Interpolative Contraction in the Framework of Super Metric Spaces

ABSTRCT: In this paper, we prove a fixed-point theorem for (λ, α)-interpolative Kannan contraction in super metric space.

Keywords: fixed point; iterative methods; (λ, α)-interpolative Kannan contraction, super metric space.

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1. Introduction

One of the most beneficial and appealing areas of nonlinear functional analysis is metric fixed-point theory. In light of Banach's pioneering fixed-point theorem, several findings and publications on the topic have been made during the past 100 years. Essentially, there are two widely accepted theories about how to advance the metric fixed point: the first is changing (weakening) the constraints on the mapping of contraction, and the second is altering the abstract structure. Metric spaces have already seen a number of generalizations and extensions. These include the quasi-metric space, the b-metric space, the symmetric space, the fuzzy metric space, the dislocated metric space, the partial metric space, the 2-metric space, the modular metric space, the cone metric space, the ultrametric space, and a variety of other combinations of these.

It is important to note that the fixed-point theory is very practical and helpful in finding solutions to numerous issues in a variety of industries. As a result, this topic has been the subject of extensive research, the findings of which have been disseminated in the form of articles and books. On the other hand, observational articles published in recent years have shown that a considerable proportion of publications' results either coincide with, overlap with, or are equivalent to other results in the literature. These discoveries highlight the fact that the fixed-point theory is congested and constrained. As an illustration, most of the fixed results for cone metric spaces are equivalent to the comparable results when standard metric space is used. Regarding the G-metric space, the same result may be drawn. Erdal Karapinar and Andreea Fulga [5] introduced supermetric space. We were able to derive some fixed-point theorems in this structure, and we believe that this method could assist in alleviating the congestion and squeezing problems noted before.

We prove a fixed-point theorem for (λ, α)-interpolative Kannan contraction in super metric space. Our findings extend the contractions of the metric space to a super metric space by Banach's contraction and Kannan's contraction.

2. Preliminaries

We begin this section with the definition of the super metric.

***Definition* 2.1** (see [5]) Let $\mathfrak{D}$ is a non-empty set. We say that a function $\eta: \mathfrak{D} \times \mathfrak{D} \to [0, +\infty)$ is a super metric if it satisfies the following axioms:

(s1). $\forall\, \sigma, \varsigma \in \mathfrak{D}$, if $\eta(\sigma, \varsigma) = 0$, then $\eta(\sigma, \varsigma) = 0$.

(s2). $\forall\, \sigma, \varsigma \in \mathfrak{D}, \eta(\sigma, \varsigma) = \eta(\varsigma, \sigma)$.

(s3). *There exists* $s \geq 1$ *such that for every* $\varsigma \in \mathfrak{D}$, *there exist distinct sequences* $\{\sigma_n\}, \{\varsigma_n\} \subset \mathfrak{D}$, *with* $\eta(\sigma_n, \varsigma_n) \to 0$ *when* $n \to \infty$, *such that*

$$\limsup_{n\to\infty} \eta(\varsigma_n, \varsigma) \leq s \limsup_{n\to\infty} \eta(\sigma_n, \varsigma)$$

The tripled $(\mathfrak{D}, \eta, s)$ is called a super metric space.

Definition 2.2 (see [5]) *On a super metric space* $(\mathfrak{D}, \eta, s)$, *a sequence* $\{\sigma_n\}$:

(i). *converges to* σ *in* $\mathfrak{D}$ *if and only if* $\lim_{n\to\infty} \eta(\sigma_n, \sigma) = 0$

(ii). *is a Cauchy sequence in* $\mathfrak{D}$ *if and only if* $\lim_{n\to\infty} \sup\{\eta(\sigma_n, \sigma_p): p > n\} = 0$.

Proposition 2.3 (see [5]) *On a super metric space, the limit of a convergent sequence is unique.*

Definition 2.4 (see [5]) *We say that a super metric space* $(\mathfrak{D}, \eta, s)$ *is* complete *if and only if every Cauchy sequence is convergent in* $\mathfrak{D}$.

Example 2.5 (see [5]) *Let the set* $\mathfrak{D} = \mathbb{R}, s = 2$, *and* $\eta: \mathfrak{D} \times \mathfrak{D} \to [0, +\infty)$ *be an application defined as follows:*

$$\eta(\sigma, \varsigma) = (\sigma - \varsigma)^2, \text{ for } \sigma, \varsigma \in \mathbb{R}\backslash\{1\}$$

$$\eta(1, \varsigma) = \eta(\varsigma, 1) = (1 - \varsigma^3)^2, \text{ for } \varsigma \in \mathbb{R}.$$

Then, the *tripled* $(\mathfrak{D}, \eta, s)$ *forms a super metric space.*

Example 2.6 (see [5]) *Let the set* $\mathfrak{D} = [0, +\infty]$ *and* $\eta: \mathfrak{D} \times \mathfrak{D} \to [0, +\infty)$ *be a function, defined as follows:*

$$\eta(\sigma, \varsigma) = \frac{|\sigma\varsigma - 1|}{\sigma + \varsigma + 1}, \textit{for } \sigma, \varsigma \in [0,1) \cup (1, +\infty], \sigma \neq \varsigma,$$

$$\eta(\sigma, \varsigma) = 0, \textit{for } \sigma, \varsigma \in [0, +\infty), \sigma = \varsigma,$$

$$\eta(\sigma, 1) = \eta(1, \sigma) = |\sigma - 1|, \textit{for } \sigma \in [0, +\infty].$$

We can easily see that η forms a super metric on $\mathfrak{D}$.

Proposition 2.7 (see [5]) *Let* $\Gamma: \mathfrak{D} \to \mathfrak{D}$ *be an asymptotically regular mapping on a complete supermetric space* $(\mathfrak{D}, \eta, s)$. *Then, the Picard iteration* $\{\Gamma^n \sigma\}$ *for the initial point* $\sigma \in \mathbb{R}$ *is a convergent sequence on* $\mathfrak{D}$.

Theorem 2.8 (see [5]) Let $(\mathfrak{D}, \eta, s)$ be a complete super-metric space and let $\Gamma: \mathfrak{D} \to \mathfrak{D}$ be a mapping. Suppose that $0 < \alpha < 1$ such that

$$\eta(\Gamma\sigma, \Gamma\varsigma) \leq \eta(\sigma, \varsigma)$$

for all $(\sigma, \varsigma) \in \mathfrak{D}$. Then Γ has a unique fixed point in $\mathfrak{D}$.

Theorem 2.9 (see [5]) *Let* $(\mathfrak{D}, \eta, s)$ *be a complete super metric space and* $\Gamma: \mathfrak{D} \to \mathfrak{D}$ *be a mapping, such that there exist* $\alpha \in [0, 1)$ *and that*

$$\eta(\Gamma\sigma, \Gamma\varsigma) \leq k \max\left\{\eta(\sigma, \varsigma), \frac{\eta(\sigma, \Gamma\sigma)\eta(\varsigma, \Gamma\varsigma)}{\eta(\sigma, \varsigma) + 1}\right\}$$

Then, Γ *has a unique fixed point.*

3. Main Results

We start with the following definition.

Definition 3.1 Let $(\mathfrak{D}, \eta, s)$ be a *super* metric space and $\Gamma: \mathfrak{D} \to \mathfrak{D}$ a self-map. We shall call Γ a (λ, α)-interpolative Kannan contraction, if there exist $\lambda \in [0,1), \alpha \in (0,1)$ such that

$$\eta(\Gamma p, \Gamma q) \leq \lambda\big(\eta(p, \Gamma p)\big)^{\alpha}\big(\eta(\text{q}, \Gamma q)\big)^{1-\alpha} \tag{3.1}$$

for all $p, q \in \mathfrak{D}$, with $p \neq q$.

Theorem 3.2 Let $(\mathfrak{D}, \eta, s)$ be a complete *super* metric space and $\Gamma: \mathfrak{D} \to \mathfrak{D}$ a self-map continuous (λ, α)-interpolative Kannan contraction with $\lambda \in [0,1), \alpha \in (0,1)$. Then, Γ has a unique fixed point.

Proof Let $p_0 \in \mathfrak{D}$ and let $\Gamma p_0 = p_1$. If $p_0 = p_1$ then $p_1$is the fixed point and the proof is completed. So, suppose that $p_0 \neq p_1$. Thus, $\eta(p_0, p_1) > 0$. Thus, without loss of generality, for each nonnegative integer n, we can define $p_{n+1} = \Gamma p_n$ such that $p_{n+1} \neq \Gamma p_n$. So $\eta(p_n, p_{n+1}) > 0$, for all $n \in \mathbb{N}$. So, we have

$$\begin{aligned}
\eta(p_n, p_{n+1}) &= \eta(\Gamma p_{n-1}, \Gamma p_n) \\
&\leq \lambda\big(\eta(p_{n-1}, \Gamma p_{n-1})\big)^{\alpha}\big(\eta(p_n, \Gamma p_n)\big)^{1-\alpha} \\
&= \lambda\big(\eta(p_{n-1}, p_n)\big)^{\alpha}\big(\eta(p_n, p_{n+1})\big)^{1-\alpha}
\end{aligned}$$

Thus,

$$\left(\eta(p_n, p_{n+1})\right)^{\alpha} \leq \lambda\left(\eta(p_{n-1}, p_n)\right)^{\alpha}$$

i.e.

$$\eta(p_n, p_{n+1}) \leq \lambda^{\frac{1}{\alpha}}\eta(p_{n-1}, p_n) \leq \lambda\eta(p_{n-1}, p_n) \tag{3.3}$$

Therefore,

$$\begin{aligned}\eta(p_n, p_{n+1}) &\leq \lambda\eta(p_{n-1}, p_n)\\ &\leq \lambda^2\eta(p_{n-1}, p_n)\\ &\vdots\\ &\leq \lambda^n\eta(p_0, p_1)\end{aligned} \tag{3.4}$$

and in taking the limit from the above inequality, we get

$$\lim_{n\to\infty} \eta(p_n, p_{n+1}) = 0. \tag{3.5}$$

Now suppose that, $m, n \in \mathbb{N}$ and $m > n$. If $p_n = p_m$, we have $\Gamma^m p_0 = \Gamma^n p_0$. Thus, we have $\Gamma^{m-n}(\Gamma^n p_0) = \Gamma^n p_0$. Thus, we have $\Gamma^n p_0$ is the fixed point of Γ^{m-n}. Also,

$$\Gamma\left(\Gamma^{m-n}(\Gamma^n p_0)\right) = \Gamma^{m-n}\left(\Gamma(\Gamma^n p_0)\right) = \Gamma(\Gamma^n p_0) \tag{3.6}$$

It means that, $\Gamma(\Gamma^n p_0)$ is the fixed point of Γ^{m-n}. Thus, $\Gamma(\Gamma^n p_0) = \Gamma^n p_0$. So, $\Gamma^n p_0$ is the fixed point of Γ. So, without loss of generality we can suppose that $p_n \neq p_m$. Therefore,

$$\limsup_{n\to\infty} \eta(p_n, p_{n+2}) \leq s \limsup_{n\to\infty} \eta(p_{n+1}, p_{n+2}) \leq s \limsup_{n\to\infty}\{\lambda^{n+1}\eta(p_0, p_1)\} = 0. \tag{3.7}$$

Thus, $\limsup_{n\to\infty} \eta(p_n, p_{n+2}) = 0$, we

$$\limsup_{n\to\infty} \eta(p_n, p_{n+3}) \leq s \limsup_{n\to\infty} \eta(p_{n+2}, p_{n+3}) \leq s \limsup_{n\to\infty}\{\lambda^{n+2}\eta(p_0, p_1)\} = 0 \tag{3.8}$$

Inductively, one can conclude that $\limsup_{n\to\infty}\{\eta(p_n, p_m): m > n\} = 0$. It means that $\{p_n\}$ is a Cauchy sequence. Since $(\mathfrak{D}, \eta, s)$ is a complete super metric space, the sequence $\{p_n\}$ converges to $p^* \in \mathfrak{D}$. We claim that p^* is the fixed point of Γ. On the contrary, assume $\eta(p^*, \Gamma p^*) > 0$. Note that

$$\begin{aligned}\eta(p_{n+1}, \Gamma p^*) &= \eta(\Gamma p_n, \Gamma p^*)\\ &\leq \lambda\left(\eta(p^*, \Gamma p^*)\right)^{\alpha}\left(\eta(p_n, \Gamma p_n)\right)^{1-\alpha}\\ &= \lambda\left(\eta(p^*, \Gamma p^*)\right)^{\alpha}\left(\eta(p_n, p_{n+1})\right)^{1-\alpha} \to 0 \quad \text{as } n \to 0.\end{aligned} \tag{3.9}$$

Thus, $\limsup_{n\to\infty} \eta(p_{n+1}, \Gamma p^*) = 0$. If there $N > 0$ such that for all $n > N$, $p_{N+1} = p^*$, (3.9) concludes that $\eta(p^*, \Gamma p^*) = 0$ and so we have p^* is the fixed point for Γp^* Otherwise, suppose that for all $n \in \mathbb{N}, p_n \neq p^*$. Thus, we have,

$$\eta(p^*, \Gamma p^*) \leq \limsup_{n\to\infty} \eta(p_{n+1}, \Gamma p^*) \qquad (3.10)$$

and one can conclude that $\eta(p^*, \Gamma p^*) = 0$, which is a contradiction. Thus, $p^* = \Gamma p^*$ is the fixed point of Γ in $\mathfrak{D}$. We shall now prove the uniqueness of the fixed point. If $q^* \in \mathfrak{D}$ is another fixed point of Γ, that is, $\Gamma q^* = q^*$, then we get

$$\eta(p^*, q^*) = \eta(\Gamma p^*, \Gamma q^*) \leq \lambda\big(\eta(p^*, \Gamma p^*)\big)^{\alpha}\big(\eta(q^*, \Gamma q^*)\big)^{1-\alpha} \leq 0$$

which is a contradiction, and hence, $p^* = q^*$.

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