A New Approach to Dual Jacobsthal Split Quaternions with Different Polar Representation

Abstract

In this paper, we introduce split quaternions with components including dual Jacobsthal and dual Jacobsthal-Lucas number sequences. By using Binet's formulas of these type split quaternions we give an explicit form of classic polar representations of them, after that we demonstrated a new polar representation by using Cayley-Dikson's notation of split quaternions which is based on two complex numbers. Some fundamental properties and identities for these type of split quaternions are studied.

Keywords: Dual Jacobsthal split quaternions, Dual Jacobsthal-Lucas split quaternions, Polar representation

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1 Introduction

Quaternions, extension of complex numbers was introduced by W. R. Hamilton as the following quadruple

$$Q = \gamma_0 + \gamma_1 i_1 + \gamma_2 i_2 + \gamma_3 i_3$$

where $\gamma_0, \gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ and $i_1^2 = i_2^2 = i_3^2 = -1, i_1 i_2 = -i_2 i_1 = i_3$. Any quaternion can be write as $Q = (\gamma_0 + \gamma_1 i_1) + i_2(\gamma_2 - \gamma_3 i_1)$, where $\gamma_0 + \gamma_1 i_1$ and $\gamma_2 - \gamma_3 i_1$ are complex numbers, therefore quaternions are one of the hyper complex numbers, [1,7,9,13,14].

J. Cokle defined the split quaternion as $\hat{Q} = \delta_0 + \delta_1 i_1 + \delta_2 i_2 + \delta_3 i_3$ with $\delta_0, \delta_1, \delta_2, \delta_3 \in \mathbb{R}$ and quatenionic units i_1, i_2, i_3 satisfy

$$i_1^2 = -i_2^2 = -i_3^2 = -1,\ i_1 i_2 = -i_2 i_1 = i_3 \tag{1.1}$$

and can be shown as $\hat{Q} = S_{\hat{Q}} + V_{\hat{Q}}$, $\hat{Q}$ with $S_{\hat{Q}} = 0$ is a pure split quaternion, [5,6,15,17,18]. Let $\hat{Q}_1 = S_{\hat{Q}_1} + V_{\hat{Q}_1}$ and $\hat{Q}_2 = S_{\hat{Q}_2} + V_{\hat{Q}_2}$ be split quaternions, then addition and multiplication are

$$\hat{Q}_1 + \hat{Q}_2 = (S_{\hat{Q}_1} + S_{\hat{Q}_2}) + (V_{\hat{Q}_1} + V_{\hat{Q}_2})$$
$$\hat{Q}_1 \hat{Q}_2 = S_{\hat{Q}_1} S_{\hat{Q}_2} + \langle V_{\hat{Q}_1}, V_{\hat{Q}_2} \rangle + S_{\hat{Q}_1} V_{\hat{Q}_1} + S_{\hat{Q}_2} V_{\hat{Q}_2} + V_{\hat{Q}_1} \times V_{\hat{Q}_2}$$

respectively, where $\langle \, , \rangle$ and $\times$ are inner and cross products in Minkowsky space $\mathbb{E}_1^3$. The conjugate and norm of $\hat{Q} = \delta_0 + \delta_1 i_1 + \delta_2 i_2 + \delta_3 i_3$ are respectively as $\overline{\hat{Q}} = \delta_0 - \delta_1 i_1 - \delta_2 i_2 - \delta_3 i_3$ and

$$N(\hat{Q}) = \sqrt{|\hat{Q}\overline{\hat{Q}}|} = \sqrt{|\delta_0^2 + \delta_1^2 - \delta_2^2 - \delta_3^2|} \tag{1.2}$$

if $N(\hat{Q}) = 1$, then $\hat{Q}$ is a unit split quaternion, for any split quaternion $\hat{Q}$ with $N(\hat{Q}) \neq 0$, $\frac{\hat{Q}}{N(\hat{Q})}$ is a unit split quaternion. Let $I_{\hat{Q}} = \delta_0^2 + \delta_1^2 - \delta_2^2 - \delta_3^2$, then $\hat{Q}$ is space-like, time-like and light-like if $I_{\hat{Q}} > 0$, $I_{\hat{Q}} < 0$ and $I_{\hat{Q}} = 0$ respectively, the multiplicative inverse of $\hat{Q}$ is $\hat{Q}^{-1} = \frac{\overline{\hat{Q}}}{N(\hat{Q})^2}$ and there is no inverse for light-like split quaternion. The Cayley-Dickson's form of a split quaternion $\hat{Q}$ is $\hat{Q} = (\delta_0 + \delta_1 i_1) + (\delta_2 + \delta_3 i_1) i_2$ which is based on two complex numbers.

Dual numbers, extension of real numbers, first were defined by W. K. Clifford in 1873 as:

$$\mathbb{D} = \{A = a_1 + \varepsilon a_2 \mid a_1, a_2 \in \mathbb{R}, \varepsilon \neq 0, \varepsilon^2 = 0\}$$

This new number system is commutative and associative algebra over real numbers wich has dimension two. Kotelnikov initiated the study of the first applications of dual numbers. Eduard Study used these numbers and associated vectors in line geometry and kinematics,[8,20].

According to the following operation

$$(a_1 + \varepsilon a_2) + (b_1 + \varepsilon b_2) = (a_1 + b_1) + \varepsilon(a_2 + b_2)$$

$$(a_1 + \varepsilon a_2)(b_1 + \varepsilon b_2) = a_1 b_1 + \varepsilon(a_1 b_2 + a_2 b_1)$$

algebra of dual numbers is a commutative ring, but not a field. The multiplicative inverse of A is

$$A^{-1} = \frac{1}{a_1} - \varepsilon \frac{a_2}{a_1^2}, \; a_1 \neq 0$$

and there is no inverse for pure dual numbers and hence this algebra of numbers is not field over real numbers. By using inverse of dual numbers we can define division operation of two dual numbers A_1 and B_1 as $A_1 B_1^{-1}$ where B_1 is not a pure dual number and $B_1 \neq 0$. Dual angle between lines d_1 and d_2 in $\mathbb{R}^3$ is defined as $\Phi = \phi + \varepsilon \phi^*$, where ϕ is angle and ϕ^* is the shortest distance between these lines. If $f(x + \varepsilon x^*)$ is a dual function, then by Taylor expansion we have

$$f(x + \varepsilon x^*) = f(x) + \varepsilon x^* f'(x) \tag{1.3}$$

where $f'(x)$ is the first derivative of $f(x)$, using Equality (1.3) we can write

$$\sqrt{a_1 + \varepsilon a_2} = \sqrt{a_1} + \varepsilon \frac{a_1}{2\sqrt{a_2}} \tag{1.4}$$

Dual split quaternion $\mathbb{Q}$ is defined as

$$\mathbb{Q} = A + B i_1 + C i_2 + D i_3$$

where A, B, C, D are dual numbers and i_1, i_2, i_3 follows rules in (1.1), $\mathbb{Q}$ can be write as $\mathbb{Q} = \hat{Q} + \varepsilon \hat{Q}^*$, where $\hat{Q}, \hat{Q}^*$ are split quaternions and $\varepsilon^2 = 0$. Let $\mathbb{Q}_1 = \hat{Q}_1 + \varepsilon \hat{Q}_1^*$ and $\mathbb{Q}_2 = \hat{Q}_2 + \varepsilon \hat{Q}_2^*$ be dual split quaternions, then

$$\mathbb{Q}_1 + \mathbb{Q}_2 = (\hat{Q}_1 + \hat{Q}_2) + \varepsilon(\hat{Q}_1^* + \hat{Q}_2^*)$$
$$\mathbb{Q}_1 \mathbb{Q}_2 = \hat{Q}_1 \hat{Q}_2 + \varepsilon(\hat{Q}_1 \hat{Q}_2^* + \hat{Q}_1^* \hat{Q}_2)$$

norm of $\mathbb{Q}$ is defined as $N(\mathbb{Q}) = \sqrt{A^2 + B^2 - C^2 - D^2}$, and dual split quaternion with $N(\mathbb{Q}) = 1$ is called unit dual split quaternion. Dual split quaternion $\mathbb{Q} = \hat{Q} + \varepsilon \hat{Q}^*$ is spacelike, timelike or lightlike if and only $\hat{Q}$ is spacelike, timelike and lightlike, respectively. Suppose $\Phi = \phi + \varepsilon\phi^*$ be dual angle then the polar representation for spacelike $\mathbb{Q}$ is as

$$\mathbb{Q} = N(\mathbb{Q})(\sinh\Phi + \nu\cosh\Phi) \tag{1.5}$$

where $\nu = \frac{Bi_{+}Ci_2+Di_3}{\sqrt{-B^2+C^2+D^2}}$ is a unit pure dual split quaternion, $\sinh\Phi = \frac{D}{N(\mathbb{Q})}$ and $\cosh\Phi = \frac{\sqrt{|B^2-C^2-D^2|}}{N(\mathbb{Q})}$. The polar representation for timelike $\mathbb{Q}$ with spacelike vector part is

$$\mathbb{Q} = N(\mathbb{Q})(\cosh\Phi + \nu\sinh\Phi) \tag{1.6}$$

where $\nu = \frac{Bi_{+}Ci_2+Di_3}{\sqrt{-B^2+C^2+D^2}}$ is a unit pure dual split quaternion, $\cosh\Phi = \frac{D}{N(\mathbb{Q})}$ and $\sinh\Phi = \frac{\sqrt{|B^2-C^2-D^2|}}{N(\mathbb{Q})}$ and the polar representation for timelike $\mathbb{Q}$ with timelike vector part is

$$\mathbb{Q} = N(\mathbb{Q})(\cos\Phi + \nu\sin\Phi) \tag{1.7}$$

where $\nu = \frac{Bi_{+}Ci_2+Di_3}{\sqrt{B^2-C^2-D^2}}$ is a unit pure dual split quaternion, $\cos\Phi = \frac{D}{N(\mathbb{Q})}$ and $\sin\Phi = \frac{\sqrt{|B^2-C^2-D^2|}}{N(\mathbb{Q})}$, [16,19].

Any dual split quaternion can be written in the form $\mathbb{Q} = Ae^{Bi_2}$, where $A = A_0 + A_1i_1$ and $B = A_2 + A_3i_1$ with A_0, A_1, A_2, A_3 dual numbers, [2].

The Jacobsthal J_n and Jacobsthal-Lucas j_n number sequences are defined by

$$\begin{aligned} J_n &= J_{n-1} + 2J_{n-2}, \qquad J_0 = 0, J_1 = 1, n \geq 2 \\ j_n &= j_{n-1} + 2j_{n-2}, \qquad j_0 = 2, j_1 = 1, n \geq 2 \end{aligned}$$

The characteristic equation of these number sequences is $x^2 - x - 2 = 0$, with roots $\alpha = 2$, $\beta = -1$ and corresponding Binet's forms are

$$J_n = \frac{2^n - (-1)^n}{3} \tag{1.8}$$

$$j_n = 2^n + (-1)^n \tag{1.9}$$

For more details and relations about these number sequences see [4,10,11,12]. The n^{th} dual Jacobsthal $\hat{J}_n$ and Jacobsthal-Lucas $\hat{j}_n$ number sequences are defined as [3]

$$\begin{aligned} \hat{J}_n &= J_n + \varepsilon J_{n+1} \\ \hat{j}_n &= j_n + \varepsilon j_{n+1} \end{aligned}$$

where J_n and j_n are the n^{th} Jacobsthal and Jacobsthal-Lucas numbers, the Binet's formulas for $\hat{J}_n$ and $\hat{j}_n$ are

$$\hat{J}_n = \frac{2^n\underline{\alpha} - (-1)^n\underline{\beta}}{3} \tag{1.10}$$

$$\hat{j}_n = 2^n\underline{\alpha} + (-1)^n\underline{\beta} \tag{1.11}$$

where

$$\begin{aligned} \underline{\alpha} &= 1 + 2\varepsilon \\ \underline{\beta} &= 1 - \varepsilon \end{aligned}$$

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2 Main results

In this section we defined dual Jacobsthal and dual Jacobsthal-Lucas split quaternions, polar representations and some properties of these split quaternions are discussed.

Definition 2.1. The dual Jacobsthal and dual Jacbosthal-Lucas split quaternions are defined as

$$\mathcal{J}_n = \hat{J}_n + \hat{J}_{n+1}i_1 + \hat{J}_{n+2}i_2 + \hat{J}_{n+3}i_3 \tag{2.1}$$

$$\widetilde{\mathcal{J}}_n = \hat{j}_n + \hat{j}_{n+1}i_1 + \hat{j}_{n+2}i_2 + \hat{j}_{n+3}i_3 \tag{2.2}$$

respectively, where $\hat{J}_n$ is n^{th} dual jacobsthal, $\hat{j}_n$ is n^{th} dual jacobsthal-Lucas number and i_1, i_2, i_3 follow the rules in (1.1).

From definition, the following recurrence relation can be prove easily

$$\mathcal{J}_n = \mathcal{J}_{n-1} + 2\mathcal{J}_{n-2},\ n \geq 2$$

and

$$\widetilde{\mathcal{J}}_n = \widetilde{\mathcal{J}}_{n-1} + 2\widetilde{\mathcal{J}}_{n-2},\ n \geq 2$$

Theorem 2.1. *The Binet's formulas for dual Jacobsthal and dual Jacobsthal-Lucas split quaternions are*

$$\mathcal{J}_n = \frac{2^n\hat{\alpha} - (-1)^n\hat{\beta}}{3} \tag{2.3}$$

$$\widetilde{\mathcal{J}}_n = 2^n\hat{\alpha} + (-1)^n\hat{\beta} \tag{2.4}$$

respectively, where

$$\hat{\alpha} = (1 + 2\varepsilon)(1 + 2i_1 + 4i_2 + 8i_3)$$
$$\hat{\beta} = (1 - \varepsilon)(1 - i_1 + i_2 - i_3)$$

Proof. The proof can be done directly by using Equalities (1.10),(1.11),(2.1),(2.2). ☐

Proposition 2.1. *The norm of dual Jacobsthal and dual Jacobsthal-Lucas split quaternions is*

$$N(\mathcal{J}_n) = \sqrt{(2^{2n+3} + \frac{1}{3}(j_n^2 - 1)) + \varepsilon(2^{2n+5} + j_{2n} + \frac{1}{3}(j_n^2 - 4))}$$
$$N(\widetilde{\mathcal{J}}_n) = 3\sqrt{(2^{2n+3} + 3J_n^2 - \frac{1}{3}) + \varepsilon(2^{2n+5} + 3J_n^2 + j_{2n} - \frac{4}{3})}$$

where J_n and j_n are n^{th} Jacobsthal and Jacobsthal-Lucas numbers, respectively.

Proof. From Equality (1.2), we have

$$N(\mathcal{J}_n) = \sqrt{|\hat{J}_n^2 + \hat{J}_{n+1}^2 - \hat{J}_{n+2}^2 - \hat{J}_{n+3}^2|}$$

and by using Equality (1.10), we obtain

$$\begin{aligned}\hat{J}_n^2 + \hat{J}_{n+1}^2 - \hat{J}_{n+2}^2 - \hat{J}_{n+3}^2 &= \left(\frac{2^n\underline{\alpha} - (-1)^n\underline{\beta}}{3}\right)^2 + \left(\frac{2^{n+1}\underline{\alpha} - (-1)^{n+1}\underline{\beta}}{3}\right)^2 \\ &- \left(\frac{2^{n+2}\underline{\alpha} - (-1)^{n+2}\underline{\beta}}{3}\right)^2 - \left(\frac{2^{n+3}\underline{\alpha} - (-1)^{n+3}\underline{\beta}}{3}\right)^2 \\ &= -\frac{1}{3}(25 \cdot 2^{2n}(1 + 4\varepsilon) + (-1)^n 2^{n+1}(1 + \varepsilon))\end{aligned}$$

Finally by using Equality (1.8) and doing basic calculations, we get

$$N(\mathcal{J}_n) = \sqrt{(2^{2n+3} + \frac{1}{3}(j_n^2 - 1)) + \varepsilon(2^{2n+5} + j_{2n} + \frac{1}{3}(j_n^2 - 4))}$$

On the other hand for $N(\widetilde{\mathcal{J}}_n)$, by using Equality (1.11) we have

$$\begin{aligned} \hat{j}_n^2 + \hat{j}_{n+1}^2 - \hat{j}_{n+2}^2 - \hat{j}_{n+3}^2 &= \left(2^n \underline{\alpha} + (-1)^n \underline{\beta}\right)^2 + \left(2^{n+1} \underline{\alpha} + (-1)^{n+1} \underline{\beta}\right)^2 \\ &\quad \left(2^{n+2} \underline{\alpha} + (-1)^{n+2} \underline{\beta}\right)^2 - \left(2^{n+3} \underline{\alpha} + (-1)^{n+3} \underline{\beta}\right)^2 \\ &= -3(25 \cdot 2^{2n}(1 + 4\varepsilon) + (-1)^{n+1} 2^{n+1}(1 + \varepsilon)) \end{aligned}$$

And by doing some necessary calculations, we get

$$N(\widetilde{\mathcal{J}}_n) = 3\sqrt{(2^{2n+3} + 3J_n^2 - \frac{1}{3}) + \varepsilon(2^{2n+5} + 3J_n^2 + j_{2n} - \frac{4}{3})}$$

□

Corollary 2.2. *The dual Jacobsthal and dual Jacobsthal-Lucas split quaternions are spacelike split quaternions with spacelike vector part.*

Theorem 2.3. *The classical polar representation of dual Jacobsthal split quaternion is*

$$\mathcal{J}_n = N(\mathcal{J}_n)(\sinh \Phi + \nu \cosh \Phi)$$

where

$$\nu = \frac{2^{n+1}(1 + 2\varepsilon)(i_1 + 2i_2 + 4i_3) + (-1)^n (1 - \varepsilon)(i_1 - i_2 + i_3)}{3\sqrt{(J_{n+1}^2 + 2^{2n+3}) + \varepsilon(J_{n+1}^2 + J_{2n+2} + 2^{2n+5})}}$$

is a pure unit dual split quaternion and Φ is a dual angle such that

$$\Phi = \tanh^{-1}\left(\frac{J_n}{\sqrt{J_{n+1}^2 + 2^{2n+3}}}\right) + \varepsilon \frac{j_n J_{n+1}^2 + 2^{2n+4}(-1)^n - J_n J_{2n+2}}{\sqrt{J_{n+1}^2 + 2^{2n+3}}\,(2^{2n+4} + \frac{2}{3}(j_n^2 - 1))}$$

Proof. Since $\mathcal{J}_n$ is a spacelike dual split quaternion, then by using Equality (1.5), the polar representation is

$$\mathcal{J}_n = N(\mathcal{J}_n)(\sinh \Phi + \nu \cosh \Phi)$$

where

$$\begin{aligned} \nu &= \frac{\hat{J}_{n+1} i_1 + \hat{J}_{n+2} i_2 + \hat{J}_{n+3} i_3}{\sqrt{-\hat{J}_{n+1}^2 + \hat{J}_{n+2}^2 + \hat{J}_{n+3}^2}} \\ \sinh \Phi &= \frac{\hat{J}_n}{N(\mathcal{J}_n)} \\ \cosh \Phi &= \frac{\sqrt{|\hat{J}_{n+1}^2 - \hat{J}_{n+2}^2 - \hat{J}_{n+3}^2|}}{N(\mathcal{J}_n)} \end{aligned}$$

therefore $\tanh \Phi = \frac{\hat{J}_n}{\sqrt{-\hat{J}_{n+1}^2 + \hat{J}_{n+2}^2 + \hat{J}_{n+3}^2}}$ and by using Equality (1.10) we get

$$\nu = \frac{2^{n+1}(1 + 2\varepsilon)(i_1 + 2i_2 + 4i_3) - (-1)^n (1 - \varepsilon)(i_1 - i_2 + i_3)}{\sqrt{(J_{n+1}^2 + 2^{2n+3}) + \varepsilon(J_{n+1}^2 + 2^{2n+5} + J_{2n+2})}}$$

and

$$\tanh\Phi = \frac{J_n + \varepsilon J_{n+1}}{\sqrt{(J_{n+1}^2 + 2^{2n+3}) + \varepsilon(J_{n+1}^2 + 2^{2n+5} + J_{2n+2})}}$$

Using Equality (1.4) and doing necessary calcualtions we will have

$$\Phi = \tanh^{-1}\left(\frac{J_n}{\sqrt{J_{n+1}^2 + 2^{2n+3}}} + \varepsilon\frac{\jmath_n J_{n+1}^2 + 2^{2n+4}(-1)^n - J_n J_{2n+2}}{2(J_{n+1}^2 + 2^{2n+3})^{\frac{3}{2}}}\right)$$

Finally by using Equality (1.3) we obtain

$$\Phi = \tanh^{-1}\left(\frac{J_n}{\sqrt{J_{n+1}^2 + 2^{2n+3}}}\right) + \varepsilon\frac{\jmath_n J_{n+1}^2 + 2^{2n+4}(-1)^n - J_n J_{2n+2}}{\sqrt{J_{n+1}^2 + 2^{2n+3}}\,(2^{2n+4} + \frac{2}{3}(\jmath_n^2 - 1))}$$

☐

Corollary 2.4. *The classical polar representation of dual Jacobsthal-Lucas split quaternions is*

$$\widetilde{\mathcal{J}}_n = N(\widetilde{\mathcal{J}}_n)(\sinh\hat{\Phi} + \hat{\nu}\cosh\hat{\Phi})$$

where

$$\hat{\nu} = \frac{2^{n+1}(1+2\varepsilon)(i_1 + 2i_2 + 4i_3) - (-1)^n(1-\varepsilon)(i_1 - i_2 + i_3)}{3\sqrt{2^{2n+3} + (\frac{\jmath_{n+1}}{3})^2 + \varepsilon(2^{2n+5} + J_{2n+2} + (\frac{\jmath_{n+1}}{3})^2)}}$$

is a pure unit dual split quaternion and $\hat{\Phi}$ is a dual angle such that

$$\hat{\Phi} = \tanh^{-1}\left(\frac{\jmath_n}{\sqrt{\jmath_{n+1}^2 + 9\cdot 2^{2n+3}}}\right) + \varepsilon\frac{J_n \jmath_{n+1}^2 - 3\cdot 2^{2n+4}(-1)^n - \jmath_n J_{2n+2}}{\sqrt{\jmath_{n+1}^2 + 9\cdot 2^{2n+3}}\,(2^{2n+4} + 6J_n^2 - \frac{2}{3})}$$

Proof. The Proof is similar to Theorem 2.3. ☐

Proposition 2.2. *Let $P = Ai_2 + Bi_3 = (A + Bi_1)i_2$ be an arbitrary dual split quaternion, if P is spacelike then its exponential form is*

$$e^P = \sinh|P| + \frac{A}{|P|}\cosh|P|i_2 + \frac{B}{|P|}\cosh|P|i_3 = \alpha_0 + \alpha_2 i_2 + \alpha_3 i_3$$

and if P is timelike, then

$$e^P = \cosh|P| + \frac{A}{|P|}\sinh|P|i_2 + \frac{B}{|P|}\sinh|P|i_3 = \beta_0 + \beta_2 i_2 + \beta_3 i_3$$

That is, it is a dual split quaternions which does not contain i_1's term.

Proof. Suppose μ is a spacelike unit dual split quaternion, that is $N(\mu) = 1$, then from Equality (1.5) we have

$$e^{\mu\theta} = \sinh\theta + \mu\cosh\theta$$

if we rewrite $P = |P|\frac{P}{|P|}$, then by taking $\mu = \frac{P}{|P|}$ and $\theta = |P|$ we get the result, we can prove similarly for timelike P by using Equality (1.6). ☐

Now we give the new polar representations for dual Jacobsthal and dual Jacobsthal-Lucas split quaternions by using Cayley-Dikson's form for split quatenions.

Theorem 2.5. *Every dual Jacobsthal split quaterninon* $\mathcal{J}_n = \hat{J}_n + \hat{J}_{n+1}i_1 + \hat{J}_{n+2}i_2 + \hat{J}_{n+3}i_3$ *can be given in the form* $\mathcal{J}_n = \mathbb{A}e^{\mathbb{B}i_2}$*, where* $\mathbb{A}$ *and* $\mathbb{B}$ *are dual Jacobsthal complex numbers, that is*

$$\mathbb{A} = \frac{\hat{J}_n + \hat{J}_{n+1}i_1}{\sqrt{\hat{J}_n^2 + \hat{J}_{n+1}^2}}$$

$$\mathbb{B} = \tanh^{-1}\left(\frac{\sqrt{\hat{J}_n^2 + \hat{J}_{n+1}^2}}{\sqrt{\hat{J}_{n+2}^2 + \hat{J}_{n+3}^2}}\right)\frac{(\hat{J}_n\hat{J}_{n+2} + \hat{J}_{n+1}\hat{J}_{n+3}) + (\hat{J}_n\hat{J}_{n+3} - \hat{J}_{n+1}\hat{J}_{n+2})i_1}{\sqrt{(\hat{J}_n^2 + \hat{J}_{n+1}^2)(\hat{J}_{n+2}^2 + \hat{J}_{n+3}^2)}}$$

Proof. Suppose that $\mathbb{A} = a + bi_1$ and $e^{\mathbb{B}i_2} = \alpha_0 + \alpha_2 i_2 + \alpha_3 i_3$, then

$$\mathcal{J}_n = \mathbb{A}e^{\mathbb{B}i_2} = a\alpha_0 + b\alpha_0 i_1 + (a\alpha_2 - b\alpha_3)i_2 + (a\alpha_3 + b\alpha_2)i_3$$

if $\alpha_0 = 0$, then we can select $a = 1$ and $b = 0$, we will get $\mathbb{A} = 1$. For $\alpha_0 \neq 0$, we construct a complex number $\psi = a\alpha_0 + b\alpha_0 i_1 = \hat{J}_n + \hat{J}_{n+1}i_1$ and then $\mathbb{A} = \frac{\psi}{|\psi|}$, by using Equality (1.10) the explicit form of $\mathbb{A}$ is

$$\mathbb{A} = \frac{2^n(1+2\varepsilon)(1+2i_1) + (-1)^{n+1}(1-\varepsilon)(1-i_1)}{\sqrt{j_n^2 + 2^{2n+2} + \varepsilon(j_n^2 + 9J_{2n} + 2^{2n+4})}}$$

Since $\mathbb{A}$ is a unit dual complex number then $\mathbb{A}^{-1} = \overline{\mathbb{A}} = \frac{\hat{J}_n - \hat{J}_{n+1}i_1}{\sqrt{\hat{J}_n^2 + \hat{J}_{n+1}^2}}$, where $\overline{\mathbb{A}}$ is conjugate of $\mathbb{A}$ and

$$e^{\mathbb{B}i_2} = \overline{\mathbb{A}}\mathcal{J}_n = \frac{(\hat{J}_n^2 + \hat{J}_{n+1}^2) + (\hat{J}_n\hat{J}_{n+2} + \hat{J}_{n+1}\hat{J}_{n+3})i_2 + (\hat{J}_n\hat{J}_{n+3} - \hat{J}_{n+1}\hat{J}_{n+2})i_3}{\sqrt{\hat{J}_n^2 + \hat{J}_{n+1}^2}}$$

The norm of $e^{\mathbb{B}i_2}$ is

$$|e^{\mathbb{B}i_2}| = \sqrt{\hat{J}_{n+2}^2 + \hat{J}_{n+3}^2 - \hat{J}_n^2 - \hat{J}_{n+1}^2}$$

Since $\frac{e^{\mathbb{B}i_2}}{|e^{\mathbb{B}i_2}|}$ is a unit spacelike dual split quaternion, then its classical polar form is

$$\frac{e^{\mathbb{B}i_2}}{|e^{\mathbb{B}i_2}|} = \sinh\hat{\Phi} + \hat{\mu}\cosh\hat{\Phi}$$

where $\hat{\Phi} = \phi + \varepsilon\phi^*$ is dual angle, then we can write

$$\sinh\hat{\Phi} = \frac{\sqrt{\hat{J}_n^2 + \hat{J}_{n+1}^2}}{\sqrt{\hat{J}_{n+2}^2 + \hat{J}_{n+3}^2 - \hat{J}_n^2 - \hat{J}_{n+1}^2}}$$

$$\cosh\hat{\Phi} = \frac{\sqrt{\hat{J}_{n+2}^2 + \hat{J}_{n+3}^2}}{\sqrt{\hat{J}_{n+2}^2 + \hat{J}_{n+3}^2 - \hat{J}_n^2 - \hat{J}_{n+1}^2}}$$

$$\hat{\mu} = \frac{(\hat{J}_n\hat{J}_{n+2} + \hat{J}_{n+1}\hat{J}_{n+3})i_2 + (\hat{J}_n\hat{J}_{n+3} - \hat{J}_{n+1}\hat{J}_{n+2})i_3}{\sqrt{(\hat{J}_n^2 + \hat{J}_{n+1}^2)(\hat{J}_{n+2}^2 + \hat{J}_{n+3}^2)}}$$

which gives

$$\tanh\hat{\Phi} = \frac{\sqrt{\hat{J}_n^2 + \hat{J}_{n+1}^2}}{\sqrt{\hat{J}_{n+2}^2 + \hat{J}_{n+3}^2}}$$

Finally from $\mathbb{B}i_2 = \hat{\mu}\hat{\Phi}$ we get the result and by using (1.10) the explicit form of $\mathbb{B}$ can be write easily. $\square$

Corollary 2.6. *Every dual Jacobsthal-Lucas split quaterninon $\widetilde{\mathcal{J}}_n = \hat{j}_n + \hat{j}_{n+1}i_1 + \hat{j}_{n+2}i_2 + \hat{j}_{n+3}i_3$ can be given in the form $\widetilde{\mathcal{J}}_n = \mathbb{A}e^{\mathbb{B}i_2}$, where $\mathbb{A}$ and $\mathbb{B}$ are dual Jacobsthal-Lucas complex numbers, that is*

$$\mathbb{A} = \frac{\hat{j}_n + \hat{j}_{n+1}i_1}{\sqrt{\hat{j}_n^2 + \hat{j}_{n+1}^2}}$$

$$\mathbb{B} = \tanh^{-1}\left(\frac{\sqrt{\hat{j}_n^2 + \hat{j}_{n+1}^2}}{\sqrt{\hat{j}_{n+2}^2 + \hat{j}_{n+3}^2}}\right)\frac{(\hat{j}_n\hat{j}_{n+2} + \hat{j}_{n+1}\hat{j}_{n+3}) + (\hat{j}_n\hat{j}_{n+3} - \hat{j}_{n+1}\hat{j}_{n+2})i_1}{\sqrt{(\hat{j}_n^2 + \hat{j}_{n+1}^2)(\hat{j}_{n+2}^2 + \hat{j}_{n+3}^2)}}$$

Proof. The proof can be done similar to Theorem 2.5. ☐

Example 2.7. *Find the new polar representation for $\mathcal{J}_1 = (1+\varepsilon) + (1+3\varepsilon)i_1 + (3+5\varepsilon)i_2 + (5+11\varepsilon)i_3$.*
We have $\mathcal{J}_1 = \mathbb{A}e^{\mathbb{B}i_2}$, where

$$\mathbb{A} = \frac{\sqrt{2}}{2}\left((1-\varepsilon) + (1+\varepsilon)i_1\right)$$

and

$$\mathbb{B} = \frac{1}{\sqrt{17}}\tanh^{-1}\left(\frac{1}{\sqrt{17}} + \varepsilon\frac{\sqrt{17} - 35\sqrt{2}}{17}\right)\left((4 + \frac{13}{17}\varepsilon) + (1 - \frac{52}{17}\varepsilon)i_1\right)$$

Theorem 2.8 (Catalan's identities)**.** *For positive integers n and r with $n \geq r \geq 1$, we have*

$$\mathcal{J}_{n+r}\mathcal{J}_{n-r} - \mathcal{J}_n^2 = \frac{1}{3}(-1)^{n-r}2^{n-r}J_r(1+\varepsilon)\left(j_r(1 - 13i_1 + i_2 - 13i_3) + 2(-1)^r(-1 + i_1 + 5i_2 + 7i_3)\right)$$
$$\widetilde{\mathcal{J}}_{n+r}\widetilde{\mathcal{J}}_{n-r} - \widetilde{\mathcal{J}}_n^2 = 3(-1)^{n-r}2^{n-r}J_r(1+\varepsilon)\left(j_r(-1 + 13i_1 - i_2 + 13i_3) - 2(-1)^r(-1 + i_1 + 5i_2 + 7i_3)\right)$$

where J_n and j_n are n^{th} jacobsthal and Jacobsthal-Lucas numbers.

Proof. By using (2.3), we get

$$\begin{aligned}
\mathcal{J}_{n+r}\mathcal{J}_{n-r} - \mathcal{J}_n^2 &= \frac{1}{3}\left(2^{n+r}\hat{\alpha} - (-1)^{n+r}\hat{\beta}\right)\frac{1}{3}\left(2^{n-r}\hat{\alpha} - (-1)^{n-r}\hat{\beta}\right) \\
&- \frac{1}{3}\left(2^n\hat{\alpha} - (-1)^n\hat{\beta}\right)\frac{1}{3}\left(2^n\hat{\alpha} - (-1)^n\hat{\beta}\right) \\
&= \frac{1}{9}2^n(-1)^n\left(\hat{\alpha}\hat{\beta}\left(1 - 2^r(-1)^{-r}\right) + \hat{\beta}\hat{\alpha}\left(1 - 2^{-r}(-1)^r\right)\right) \\
&= \frac{1}{3}2^n(-1)^nJ_r\left((-1)^{r+1}\hat{\alpha}\hat{\beta} + 2^{-r}\hat{\beta}\hat{\alpha}\right) \\
&= \frac{1}{3}2^{n-r}(-1)^{n-r}J_r\left((-1)^r\hat{\beta}\hat{\alpha} - 2^r\hat{\alpha}\hat{\beta}\right)
\end{aligned}$$

from using

$$\hat{\alpha}\hat{\beta} = (1+\varepsilon)(-1 + 13i_1 - i_2 + 13i_3) \tag{2.5}$$
$$\hat{\beta}\hat{\alpha} = (1+\varepsilon)(-1 - 11i_1 + 11i_2 + i_3) \tag{2.6}$$

we get the result. The proof for Jacobsthal-Lucas can be done similarly by using Equalities (2.4), (2.5) and (2.6). ☐

Corollary 2.9 (Cassini's identities)**.** *For positive integer n, the following identities hold*

$$\mathcal{J}_{n+1}\mathcal{J}_{n-1} - \mathcal{J}_n^2 = (-1)^n2^{n-1}(1+\varepsilon)\left(-1 + 5i_1 + 3i_2 + 9i_3\right)$$
$$\widetilde{\mathcal{J}}_{n+1}\widetilde{\mathcal{J}}_{n-1} - \widetilde{\mathcal{J}}_n^2 = 9(-1)^{n-1}2^{n-1}(1+\varepsilon)\left(-1 + 5i_1 + 3i_2 + 9i_3\right)$$

Proof. The proof can be done by taking $r = 1$ in Theorem 2.8. ☐

Theorem 2.10 (d'Ocagne's identities)**.** *For positive integers n and m with $n \geq m$ we have*

$$\mathcal{J}_{m+1}\mathcal{J}_n - \mathcal{J}_m\mathcal{J}_{n+1} = \frac{1+\varepsilon}{3}\left(2^n(-1)^m(-1-11i_1+11i_2+i_3) - 2^m(-1)^n(-1+13i_1-i_2+13i_3)\right)$$

$$\widetilde{\mathcal{J}}_{m+1}\widetilde{\mathcal{J}}_n - \widetilde{\mathcal{J}}_m\widetilde{\mathcal{J}}_{n+1} = 3(1+\varepsilon)\left(2^m(-1)^n(-1+13i_1-i_2+13i_3) - 2^n(-1)^m(-1-11i_1+11i_2+i_3)\right)$$

Proof. By using Equality (2.3) we have

$$\begin{aligned}
\mathcal{J}_{m+1}\mathcal{J}_n - \mathcal{J}_m\mathcal{J}_{n+1} &= \frac{1}{3}\left(2^{m+1}\hat{\alpha} - (-1)^{m+1}\hat{\beta}\right)\frac{1}{3}\left(2^n\hat{\alpha} - (-1)^n\hat{\beta}\right) \\
&- \frac{1}{3}\left(2^m\hat{\alpha} - (-1)^m\hat{\beta}\right)\frac{1}{3}\left(2^{n+1}\hat{\alpha} - (-1)^{n+1}\hat{\beta}\right) \\
&= \frac{1}{9}\left(-\hat{\alpha}\hat{\beta}\,2^{m+1}(-1)^n - \hat{\beta}\hat{\alpha}\,(-1)^{m+1}2^n + \hat{\alpha}\hat{\beta}\,(-1)^{n+1}2^m + \hat{\beta}\hat{\alpha}\,2^{n+1}(-1)^m\right) \\
&= \frac{1}{3}\left(\hat{\beta}\hat{\alpha}\,2^n(-1)^m - \hat{\alpha}\hat{\beta}\,2^m(-1)^n\right)
\end{aligned}$$

Finally by using Equalities (2.5) and (2.6) the result is clear. The proof for Jacobthal-Lucas can be done similarly by using Equalities (2.4), (2.5) and (2.6). ☐

3 CONCLUSIONS

In this paper, we define dual Jacobstahl and dual Jacobsthal-Lucas split quaternions. The polar representations of these split quaternions have been obtained similar to the real quaternions. For this, the modulus and argument have been calculated from an arbitrary split quaternion. In further the current paper, it would be valuable to replicate similar approaches in dual split quaternions with Jacobsthal and Jacobsthal-Lucas number sequences.

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