ is equal to the summation of the squares of eigenvalues of $\mathbb{A}_{sd}(H_n)$.
$$\sum_{i=1}^{n} \alpha_i^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} e_{ij} e_{ij} = \sum_{i=1}^{n} (e_{ii})^2 + \sum_{i \neq j} e_{ij} e_{ij} = \sum_{i=1}^{n} (e_{ii})^2 + 2 \sum_{i<j} (e_{ij})^2$$
$$\sum_{i=1}^{n} \alpha_i^2 = |D| + \sum_{i=1}^{n} |E_{sn}(v_i)| \quad \left[Since, 2 \sum_{i<j} (e_{ij})^2 = \sum_{i=1}^{n} |E_{sn}(v_i)|\right]$$

Theorem 3.2: For a star graph S_n, cocktail party and complete graph, if D be the minimum superior dominating set and $W = |det\ \mathbb{A}_{sd}(G)|$ then
$$\sqrt{|D| + \sum_{i=1}^{n} |E_{sn}(v_i)| + (n-1) + n(n-1) W^{\frac{2}{n}}} \leq \mathbb{E}_{sd}(G) \leq \sqrt{n\left(\sum_{i=1}^{n} |E_{sn}(v_i)| + (n-1) + |D|\right)}$$
Proof: By Cauchy schwarz inequality $(\sum_{i=1}^{n} a_i b_i)^2 \leq (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2)$. If $a_i = 1$ and $b_i = \alpha_i$ then
$$\left(\sum_{i=1}^{n} |\alpha_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} \alpha_i^2\right)$$
$$(\mathbb{E}_{sd}(G))^2 \leq n\left(|D| + \sum_{i=1}^{n} |E_{sn}(v_i)| + (n-1)\right)$$
$$\Rightarrow \mathbb{E}_{sd}(G) \leq \sqrt{n\left(|D| + \sum_{i=1}^{n} |E_{sn}(v_i)| + (n-1)\right)}$$
Since the arithmetic mean is not smaller than geometric mean

$$\frac{1}{n(n-1)}\sum_{i\neq j}|\alpha_i|\,|\alpha_j| \geq \left[\prod_{i\neq j}|\alpha_i||\alpha_j|\right]^{\frac{1}{n(n-1)}} = \left[\prod_{i=1}^{n}|\alpha_i|^{2(n-1)}\right]^{\frac{1}{n(n-1)}} = \left[\prod_{i=1}^{n}|\alpha_i|\right]^{\frac{2}{n}} = \left[\prod_{i=1}^{n}\alpha_i\right]^{\frac{2}{n}}$$

$$\frac{1}{n(n-1)}\sum_{i\neq j}|\alpha_i|\,|\alpha_j| = |\det\,\mathbb{A}_{sd}(G)|^{\frac{2}{n}} = W^{\frac{2}{n}}$$

$$\sum_{i\neq j}|\alpha_i|\,|\alpha_j| \geq n(n-1)W^{\frac{2}{n}}$$

Now consider

$$\left(\mathbb{E}_{sd}(G)\right)^2 = \left(\sum_{i=1}^{n}|\alpha_i|\right)^2 = \left(\sum_{i=1}^{n}|\alpha_i|\right)^2 + \sum_{i\neq j}|\alpha_i|\,|\alpha_j|$$

$$\left(\mathbb{E}_{sd}(G)\right)^2 = \left(|D| + \sum_{i=1}^{n}|E_{sn}(v_i)| + (n-1)\right) + n(n-1)W^{\frac{2}{n}}$$

$$\mathbb{E}_{sd}(G) \geq \sqrt{\left(|D| + \sum_{i=1}^{n}|E_{sn}(v_i)| + (n-1)\right) + n(n-1)W^{\frac{2}{n}}}$$

Theorem 3.3: For a crown graph H_n, if D be the minimum superior dominating set and $W = |\det\,\mathbb{A}_{sd}(H_n)|$ then

$$\sqrt{|D| + \sum_{i=1}^{n}|E_{sn}(v_i)| + n(n-1)W^{\frac{2}{n}}} \leq \mathbb{E}_{sd}(H_n) \leq \sqrt{n\left(\sum_{i=1}^{n}|E_{sn}(v_i)| + |D|\right)}$$

Proof: The proof is similar to Theorem 3.2.

Theorem 3.4: If $\alpha_1(G)$ is the largest minimum superior dominating eigenvalue of $\mathbb{A}_{sd}(G)$ then

1. For a star graph S_n, cocktail party and complete graph, $\alpha_1(G) \geq \frac{|D|+\sum_{i=1}^{n}|E_{sn}(v_i)|+(n-1)}{n}$
2. For a crown graph H_n, $\alpha_1(H_n) \geq \frac{|D|+\sum_{i=1}^{n}|E_{sn}(v_i)|}{n}$

Proof:

1. In a star graph S_n, cocktail party and complete graph, let Y be a non-zero vector, then by ref.[6], $\alpha_1(\mathbb{A}_{sd}(G)) = \max_{Y\neq 0}\frac{Y'\,\mathbb{A}_{sd}(G)\,Y}{Y'\,Y}$
   $\alpha_1(\mathbb{A}_{sd}(G)) \geq \frac{U'\,\mathbb{A}_{sd}(G)\,U}{U'\,U} = \frac{|D|+\sum_{i=1}^{n}|E_{sn}(v_i)|+(n-1)}{n}$ where U is the unit matrix.
2. In a crown graph H_n, let Y be a non-zero vector, then by ref.[6], $\alpha_1(\mathbb{A}_{sd}(H_n)) = \max_{Y\neq 0}\frac{Y'\,\mathbb{A}_{sd}(H_n)\,Y}{Y'\,Y}$
   $\alpha_1(\mathbb{A}_{sd}(H_n)) \geq \frac{U'\,\mathbb{A}_{sd}(H_n)\,U}{U'\,U} = \frac{|D|+\sum_{i=1}^{n}|E_{sn}(v_i)|}{n}$ where U is the unit matrix.

4. CONCLUSION

In this paper minimum superior dominating energy of graph is introduced. The minimum superior dominating energy for star and crown graphs are computed. The properties of eigenvalues of $\mathbb{A}_{sd}(G)$ for star, cocktail party, complete and crown graphs are discussed.Results related to the upper and lower bound of the energy of standard graphs is stated and proved.

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