Minimum Injective Eccentric Dominating Energy of Graphs

ABSTRACT

The minimum injective eccentric dominating energy of a graph is defined. The minimum injective eccentric dominating energy for star, complete and cocktail party graphs are computed. Inspired by McClelland's bounds for energy of a graph, the upper and lower bounds of minimum injective eccentric dominating energy for star, complete and cocktail party graphs are discussed. Properties of eigenvalues of minimum injective eccentric dominating matrix for star, complete and cocktail party graphs are discussed.

Keywords: Domination, Eccentric domination, Injective eccentric dominating eigenvalues, Minimum injective eccentric dominating set, Minimum injective eccentric dominating energy.

1. INTRODUCTION

Ore[13] and Berge[14] introduced domination in graphs.In 1978 Ivan Gutman[1] introduced "the concept energy of a graph. Inspired by Ivan Gutman many authors have explored different types of energy in graph theory". "The concept of minimum dominating energy of a graph was introduced" by M.R. Rajesh Kanna et al[2]. "The concept of eccentric domination was introduced" by T. N. Janakiraman et al[4] in 2010. "The concept of minimum eccentric dominating energy of graphs was introduced" byTejaskumar R, A Mohamed Ismayil and Ivan Gutman[3]. Anwar Alwardi et al[5] introduced "injective domination of graphs". Riyaz Ur Rehman A et al[9] introduced injective eccentric domination in graphs. Inspired by Tejaskumar et al[3]"minimum injective eccentric dominating energy $\mathbb{E}_{ined}(G)$ of graphs is introduced".

2. PRELIMINARIES

Definition 2.1:[11]A subset D of V is said to be a dominating set, if every vertex not in D is adjacent to at least one vertex in D.

Definition 2.2:[4]The eccentricity $e(v)$ of v is the distance to a vertex farthest from v. Thus $e(v) = \max\{d(u,v): u \in V\}$. For a vertex v, each vertex at a distance $e(v)$ from v is an eccentric vertex. Eccentric set of a vertex v is defined as $E(v) = \{u \in V(G): d(u,v) = e(v)\}$.

Definition 2.3:[4]A set $S \subseteq V(G)$ is an eccentric dominating set (ED set) if S is a dominating set of G and for every vertex $v \in V - D$, there exists at least one eccentric vertex of v in S.

Definition 2.4:[9]An eccentric dominating set (ED set)S is called an injective eccentric dominating set (INED set) if for every vertex $v \in V - S$ there exists a vertex $u \in S$ such that $|\Gamma(v,u)| \geq 1$ where $\Gamma(v,u)$ is the set of vertices different from v and u, that are adjacent to both v and u.

Definition 2.5:[2] For $G = (V, E)$, let $\mathbb{M} = (m_{ij})$ be a minimum dominating matrix defined by

$$(m_{ij}) = \begin{cases} 1, & if\ v_i v_j \in E, \\ 1, & if\ i = j\ and\ v_i \in D, \\ 0, & otherwise \end{cases}$$

and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $\mathbb{M}$. The minimum dominating energy $\mathbb{E}_D = \sum_{i=1}^{n} |\lambda_i|$.

3. THE MINIMUM INJECTIVE ECCENTRIC DOMINATING ENERGY

In this section the minimum injective eccentric dominating energy is introduced. The minimum injective eccentric dominating energy for star, complete and cocktail party graphs are computed. Properties, upper and lower bounds of minimum injective eccentric dominating energy for some class of graphs are discussed. The eccentricity and eccentric vertices of antenna graph are tabulated in Table 1. The minimum injective eccentric dominating energy of various standard graphs along with their characteristic equation, eigenvalues and minimum injective eccentric dominating energy have been tabulated in Table 2. For definitions regarding standard graphs refer the textbook 'Graph Classes- A Survey'[10].

Definition 3.1: For $G = (V, E)$, let D be a minimum injective eccentric dominating set (INED set) of G then the minimum injective eccentric dominating (INED) matrix of G is a $n \times n$ defined by $\mathbb{M}_{ined}(G) = (m_{ij})$, where

$$(m_{ij}) = \begin{cases} 1, & if\ |\Gamma(v_i, v_j)| \geq 1\ and\ either\ v_i \in E(v_j)\ or\ v_j \in E(v_i), \\ 1, & if\ i = j\ and\ v_i \in D, \\ 0, & otherwise \end{cases}$$

Definition 3.2: The characteristic polynomial of the minimum INED matrix $\mathbb{M}_{ined}(G)$ is denoted by $\mathcal{F}_n(G, \wp) = \det(\mathbb{M}_{ined}(G) - \wp I)$, where I is the identity matrix.

Definition 3.3: The eigenvalues of $\mathbb{M}_{ined}(G)$ is defined by the minimum INED eigenvalues of G.

Remark 3.1: Since $\mathbb{M}_{ined}(G)$ is symmetric, the eigenvalues are real. We label the eigenvalues in non-increasing order $\wp_1 \geq \wp_2 \geq \cdots \geq \wp_n$.

Definition 3.4: The minimum INED energy of G is defined by $\mathbb{E}_{ined}(G) = \sum_{i=1}^{n} |\wp_i|$, where $\wp_i$ is the eigenvalues of $\mathbb{E}_{ined}(G)$.

Remark 3.2: The trace of $\mathbb{M}_{ined}(G)$= Injective eccentric domination number.

Example 3.1: In Fig.1 antenna graph has 6 vertices and 7 edges.

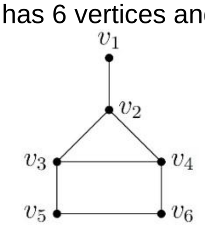


Fig. 1.Antenna graph G

Table 1. Eccentricity and eccentric vertices of antenna graph.

Vertex	Eccentricity $e(v)$	Eccentric vertex $E(v)$
v_1	3	$\{v_5, v_6\}$
v_2	2	$\{v_5, v_6\}$
v_3	2	$\{v_1, v_6\}$

v_4	2	$\{v_1, v_5\}$
v_5	3	$\{v_1\}$
v_6	3	$\{v_1\}$

The minimum INED sets of an antenna graph given in Fig. 1 are $D_1 = \{v_1, v_2, v_5\}$, $D_2 = \{v_1, v_2, v_6\}$, $D_3 = \{v_1, v_3, v_5\}$, $D_4 = \{v_1, v_4, v_6\}$ and $D_5 = \{v_1, v_5, v_6\}$.

1. $D_1 = \{v_1, v_2, v_5\}$,

$$\mathbb{M}_{ined}(G) = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The characteristic polynomial $\mathcal{F}_n(G, \wp) = \wp^6 - 3\wp^5 - 3\wp^4 + 11\wp^3 + 2\wp^2 - 8\wp - 2$.
Minimum injective eccentric dominating eigenvalues are $\wp_1 \approx 2.6129, \wp_2 \approx 1.808, \wp_3 \approx 1.2582, \wp_4 \approx -0.2582, \wp_5 \approx -0.808, \wp_6 \approx -1.6129$.
Minimum injective eccentric dominating energy $\mathbb{E}_{ined}(G) \approx 8.3582$.

2. $D_2 = \{v_1, v_2, v_6\}$,

$$\mathbb{M}_{ined}(G) = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

The characteristic polynomial $\mathcal{F}_n(G, \wp) = \wp^6 - 3\wp^5 - 3\wp^4 + 11\wp^3 + 2\wp^2 - 8\wp - 2$.
Minimum injective eccentric dominating eigenvalues are $\wp_1 \approx 2.6129, \wp_2 \approx 1.808, \wp_3 \approx 1.2582, \wp_4 \approx -0.2582, \wp_5 \approx -0.808, \wp_6 \approx -1.6129$.
Minimum injective eccentric dominating energy $\mathbb{E}_{ined}(G) \approx 8.3582$.

3. $D_3 = \{v_1, v_3, v_5\}$,

$$\mathbb{M}_{ined}(G) = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The characteristic polynomial $\mathcal{F}_n(G, \wp) = \wp^6 - 3\wp^5 - 3\wp^4 + 11\wp^3 + 2\wp^2 - 8\wp - 2$.
Minimum injective eccentric dominating eigenvalues are $\wp_1 \approx 2.6129, \wp_2 \approx 1.808, \wp_3 \approx 1.2582, \wp_4 \approx -0.2582, \wp_5 \approx -0.808, \wp_6 \approx -1.6129$.
Minimum injective eccentric dominating energy $\mathbb{E}_{ined}(G) \approx 8.3582$.

4. $D_4 = \{v_1, v_4, v_6\}$,

$$\mathbb{M}_{ined}(G) = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

The characteristic polynomial $\mathcal{F}_n(G, \wp) = \wp^6 - 3\wp^5 - 3\wp^4 + 11\wp^3 + 2\wp^2 - 8\wp - 2$.

Minimum injective eccentric dominating eigenvalues are $\wp_1 \approx 2.6129, \wp_2 \approx 1.808, \wp_3 \approx 1.2582, \wp_4 \approx -0.2582, \wp_5 \approx -0.808, \wp_6 \approx -1.6129$.
Minimum injective eccentric dominating energy $\mathbb{E}_{ined}(G) \approx 8.3582$.

5. $D_5 = \{v_1, v_5, v_6\}$,

$$\mathbb{M}_{ined}(G) = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

The characteristic polynomial $\mathcal{F}_n(G, \wp) = \wp^6 - 3\wp^5 - 3\wp^4 + 11\wp^3 + 3\wp^2 - 9\wp - 4$.
Minimum injective eccentric dominating eigenvalues are $\wp_1 \approx 2.5616, \wp_2 \approx 1.618, \wp_3 \approx 1.618, \wp_4 \approx -0.618, \wp_5 \approx -0.618, \wp_6 \approx -1.5616$.
Minimum injective eccentric dominating energy $\mathbb{E}_{ined}(G) \approx 8.5952$.

Observation 3.1: The energy of antenna graph G given in the Example 1 varies for different minimum INEDsets.
For the set $D_1, D_2, D_3, D_4, \mathbb{E}_{ined}(G) \approx 8.3582$,
For the set $D_5, \mathbb{E}_{ined}(G) \approx 8.5952$.

Remark 3.3: The minimum INED energy depends on the different minimum INEDset. The examples can be seen in Table 2.

Theorem 3.1: For a star graph S_n where $n > 2$ the minimum INEDenergy of star
$\mathbb{E}_{ined}(S_n) = 1 + (n-3) + \left|\frac{(n-2)+\sqrt{(n-2)^2+4}}{2}\right| + \left|\frac{(n-2)-\sqrt{(n-2)^2+4}}{2}\right|$.

Proof: Consider a star graph S_n with the vertex set $V = \{v_1, v_2, \ldots, v_k, \ldots v_n\}$ where v_k is the central vertex. The minimum INED set is $D = \{v_1, v_k\}$ then

$$\mathbb{M}_{ined}(S_n) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 0 & \cdots & 1 & 1 & 1 \\ 1 & 0 & 1 & \cdots & 0 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 0 & \cdots & 0 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & \cdots & 0 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 0 & \cdots & 1 & 0 & 1 \\ 1 & 1 & 1 & \cdots & 0 & \cdots & 1 & 1 & 0 \end{pmatrix}_{n \times n}$$

Characteristic polynomial is $\mathcal{F}_n(S_n, \wp) = \det(\mathbb{M}_{ined}(S_n) - \wp I)$.

$$= \begin{vmatrix} 1-\wp & 1 & 1 & \cdots & 0 & \cdots & 1 & 1 & 1 \\ 1 & -\wp & 1 & \cdots & 0 & \cdots & 1 & 1 & 1 \\ 1 & 1 & -\wp & \cdots & 0 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-\wp & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & \cdots & -\wp & 1 & 1 \\ 1 & 1 & 1 & \cdots & 0 & \cdots & 1 & -\wp & 1 \\ 1 & 1 & 1 & \cdots & 0 & \cdots & 1 & 1 & -\wp \end{vmatrix}$$

The characteristic polynomial $\mathcal{F}_n(S_n, \wp) = (-1)^n(\wp - 1)(\wp + 1)^{n-3}(\wp^2 - (n-2)\wp - 1)$.
The minimum injective eccentric dominating eigenvalues are

$$\wp = 1,$$
$$\wp = -1\ ((n-3)\ \ times),$$

$$\wp = \frac{(n-2)+\sqrt{(n-2)^2+4}}{2},$$
$$\wp = \frac{(n-2)-\sqrt{(n-2)^2+4}}{2}.$$

The minimum INED energy of the star S_n is given by

$$\mathbb{E}_{ined}\,(S_n) = 1 + |(-1)|(n-3) + \left|\frac{(n-2)+\sqrt{(n-2)^2+4}}{2}\right| + \left|\frac{(n-2)-\sqrt{(n-2)^2+4}}{2}\right|.$$
$$\mathbb{E}_{ined}\,(S_n) = 1 + (n-3) + \left|\frac{(n-2)+\sqrt{(n-2)^2+4}}{2}\right| + \left|\frac{(n-2)-\sqrt{(n-2)^2+4}}{2}\right|.$$

Remark 3.4: The number of pairs of vertices where the common neighbourhood exists in a star graph S_n is given by $(n-1)(n-2)$.

Theorem 3.2: For a graph K_n where $n > 2$ the minimum INEDenergy of complete graph

$$\mathbb{E}_{ined}\,(K_n) = (n-2) + \left|\frac{(n-1)+\sqrt{(n-1)^2+4}}{2}\right| + \left|\frac{(n-1)-\sqrt{(n-1)^2+4}}{2}\right|.$$

Proof: Let $V = \{v_1, v_2, \dots v_n\}$ be the vertex set of a complete graph K_n. The minimum INED set is $D = \{v_1\}$ then

$$\mathbb{M}_{ined}\,(K_n) = \begin{pmatrix} 1 & 1 & 1 & & 1 & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 0 & & 1 & 1 & 1 \\ & \vdots & & \ddots & & \vdots & \\ 1 & 1 & 1 & & 0 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 1 \\ 1 & 1 & 1 & & 1 & 1 & 0 \end{pmatrix}_{n\times n}$$

Characteristic polynomial is $\mathcal{F}_n(K_n, \wp) = \det(\mathbb{M}_{ined}\,(K_n) - \wp I)$.

$$= \begin{vmatrix} 1-\wp & 1 & 1 & & 1 & 1 & 1 \\ 1 & -\wp & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & -\wp & & 1 & 1 & 1 \\ & \vdots & & \ddots & & \vdots & \\ 1 & 1 & 1 & & -\wp & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & -\wp & 1 \\ 1 & 1 & 1 & & 1 & 1 & -\wp \end{vmatrix}$$

The characteristic polynomial $\mathcal{F}_n(K_n, \wp) = (-1)^{n-2}(\wp+1)^{n-2}(\wp^2 - (n-1)\wp - 1)$.

The minimum INED eigenvalues are

$$\wp = -1\ ((n-2)\ \ times),$$
$$\wp = \frac{(n-1)+\sqrt{(n-1)^2+4}}{2},$$
$$\wp = \frac{(n-1)-\sqrt{(n-1)^2+4}}{2}.$$

The minimum INED energy of the complete graph K_n is given by

$$\mathbb{E}_{ined}\,(K_n) = |(-1)|(n-2) + \left|\frac{(n-1)+\sqrt{(n-1)^2+4}}{2}\right| + \left|\frac{(n-1)-\sqrt{(n-1)^2+4}}{2}\right|.$$
$$\mathbb{E}_{ined}\,(K_n) = (n-2) + \left|\frac{(n-1)+\sqrt{(n-1)^2+4}}{2}\right| + \left|\frac{(n-1)-\sqrt{(n-1)^2+4}}{2}\right|.$$

Theorem 3.3: For a cocktail party graph G where $n \geq 4$ the minimum INED energy of cocktail party graph $\mathbb{E}_{ined}\,(G) = \left[\left|\frac{1+\sqrt{5}}{2}\right| + \left|\frac{1-\sqrt{5}}{2}\right|\right]\frac{n}{2}$.

Proof: Let $V = \{v_1, v_2, \dots v_n\}$ be the vertex set of cocktail party graph. The minimum INED set is $D = \{v_1, v_2, \dots v_{n/2}\}$, $|D| = n/2$ then

$$\mathbb{M}_{ined}(G) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & \dots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix}_{n\times n}$$

Characteristic polynomial is $\mathcal{F}_n(G,\wp) = \det(\mathbb{M}_{ined}(G) - \wp I)$.

$$= \begin{vmatrix} 1-\wp & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 1-\wp & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 0 & 1-\wp & 0 & \dots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\wp & \dots & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & \dots & -\wp & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & -\wp & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & -\wp & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & -\wp \end{vmatrix}$$

The characteristic polynomial $\mathcal{F}_n(G,\wp) = (\wp^2 - \wp - 1)^{\frac{n}{2}}$.
The minimum injective eccentric dominating eigenvalues are

$$\wp = \frac{1+\sqrt{5}}{2}\left(\left(\frac{n}{2}\right)\ times\right),$$
$$\wp = \frac{1-\sqrt{5}}{2}\left(\left(\frac{n}{2}\right)\ times\right).$$

The minimum INED energy of the cocktail party graph G is given by
$\mathbb{E}_{ined}(G) = \left[\left|\frac{1+\sqrt{5}}{2}\right| + \left|\frac{1-\sqrt{5}}{2}\right|\right]\frac{n}{2}$.

4. PROPERTIES OF MINIMUM INJECTIVE ECCENTRIC DOMINATING EIGENVALUES

Theorem 4.1: If D is the minimum INED set and $\wp_1, \wp_2, \dots, \wp_n$ are the eigenvalues of minimum INED matrix $\mathbb{M}_{ined}(G)$ then

1. For any graph G, $\sum_{i=1}^{n} \wp_i = |D|$,
2. For a star graph S_n, $\sum_{i=1}^{n} \wp_i^2 = |D| + (n-1)(n-2)$,
3. For a complete graph K_n, $\sum_{i=1}^{n} \wp_i^2 = |D| + (n)(n-1)$,
4. For a cocktail party graph G,$\sum_{i=1}^{n} \wp_i^2 = |D| + n$.

Proof:

1. The trace of $\mathbb{M}_{ined}(G)$ is the sum of eigenvalues of $\mathbb{M}_{ined}(G)$.
$\sum_{i=1}^{n} \wp_i = \sum_{i=1}^{n} m_{ii} = |D|$.
2. For a star graph S_n, sum of the squares of eigenvalues of $\mathbb{M}_{ined}(G)$ is trace of $[\mathbb{M}_{ined}(G)]^2$

$$\sum_{i=1}^{n} \wp_i^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}m_{ij} = \sum_{i=1}^{n}(m_{ii})^2 + \sum_{i\neq j} m_{ij}m_{ij} = \sum_{i=1}^{n}(m_{ii})^2 + 2\sum_{i<j}(m_{ij})^2$$
$$\sum_{i=1}^{n} \wp_i^2 = |D| + (n-1)(n-2)$$

Since for a star graph S_n, $2\sum_{i<j}(m_{ij})^2 = (n-1)(n-2)$.

3. For a complete graph K_n sum of square of eigenvalues of $\mathbb{M}_{ined}(G)$ is trace of $[\mathbb{M}_{ined}(G)]^2$.

$$\sum_{i=1}^{n} \wp_i^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij} m_{ij} = \sum_{i=1}^{n} (m_{ii})^2 + \sum_{i\neq j} m_{ij} m_{ij} = \sum_{i=1}^{n} (m_{ii})^2 + 2\sum_{i<j} \left(m_{ij}\right)^2$$

$$\sum_{i=1}^{n} \wp_i^2 = |D| + (n)(n-1)$$

Since for a complete graph K_n, $2\sum_{i<j}\left(m_{ij}\right)^2 = (n)(n-1)$.

4. Similarly, for a cocktail graph G sum of square of eigenvalues of $\mathbb{M}_{ined}(G)$ is trace of $[\mathbb{M}_{ined}(G)]^2$.

$$\sum_{i=1}^{n} \wp_i^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij} m_{ij} = \sum_{i=1}^{n} (m_{ii})^2 + \sum_{i\neq j} m_{ij} m_{ij} = \sum_{i=1}^{n} (m_{ii})^2 + 2\sum_{i<j} \left(m_{ij}\right)^2$$

$$\sum_{i=1}^{n} \wp_i^2 = |D| + n$$

Since for a cocktail graph G, $2\sum_{i<j}\left(m_{ij}\right)^2 = n$.

Theorem 4.2: For a star graph S_n, where $n > 3$, if D be the minimum INED set and $W = |\det \mathbb{M}_{ined}(G)|$ then

$$\sqrt{|D| + (n-1)(n-2) + n(n-1)W^{\frac{2}{n}}} \leq \mathbb{E}_{ined}(G) \leq \sqrt{n((n-1)(n-2) + |D|)}$$

Proof: By Cauchy schwarz inequality $(\sum_{i=1}^{n} g_i h_i)^2 \leq (\sum_{i=1}^{n} g_i^2)(\sum_{i=1}^{n} h_i^2)$. If $g_i = 1$ and $h_i = \wp_i$ then

$$\left(\sum_{i=1}^{n} |\wp_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} \wp_i^2\right)$$

$$(\mathbb{E}_{ined}(G))^2 \leq n(|D| + (n-1)(n-2))$$

$$\Rightarrow \mathbb{E}_{ined}(G) \leq \sqrt{n(|D| + (n-1)(n-2))}$$

Since the arithmetic mean is not smaller than geometric mean

$$\frac{1}{n(n-1)}\sum_{i\neq j} |\wp_i|\,|\wp_j| \geq \left[\prod_{i\neq j} |\wp_i||\wp_j|\right]^{\frac{1}{n(n-1)}} = \left[\prod_{i=1}^{n} |\wp_i|^{2(n-1)}\right]^{\frac{1}{n(n-1)}} = \left[\prod_{i=1}^{n} |\wp_i|\right]^{\frac{2}{n}} = \left[\prod_{i=1}^{n} \wp_i\right]^{\frac{2}{n}}$$

$$\frac{1}{n(n-1)}\sum_{i\neq j} |\wp_i|\,|\wp_j| = |\det \mathbb{M}_{ined}(S_n)|^{\frac{2}{n}} = W^{\frac{2}{n}}$$

$$\sum_{i\neq j} |\wp_i|\,|\wp_j| \geq n(n-1)W^{\frac{2}{n}}$$

Now consider

$$(\mathbb{E}_{ined}(S_n))^2 = \left(\sum_{i=1}^{n} |\wp_i|\right)^2 = \left(\sum_{i=1}^{n} \wp_i\right)^2 + \sum_{i\neq j} |\wp_i||\wp_j|$$

$$(\mathbb{E}_{ined}(S_n))^2 = (|D| + (n-1)(n-2)) + n(n-1)W^{\frac{2}{n}}$$

$$\mathbb{E}_{ined}(S_n) \geq \sqrt{(|D| + (n-1)(n-2)) + n(n-1)W^{\frac{2}{n}}}$$

Theorem 4.3: For a complete graph K_n where $n > 2$, if D be the minimum INED set and $W = |\det \mathbb{M}_{ined}(G)|$ then

$$\sqrt{|D| + n(n-1) + n(n-1)W^{\frac{2}{n}}} \leq \mathbb{E}_{ined}\,(K_n) \leq \sqrt{n(n(n-1) + |D|)}$$

Proof: The proof follows on thesimilar lines to Theorem4.2.

Theorem 4.4: For a cocktail graph G where $n \geq 4$, if D be the minimum INED set and $W = |\det \mathbb{M}_{ined}\,(G)|$ then

$$\sqrt{|D| + n + n(n-1)W^{\frac{2}{n}}} \leq \mathbb{E}_{ined}\,(G) \leq \sqrt{n(n + |D|)}$$

Proof: The proof follows on the similar lines to Theorem4.2.

Theorem 4.5: If $\wp_1(G)$ is the largest minimum INEDeigenvalue of $\mathbb{M}_{ined}\,(G)$ then

1. For a star graph S_n, $\wp_1(S_n) \geq \frac{|D|+(n-1)(n-2)}{n}$
2. For a complete graph K_n, $\wp_1(K_n) \geq \frac{|D|+n(n-1)}{n}$
3. For a cocktail graph G, $\wp_1(G) \geq \frac{|D|+n}{n}$

Proof:

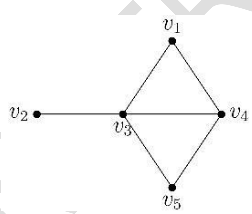
1. Let Y be a non-zero vector, then by [6],

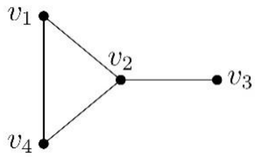
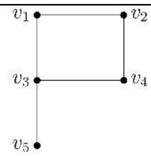
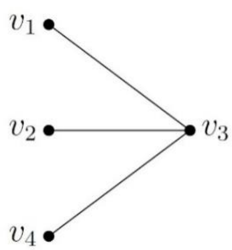
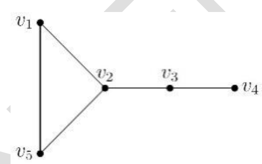
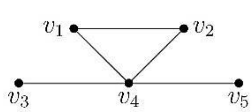
$$\wp_1\big(\mathbb{M}_{ined}\,(S_n)\big) = \max_{Y \neq 0} \frac{Y^T \mathbb{M}_{ined}\,(S_n)\ Y}{Y^T\ Y}$$

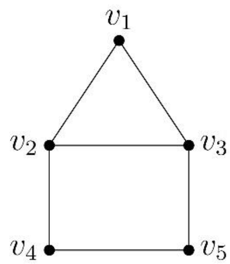
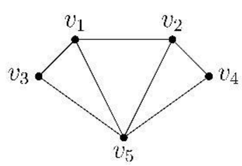
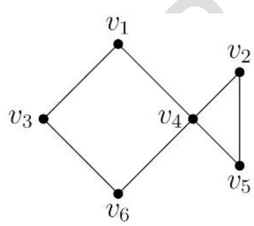
$\wp_1(\mathbb{M}_{ined}\,(S_n)) \geq \frac{U^T \mathbb{M}_{ined}\,(S_n)\ U}{U^T\ U} = \frac{|D|+(n-1)(n-2)}{n}$ where U is the unit matrix.
Analogously,

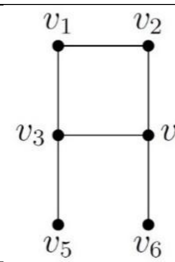
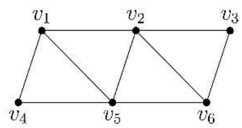
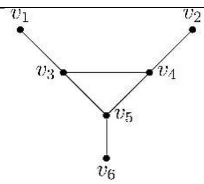
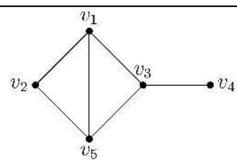
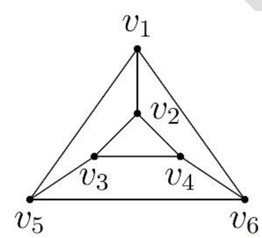
2. $\wp_1(\mathbb{M}_{ined}\,(K_n)) \geq \frac{U^T \mathbb{M}_{ined}\,(K_n)\ U}{U^T\ U} = \frac{|D|+(n-1)(n-2)}{n}$.
3. $\wp_1(\mathbb{M}_{ined}\,(G)) \geq \frac{U^T \mathbb{M}_{ined}\,(G)\,U}{U^T\ U} = \frac{|D|+n}{n}$.

Table 2. Eigenvalues and energy of minimum injective eccentric dominating matrix of various graphs.

Graph	Figure	Minimum INED set	Characteristic equation $\mathcal{F}_n(G,\wp)$	Roots $\wp(G)$	Energy $\mathbb{E}_{ined}\,(G)$
Dart graph		$\{v_2, v_3\}$,	$-\wp^5 + 2\wp^4 + 6\wp^3 - 4\wp^2 - 5\wp + 2.$	$\wp_1 = 3.3234$, $\wp_2 = 1$, $\wp_3 = 0.3579$, $\wp_4 = -1$, $\wp_5 = -1.6813$.	7.3626
		$\{v_2, v_4\}$.	$-\wp^5 + 2\wp^4 + 6\wp^3 - 5\wp^2 - 5\wp + 3.$	$\wp_1 = 3.2534$, $\wp_2 = 1$, $\wp_3 = 0.52$, $\wp_4 = -1$, $\wp_5 = -1.7734$.	7.5468
Graph	Figure	Minimum INED set	Characteristic equation $\mathcal{F}_n(G,\wp)$	Roots $\wp(G)$	Energy $\mathbb{E}_{ined}\,(G)$
		$\{v_1, v_3\}$,	$\wp^4 - 2\wp^3 - 3\wp^2 + 4\wp - 1.$	$\wp_1 = 2.618$, $\wp_2 = 0.618$, $\wp_3 = 0.382$, $\wp_4 = -1.618$.	5.236
Paw		$\{v_2, v_3\}$,	$\wp^4 - 2\wp^3 -$	$\wp_1 = 2.5616$, $\wp_2 = 1$,	5.1232

graph			$3\wp^2 + 4\wp$.	$\wp_3 = 0$, $\wp_4 = -1.5616$.	
		$\{v_3, v_4\}$.	$\wp^4 - 2\wp^3 - 3\wp^2 + 4\wp - 1$.	$\wp_1 = 2.618$, $\wp_2 = 0.618$, $\wp_3 = 0.382$, $\wp_4 = -1.618$.	5.236
Banner graph		$\{v_2, v_5\}$.	$-\wp^5 + 2\wp^4 + 3\wp^3 - 3\wp^2 - 4\wp - 1$.	$\wp_1 = 2.4142$, $\wp_2 = 1.618$, $\wp_3 = 1$, $\wp_4 = -0.4142$, $\wp_5 = -0.618$.	6.0644
Claw graph		$\{v_1, v_3\}$,	$\wp^4 - 2\wp^3 - 2\wp^2 + 2\wp + 1$.	$\wp_1 = 1$, $\wp_2 = -1$, $\wp_3 = 2.4142$, $\wp_4 = -0.4142$.	4.8284
		$\{v_2, v_3\}$,	$\wp^4 - 2\wp^3 - 2\wp^2 + 2\wp + 1$.	$\wp_1 = 1$, $\wp_2 = -1$, $\wp_3 = 2.4142$, $\wp_4 = -0.4142$.	4.8284
		$\{v_3, v_4\}$.	$\wp^4 - 2\wp^3 - 2\wp^2 + 2\wp + 1$.	$\wp_1 = 1$, $\wp_2 = -1$, $\wp_3 = 2.4142$, $\wp_4 = -0.4142$.	4.8284
(3,2) Tadpole graph		$\{v_1, v_4\}$,	$-\wp^5 + 2\wp^4 + 2\wp^3 - 4\wp^2 - \wp + 1$.	$\wp_1 = 1.8019$, $\wp_2 = 1.618$, $\wp_3 = 0.445$, $\wp_4 = -0.618$, $\wp_5 = -1.247$.	5.7299
		$\{v_4, v_5\}$.	$-\wp^5 + 2\wp^4 + 2\wp^3 - 4\wp^2 - \wp + 1$.	$\wp_1 = 1.8019$, $\wp_2 = 1.618$, $\wp_3 = 0.445$, $\wp_4 = -0.618$, $\wp_5 = -1.247$.	5.7299
Graph	Figure	Minimum INED set	Characteristic equation $\mathcal{F}_n(G, \wp)$	Roots $\wp(G)$	Energy $\mathbb{E}_{ined}(G)$
Cricket graph		$\{v_3, v_4\}$,	$-\wp^5 + 2\wp^4 + 6\wp^3 - 5\wp^2 - 4\wp$.	$\wp_1 = 3.2554$, $\wp_2 = 1.198$, $\wp_3 = 0$, $\wp_4 = -0.5345$, $\wp_5 = -1.9188$.	6.9067
		$\{v_4, v_5\}$.		$\wp_1 = 3.2554$,	6.9067

			$-\wp^5 + 2\wp^4 + 6\wp^3 - 5\wp^2 - 4\wp.$	$\wp_2 = 1.198,$ $\wp_3 = 0,$ $\wp_4 = -0.5345,$ $\wp_5 = -1.9188.$	
House graph		$\{v_2, v_4\}$, $\{v_3, v_5\}$.	$-\wp^5 + 2\wp^4 + 3\wp^3 - 5\wp^2 - 2\wp + 1.$ $-\wp^5 + 2\wp^4 + 3\wp^3 - 5\wp^2 - 2\wp + 1.$	$\wp_1 = 2.1701,$ $\wp_2 = 1.618,$ $\wp_3 = 0.3111,$ $\wp_4 = -0.618,$ $\wp_5 = -1.4812.$ $\wp_1 = 2.1701,$ $\wp_2 = 1.618,$ $\wp_3 = 0.3111,$ $\wp_4 = -0.618,$ $\wp_5 = -1.4812.$	6.1984 6.1984
Gem graph		$\{v_1, v_2\}$, $\{v_3, v_4\}$.	$-\wp^5 + 2\wp^4 + 6\wp^3 - 4\wp^2 - 8\wp.$ $-\wp^5 + 2\wp^4 + 6\wp^3 - 2\wp^2 - 5\wp.$	$\wp_1 = 3.2361,$ $\wp_2 = 1.4142,$ $\wp_3 = 0,$ $\wp_4 = -1.2361,$ $\wp_5 = -1.4142.$ $\wp_1 = 3.4495,$ $\wp_2 = 1,$ $\wp_3 = 0,$ $\wp_4 = -1,$ $\wp_5 = -1.4495.$	7.3006 6.899
Fish graph		$\{v_2, v_3\}$, $\{v_3, v_5\}$.	$\wp^6 - 2\wp^5 - 5\wp^4 + 5\wp^3 + 8\wp^2 - \wp - 2.$ $\wp^6 - 2\wp^5 - 5\wp^4 + 5\wp^3 + 8\wp^2 - \wp - 2.$	$\wp_1 = 2.8136,$ $\wp_2 = 1.618,$ $\wp_3 = 0.5293,$ $\wp_4 = -0.618,$ $\wp_5 = -1,$ $\wp_6 = -1.3429.$ $\wp_1 = 2.8136,$ $\wp_2 = 1.618,$ $\wp_3 = 0.5293,$ $\wp_4 = -0.618,$ $\wp_5 = -1,$ $\wp_6 = -1.3429.$	7.9218 7.9218
Graph	Figure	Minimum INED set	Characteristic equation $\mathcal{F}_n(G, \wp)$	Roots $\wp(G)$	Energy $\mathbb{E}_{ined}(G)$
A graph		$\{v_1, v_5, v_6\}$,	$\wp^6 - 3\wp^5 - \wp^4 + 8\wp^3 - 2\wp^2 - 5\wp + 2.$	$\wp_1 = 2,$ $\wp_2 = 1.8019,$ $\wp_3 = 0.445,$ $\wp_4 = 1,$ $\wp_5 = -1,$ $\wp_6 = -1.247.$	7.4939

		$\{v_2, v_5, v_6\}$.	$\wp^6 - 3\wp^5 - \wp^4 + 8\wp^3 - 2\wp^2 - 5\wp + 2.$	$\wp_1 = 2,$ $\wp_2 = 1.8019,$ $\wp_3 = 0.445,$ $\wp_4 = 1,$ $\wp_5 = -1,$ $\wp_6 = -1.247.$	7.4939
4 polynomial graph		$\{v_3, v_4\}$.	$\wp^6 - 2\wp^5 - 4\wp^4 + 6\wp^3 + 5\wp^2 - 2\wp - 1.$	$\wp_1 = 2.247,$ $\wp_2 = 1.8794,$ $\wp_3 = 0.555,$ $\wp_4 = -0.3473,$ $\wp_5 = -0.8019,$ $\wp_6 = -1.5321.$	7.6627
Net graph		$\{v_1, v_2, v_6\}$.	$\wp^6 - 3\wp^5 - 3\wp^4 + 11\wp^3 + 3\wp^2 - 9\wp - 4.$	$\wp_1 = 2.5616,$ $\wp_2 = 1.618,$ $\wp_3 = 1.618,$ $\wp_4 = -0.618,$ $\wp_5 = -0.618,$ $\wp_6 = -1.5616.$	8.5952
Kite graph		$\{v_2, v_4\}$.	$-\wp^5 + 2\wp^4 + 2\wp^3 - 3\wp^2 - 2\wp.$	$\wp_1 = 2,$ $\wp_2 = 1.618,$ $\wp_3 = 0,$ $\wp_4 = -0.618,$ $\wp_5 = -1.$	5.236
3 prism		$\{v_1, v_2\}$,	$\wp^6 - 2\wp^5 - 5\wp^4 + 8\wp^3 + 7\wp^2 - 6\wp - 3.$	$\wp_1 = 2.4142,$ $\wp_2 = 1.7321,$ $\wp_3 = 1,$ $\wp_4 = -0.4142,$ $\wp_5 = -1,$ $\wp_6 = -1.7321.$	8.2926
		$\{v_3, v_4\}$,	$\wp^6 - 2\wp^5 - 5\wp^4 + 8\wp^3 + 7\wp^2 - 6\wp - 3.$	$\wp_1 = 2.4142,$ $\wp_2 = 1.7321,$ $\wp_3 = 1,$ $\wp_4 = -0.4142,$ $\wp_5 = -1,$ $\wp_6 = -1.7321.$	8.2926
		$\{v_4, v_6\}$.	$\wp^6 - 2\wp^5 - 5\wp^4 + 8\wp^3 + 7\wp^2 - 6\wp - 3.$	$\wp_1 = 2.4142,$ $\wp_2 = 1.7321,$ $\wp_3 = 1,$ $\wp_4 = -0.4142,$ $\wp_5 = -1,$ $\wp_6 = -1.7321.$	8.2926

5. CONCLUSION

In this paper minimum injective eccentric dominating energy of graph is introduced. The injective eccentric dominating energy of some standard graphs are calculated. Results related to the upper and lower bound of the energy of standard graphs is stated and proved.

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