Resolution of the standard telegraph equation by the Laplace-Adomian method

Abstract

In this paper, we resarch the solution of the standard telegraph equation by the Laplace-Adomian method. The Laplace-Adomian method is based on the combination of Laplace transform and the Adomian decompositionnal method.

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Key word: Telegraph equation, Laplace transform, ADM method.

1 Introduction

In this article, we study the general solution of the standard telegraph equations by the method of Laplace-Adomian. The standard telegraph equation is an important equation arises in the propagation of electrical signals along a telegraph line, taking into consideration the inductance, capacitance and conductance of the cable. However the method of Laplace-Adomian is a numerical method based on the combination of the Laplace Transform and Adomian decompositionnal method.

2 The numerical Laplace-Adomian method

The standard telegraph equation is a partial differential equation given by :

$$\partial^2 u \partial x^2 = \alpha \partial^2 u \partial t^2 + \beta \partial u \partial t + \gamma u$$

where $u = u(t,x)$ is the resistance, and α,β and γ are constants related to the inductance, capacitance and conductance of the cable respectively.

Let us consider the following functional equation:

$$\begin{cases} \partial^2 u \partial x^2 & = & \alpha \partial^2 u \partial t^2 + \beta \partial u \partial t + \gamma u \\ u(t,0) & = & f(t) \\ \partial u \partial x (t,0) & = & g(t) \\ u(0,x) & = & h(x) \\ \partial u \partial t (0,x) & = & v(x) \end{cases} \quad (1)$$

Taking $Lu = \partial^2 u \partial x^2$ and $Ru = \alpha \partial^2 u \partial t^2 + \beta \partial u \partial t + \gamma u$
We have :

$$Lu = Ru \quad (2)$$

Where L is an invertible operator in the Adomian sense and R the linear remainder.

Applying the laplace transform to the equation (2), we obtain :

$$_x (Lu) =_x (Ru) \Leftrightarrow p^2_x (u) - pu(t,0) - \partial u \partial x (t,0) =_x (Ru) \quad (3)$$

$$p^2_x (u) = pf(t) + g(t) +_x (Ru) \quad (4)$$

Using the decomposition series for the linear term $u(t,x)$ gives

$$p^2_{n \geq 0 x} (u_n) = pf(t) + g(t) +_{n \geq 0\ x} (Ru_n) \quad (5)$$

This yields the following Adomian algorithm:

$$\begin{cases} p^2_x (u_0) = pf(t) + g(t) \\ p^2_x (u_{n+1}) =_x (Ru_n) ; n \geq 0 \end{cases} \quad (6)$$

Applying the laplace transform to the equation (2), we obtain :

$$\begin{cases} u_0 (t,x) =^{-1}_x [1p^2 (pf(t) + g(t))] \\ \ldots \\ u_{n+1} (t,x) =^{-1}_x [1{p^2}_x (Ru_n)] ; n \geq 0 \end{cases} \quad (7)$$

3 Algorithm of Laplace - ADM Convergence's

Considering the equation (1)

$$\begin{cases} \partial^2 u \partial x^2 & = & \alpha \partial^2 u \partial t^2 + \beta \partial u \partial t + \gamma u \\ u(t,0) & = & f(t) \\ \partial u \partial x (t,0) & = & g(t) \\ u(0,x) & = & h(x) \\ \partial u \partial t (0,x) & = & v(x) \end{cases}$$

With $(t, x) \in \Omega$ where $\Omega = [0; +\infty[\times [a, b]$

The application of the Laplace-ADM method gives

$$\begin{cases} u_0 (t, x) =_x^{-1} [1p^2 (pf (t) + g (t))] \\ u_{n+1} (t, x) =_x^{-1} [1p^2{}_x (Ru_n)] ; n \geq 0 \end{cases}$$

Let us suppose :
$\cdot$ (H_1)
f is continuous then there is a real M so that

$$|f (t)| \leq M forallt \in [0, T]$$

$\cdot$ (H_2)
g is continuous then there is a real M so that

$$|g (t)| \leq M' forallt \in [0, T]$$

However
R the linear remainder is continuous then there is a real $\lambda > 0$ so that

$$||Ru|| \leq \lambda ||u||$$

Indeed, we have :

$$\begin{cases} |u_0| & = & |_x^{-1} [f (t)p] +_x^{-1} [g (t)p^2]| \\ ... & ... & ... \\ |u_n| & = & |_x^{-1} [_t (Ru_{n-1})p]| \ ; \ n \geq 1 \end{cases}$$

There is a real $x_0 \in \mathbb{R}_*^+$ so that $\Re e (p) > x_0$, we deduce the following system :

$$\begin{cases} |u_0| & \leq & |_x^{-1} [f (t)p]| + |_x^{-1} [g (t)p^2]| \\ ... & ... & ... \\ |u_n| & \leq & {}_x^{-1} [|Ru_{n-1}||p^2|] \ ; \ n \geq 1 \end{cases}$$

$$\Rightarrow \begin{cases} |u_0| & \leq & M + M'b \\ ... & ... & ... \\ |u_n| & \leq & {}_x^{-1} [_x (|Ru_{n-1}|)x_0^2] \ ; \ n \geq 1 \end{cases}$$

$$\Rightarrow \begin{cases} |u_0| & \leq & M + M'b \\ ... & ... & ... \\ |u_n| & \leq & 1x_{0x}^{2-1} \left[_x \left(\left|Ru_{n-1}^1\right|\right)\right] \ ; \ n \geq 1 \end{cases}$$

$$\Rightarrow \begin{cases} |u_0| & \leq & M + M'b \\ ... & ... & ... \\ |u_n| & \leq & \lambda x_0^2 \left|\left|u_{n-1}^1\right|\right| \ ; \ n \geq 1 \end{cases}$$

Step by step, we deduce :

$$\Rightarrow \begin{cases} |u_0| & \leq & M + M'b \\ ... & ... & ... \\ |u_n| & \leq & (\lambda x_0^2)^n (M + M'b) \ ; \ n \geq 1 \end{cases}$$

With $\lambda x_0^2 < 1$ and $x_0 \neq \sqrt{\lambda}$, we obtain

$$\Rightarrow \begin{cases} |u_0| & \leq & M + M'b \\ ... & ... & ... \\ {}_{n\geq 0} |u_n| & \leq & (M + M'b)\, x_0^2 x_0^2 - \lambda \end{cases}$$

Then the series ${}_{n\geq 0} u_n$ is convergent, therefore this algorithm is convergent.

4 Applications

4.1 Example : wave equation of microstrip antenna

In this example, we study the standard linear telegraph equation presented in wave equation of microstrip antenna equation is given as

$$\begin{cases} \partial^2 u \partial x^2 + \partial^2 u \partial t^2 + 2\partial u \partial t - u & = & 0 \\ \partial u \partial x\,(t,0) & = & e^{-2t} \\ u\,(t,0) & = & e^{-2t} \end{cases}$$

Taking $Lu = \partial^2 u \partial x^2$, $Ru = -\partial^2 u \partial t^2 - 2\partial u \partial t + u$.

Where L is an invertible operator in the Adomian sense and R the linear remainder.

Applying the laplace transform to the equation (2), we obtain :

$$_x\,(Lu) =_x (Ru) \tag{8}$$

$\Leftrightarrow$

$$p_x^2\,(u) - pu\,(t,0) - \partial u \partial x\,(t,0) =_x \left(-\partial^2 u \partial t^2 - 2\partial u \partial t + u\right) \tag{9}$$

$$\left(p^2 - 1\right)_x (u) = pe^{-2t} + e^{-2t} +_x \left(-\partial^2 u \partial t^2 - 2\partial u \partial t\right) \tag{10}$$

Using the decomposition series for the linear term $u(t,x)$ gives

$$\left(p^2 - 1\right)_{n\geq 0\ x} (u_n) = pe^{-2t} + e^{-2t} +_{n\geq 0\ x} \left(-\partial^2 u_n \partial t^2 - 2\partial u_n \partial t\right) \tag{11}$$

We deduce the following Laplace-Adomian algorithm

$$\begin{cases} (p^2 - 1)_x\,(u_0) = pe^{-2t} + e^{-2t} \\ (p^2 - 1)_x\,(u_{n+1}) =_x (Ru_n)\,; n \geq 0 \end{cases} \tag{12}$$

We obtain

$$\begin{cases} u_0\,(t,x) =_x^{-1}\left[1(p^2-1)\,(pe^{-2t}+e^{-2t})\right] \\ u_{n+1}\,(t,x) =_x^{-1}\left[1(p^2-1)_x\,(Ru_n)\right];n\geq 0 \end{cases}$$

Determinate $u_n(t,x)$, **for** $n\geq 0$
$u_0(t,x) =_x^{-1}\left[1(p^2-1)\,(p+1)\,e^{-2t}\right]$
$\Rightarrow u_0(t,x) =_x^{-1}\left[1(p-1)e^{-2t}\right]$
$\Rightarrow u_0(t,x) = e^{x-2t}$
$u_1(t,x) = \mathcal{L}_t^{-1}\left[1(p^2-1)\mathcal{L}_t\,(R(u_0))\right]$
$\Rightarrow u_1(t,x) = \mathcal{L}_t^{-1}\left[1(p^2-1)\,(-4e^{x-2t}+4e^{x-2t})\right]$

$\Rightarrow u_1(t,x) = 0$

$u_2(t,x) = \mathcal{L}_t^{-1}\left[1(p^2-1)\mathcal{L}_t\,(R(u_1))\right]$
$\Rightarrow u_2(t,x) = 0$

In recursive way, we deduce

$$u_n\,(t,x) = 0 forall n \geq 1$$

Then

$$u(t,x) =_{n\geq 0} u_n(t,x) = e^{x-2t}$$

The exact solution of model is

$$u(t,x) = e^{x-2t}$$

4.2 Example : wave equation of microstrip antenna

In this example, we study the standard linear telegraph equation is given as

$$\begin{cases} \partial^2 u \partial x^2 &= \partial^2 u \partial t^2 + 4\partial u \partial t + 4u \\ \partial u \partial t\,(0,x) &= -2 \\ u\,(0,x) &= 1+e^{2x} \end{cases}$$

Taking $Lu = \partial^2 u \partial t^2$, $Ru = \partial^2 u \partial x^2 - 4\partial u \partial t - 4u$.

Where L is an invertible operator in the Adomian sense and R the linear remainder.

Applying the laplace transform to the equation (2), we obtain :

$$_t\left(Lu\right) =_t \left(Ru\right) \Leftrightarrow p_t^2\left(u\right) - pu\left(0,x\right) - \partial u \partial x\left(0,x\right) =_t \left(\partial^2 u \partial x^2 - 4\partial u \partial t - 4u\right) \quad (13)$$

$$p_x^2\left(u\right) = p\left(1+e^{2x}\right) - 2 +_x \left(\partial^2 u \partial x^2 - 4\partial u \partial t - 4u\right) \quad (14)$$

Using the decomposition series for the linear term $u(t,x)$ gives

$$p_{n\geq 0x}^2\left(u_n\right) = p - 2 + pe^{2x} +_{n\geq 0\ x} \left(\partial^2 u_n \partial x^2 - 4\partial u_n \partial t - 4u\right) \quad (15)$$

$$p_{n\geq 0x}^2\left(u_n\right) = p - 2 + pe^{2x} +_{n\geq 0\ x} \left(\partial^2 u_n \partial x^2 - 4\partial u_n \partial t - 4u\right) \quad (16)$$

We deduce the following Laplace-Adomian algorithm

$$\begin{cases} p_x^2\left(u_0\right) = p - 2 + pe^{2x} \\ p_x^2\left(u_{n+1}\right) =_x \left(Ru_n\right); n \geq 0 \end{cases} \quad (17)$$

We obtain

$$\begin{cases} u_0\left(t,x\right) =_x^{-1} \left[1p^2\left(p-2+pe^{2x}\right)\right] \\ u_{n+1}\left(t,x\right) =_x^{-1} \left[1p^2{}_x\left(Ru_n\right)\right]; n \geq 0 \end{cases}$$

Determinate $u_n(t,x)$, **for** $n \geq 0$

$u_0(t,x) =_t^{-1} \left[1p^2\left(p-2+pe^{2x}\right)\right]$

$\Rightarrow u_0(t,x) =_t^{-1} \left[1p\left(1+e^{2x}\right) - 2p^2\right]$

$\Rightarrow u_0(t,x) = e^{2x} - 2t + 1$

$u_1(t,x) =_t^{-1} \left[1p^2{}_t\left(R(u_0)\right)\right]$

$\Rightarrow u_1(t,x) =_t^{-1} \left[1p^2\left[_t\left(-8t+4\right)\right]\right]$

$\Rightarrow u_1(t,x) =_t^{-1} \left(-8p^4 + 4p^3\right) = -8t^3 3! + 4t2!$

$\Rightarrow u_1(t,x) = -(2t)^3 3! + (2t)^2 2!$

$u_2(t,x) =_t^{-1} \left[1p^2{}_t\left(R(u_1)\right)\right]$

$\Rightarrow u_2(t,x) =_t^{-1} \left[1p^2\left[_t\left(-16t - 24t^2 - 32t^3 3!\right)\right]\right]$

$\Rightarrow u_2(t,x) =_t^{-1} \left(-161p^4 - 481p^5 - 321p^6\right)$

$\Rightarrow u_2(t,x) = -2(2t)^3 3! - 3(2t)^4 4! - (2t)^5 5!$

$u_3(t,x) =_t^{-1} [1p^2{}_t(R(u_2))]$
$\Rightarrow u_3(t,x) =_t^{-1} [1p^2 [_t (2^6t^2 2! + 2^8t^3 3! + 10 \times 2^5t^4 4! + 2^7t^5 5!)]]$

$\Rightarrow u_3(t,x) =_t^{-1} (2^6 1p^5 + 2^8 1p^6 + 5 \times 2^6 1p^7 + 2^7 1p^8)$

$\Rightarrow u_3(t,x) = 4(2t)^4 4! + 8(2t)^5 5! + 5(2t)^6 6! + (2t)^7 7!$

$u_4(t,x) =_t^{-1} [1p^2{}_t(R(u_3))]$
$\Rightarrow u_4(t,x) =_t^{-1} \left[1p^2 \left[_t \left(-\frac{32}{315}t^7 - \frac{112}{45}t^6 - \frac{96}{5}t^5 - \frac{160}{3}t^4 - \frac{128}{3}t^3\right)\right]\right]$
$\Rightarrow u_4(t,x) =_t^{-1} [1p^2 [_t (-2^8t^3 3! - 5 \times 2^8t^4 4! - 9 \times 2^8t^5 5! - 7 \times 2^8t^6 6! - 2^9t^7 7!)]]$
$\Rightarrow u_4(t,x) =_t^{-1} (-2^8 1p^6 - 5 \times 2^8 1p^7 - 9 \times 2^8 1p^8 - 7 \times 2^8 1p^9 - 2^9 1p^{10})$

$\Rightarrow u_4(t,x) = -8(2t)^5 5! - 20(2t)^6 6! - 18(2t)^7 7! - 7(2t)^8 8! - (2t)^9 9!$
Step by step, we deduce

$$_{n\geq 0}u_n(t,x) = e^{2x} +_{n\geq 0} (-2t)^k n!$$

Then

$$u(t,x) =_{n\geq 0} u_n(t,x) = e^{2x} + e^{-2t}$$

The exact solution of model is

$$u(t,x) = e^{2x} + e^{-2t}$$

5 Conclusion

Laplace's Adomian numerical method allowed us to solve some linear partial differential equations by modelling the standard telegraph equation. It is therefore a very powerful numerical analysis tool to solve this type of problem, this method accelerates convergence to the solution . Our study was limited to the linear models of telegraph non-homogeneous reaction, a study of these models in non-homogeneous cases would be an important contribution to the understanding of these models.

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