# Resolution of the standard telegraph equation by the Laplace-Adomian method

**Abstract**

In this paper, we resarch the solution of the standard telegraph equation by the Laplace-Adomian method. The Laplace-Adomian method is based on the combination of Laplace transform and the Adomian decompositionnal method.

**AMS subjet classification**: 47H14, 34G20, 47J25, 65J15.

**Key word**: Telegraph equation, Laplace transform, ADM method.

## 1 Introduction

In this article, we study the general solution of the standard telegraph equations by the method of Laplace-Adomian. The standard telegraph equation is an important equation arises in the propagation of electrical signals along a telegraph line, taking into consideration the inductance, capacitance and conductance of the cable. However the method of Laplace-Adomian is a numerical method based on the combination of the Laplace Transform and Adomian decompositionnal method.

## 2 The numerical Laplace-Adomian method

The standard telegraph equation is a partial differential equation given by :

$$\partial u^2 \partial x^2 = \alpha \partial^2 u \partial t^2 + \beta \partial u \partial t + \gamma u$$

where $u = u(t, x)$ is the resistance, and $a$,$b$ and $c$ are constants related to the inductance, capacitance and conductance of the cable respectively.

Let us consider the following functional equation:

$$\left\{\begin{array}{lcl} \partial u^2 \partial x^2 & = & \alpha \partial^2 u \partial t^2 + \beta \partial u \partial t + \gamma u \\ u\left(t,0\right) & = & f\left(t\right) \\ \partial u \partial x\left(t,0\right) & = & g\left(t\right) \\ u\left(0,x\right) & = & h\left(x\right) \\ \partial u \partial t\left(0,x\right) & = & v\left(x\right) \end{array}\right. \quad (1)$$

Taking $Lu = \partial u^2 \partial x^2$ and $Ru = \alpha \partial^2 u \partial t^2 + \beta \partial u \partial t + \gamma u$
We have :

$$Lu = Ru \quad (2)$$

Where $L$ is an invertible operator in the Adomian sense and $R$ the linear remainder.

Applying the laplace transform to the equation (2), we obtain :

$$_x\left(Lu\right) =_x \left(Ru\right) \Leftrightarrow p^2_x\left(u\right) - pu\left(t,0\right) - \partial u \partial x\left(t,0\right) =_x \left(Ru\right) \quad (3)$$

$$p^2_x\left(u\right) = pf\left(t\right) + g\left(t\right) +_x \left(Ru\right) \quad (4)$$

Using the decomposition series for the linear term $u(t,x)$ gives

$$p^2_{n\geq 0 x}\left(u_n\right) = pf\left(t\right) + g\left(t\right) +_{n\geq 0\ x} \left(Ru_n\right) \quad (5)$$

This yields the following Adomian algorithm:

$$\left\{\begin{array}{l} p^2_x\left(u_0\right) = pf\left(t\right) + g\left(t\right) \\ p^2_x\left(u_{n+1}\right) =_x \left(Ru_n\right); n \geq 0 \end{array}\right. \quad (6)$$

Applying the laplace transform to the equation (2), we obtain :

$$\left\{\begin{array}{l} u_0\left(t,x\right) =^{-1}_x \left[1p^2\left(pf\left(t\right) + g\left(t\right)\right)\right] \\ u_{n+1}\left(t,x\right) =^{-1}_x \left[1p^2{}_x\left(Ru_n\right)\right]; n \geq 0 \end{array}\right. \quad (7)$$

# 3 Algorithm of Laplace - ADM Convergence's

Considering the equation (1)

$$\left\{\begin{array}{lcl} \partial u^2 \partial x^2 & = & \alpha \partial^2 u \partial t^2 + \beta \partial u \partial t + \gamma u \\ u\left(t,0\right) & = & f\left(t\right) \\ \partial u \partial x\left(t,0\right) & = & g\left(t\right) \\ u\left(0,x\right) & = & h\left(x\right) \\ \partial u \partial t\left(0,x\right) & = & v\left(x\right) \end{array}\right.$$

With $(t,x) \in \Omega$ where $\Omega = [0; +\infty[ \times [a,b]$

The application of the Laplace-ADM method gives

$$\begin{cases} u_0(t,x) =_x^{-1} [1p^2 (pf(t) + g(t))] \\ u_{n+1}(t,x) =_x^{-1} [1p^2{}_x (Ru_n)] ; n \geq 0 \end{cases}$$

**Let us suppose :**

$\cdot$ $(H_1)$

$f$ is continuous then there is a real $M$ so that

$$|f(t)| \leq M forall t \in [0,T]$$

$\cdot$ $(H_2)$

$g$ is continuous then there is a real $M$ so that

$$|g(t)| \leq M' forall t \in [0,T]$$

**However**

$R$ the linear remainder is continuous then there is a real $\alpha > 0$ so that

$$||Ru|| \leq \alpha ||u||$$

**Indeed, we have :**

$$\begin{cases} |u_0| &= |_x^{-1} [f(t)p] +_x^{-1} [g(t)p^2]| \\ |u_n| &= |_x^{-1} [_t (Ru_{n-1})p]| \; ; n \geq 1 \end{cases}$$

There is a real $x_0 \in \mathbb{R}_*^+$ so that $\Re e(p) > x_0$, we deduce the following system :

$$\begin{cases} |u_0| &\leq |_x^{-1} [f(t)p]| + |_x^{-1} [g(t)p^2]| \\ |u_n| &\leq {}_x^{-1} [|Ru_{n-1}||p^2|] \; ; n \geq 1 \end{cases}$$

$$\Rightarrow \begin{cases} |u_0| &\leq M + M'b \\ |u_n| &\leq {}_x^{-1} [_x (|Ru_{n-1}|) x_0^2] \; ; n \geq 1 \end{cases}$$

$$\Rightarrow \begin{cases} |u_0| &\leq M + M'b \\ |u_n| &\leq 1x_{0x}^{2-1} \left[_x \left(\left|Ru_{n-1}^1\right|\right)\right] \; ; n \geq 1 \end{cases}$$

$$\Rightarrow \begin{cases} |u_0| &\leq M + M'b \\ |u_n| &\leq \alpha x_0^2 \left|\left|u_{n-1}^1\right|\right| \; ; n \geq 1 \end{cases}$$

Step by step, we deduce :

$$\Rightarrow \begin{cases} |u_0| &\leq M + M'b \\ |u_n| &\leq (\alpha x_0^2)^n (M + M'b) \; ; n \geq 1 \end{cases}$$

$$\Rightarrow \begin{cases} |u_0| &\leq M + M'b \\ {}_{n\geq 0} |u_n| &\leq (M + M'b) x_0^2 x_0^2 - \alpha \end{cases}$$

Then the series ${}_{n\geq 0} u_n$ is convergent, therefore this algorithm is convergent.

# 4 Applications

## 4.1 Example : wave equation of microstrip antenna

In this example, we study the standard linear telegraph equation presented in wave equation of microstrip antenna equation is given as

$$\begin{cases} \partial u^2 \partial x^2 + \partial^2 u \partial t^2 + 2\partial u \partial t - u & = & 0 \\ \partial u \partial x\,(t,0) & = & e^{-2t} \\ u\,(t,0) & = & e^{-2t} \end{cases}$$

Taking $Lu = \partial u^2 \partial x^2$, $Ru = -\partial^2 u \partial t^2 - 2\partial u \partial t + 2u$.

Where $L$ is an invertible operator in the Adomian sense and $R$ the linear remainder.

Applying the laplace transform to the equation (2), we obtain :

$$_x\,(Lu) =_x (Ru) \tag{8}$$

$\Leftrightarrow$

$$p^2_x\,(u) - pu\,(t,0) - \partial u \partial x\,(t,0) =_x \left(-\partial^2 u \partial t^2 - 2\partial u \partial t + u\right) \tag{9}$$

$$\left(p^2 - 1\right)_x (u) = pe^{-2t} + e^{-2t} +_x \left(-\partial^2 u \partial t^2 - 2\partial u \partial t\right) \tag{10}$$

Using the decomposition series for the linear term $u(t,x)$ gives

$$\left(p^2 - 1\right)_{n\geq 0\ x} (u_n) = pe^{-2t} + e^{-2t} +_{n\geq 0\ x} \left(-\partial^2 u_n \partial t^2 - 2\partial u_n \partial t\right) \tag{11}$$

We deduce the following Laplace-Adomian algorithm

$$\begin{cases} (p^2-1)_x\,(u_0) = pe^{-2t} + e^{-2t} \\ (p^2-1)_x\,(u_{n+1}) =_x (Ru_n)\,; n \geq 0 \end{cases} \tag{12}$$

We obtain

$$\begin{cases} u_0\,(t,x) =^{-1}_x [1(p^2-1)\,(pe^{-2t} + e^{-2t})] \\ u_{n+1}\,(t,x) =^{-1}_x [1(p^2-1)_x\,(Ru_n)]\,; n \geq 0 \end{cases}$$

**Determinate** $u_n(t,x)$, **for** $n \geq 0$

$u_0(t,x) =^{-1}_x [1(p^2-1)\,(p+1)\,e^{-2t}]$
$\Rightarrow u_0(t,x) =^{-1}_x [1(p-1)e^{-2t}]$
$\Rightarrow u_0(t,x) = e^{x-2t}$
$u_1(t,x) = \mathcal{L}_t^{-1} [1(p^2-1)\mathcal{L}_t\,(R(u_0))]$
$\Rightarrow u_1(t,x) = \mathcal{L}_t^{-1} [1(p^2-1)\,(-4e^{x-2t} + 4e^{x-2t})]$

$\Rightarrow u_1(t,x) = 0$

$u_2(t,x) = \mathcal{L}_t^{-1}\left[1(p^2-1)\mathcal{L}_t\left(R(u_1)\right)\right]$
$\Rightarrow u_2(t,x) = 0$

In recursive way, we deduce

$$u_n\left(t,x\right) = 0 for all n \geq 1$$

Then

$$u(t,x) =_{n\geq 0} u_n(t,x) = e^{x-2t}$$

The exact solution of model is

$$u(t,x) = e^{x-2t}$$

### 4.2 Example : wave equation of microstrip antenna

In this example, we study the standard linear telegraph equation is given as

$$\begin{cases} \partial u^2 \partial x^2 & = & \partial^2 u \partial t^2 + 4\partial u \partial t + 4u \\ \partial u \partial t\left(0,x\right) & = & -2 \\ u\left(0,x\right) & = & 1+e^{2x} \end{cases}$$

Taking $Lu = \partial u^2 \partial t^2$, $Ru = \partial^2 u \partial x^2 - 4\partial u \partial t - 4u$.

Where $L$ is an invertible operator in the Adomian sense and $R$ the linear remainder.

Applying the laplace transform to the equation (2), we obtain :

$$_t\left(Lu\right) =_t \left(Ru\right) \Leftrightarrow p_t^2\left(u\right) - pu\left(0,x\right) - \partial u \partial x\left(0,x\right) =_t \left(\partial^2 u \partial x^2 - 4\partial u \partial t - 4u\right) \tag{13}$$

$$p_x^2\left(u\right) = p\left(1+e^{2x}\right) - 2 +_x \left(\partial^2 u \partial x^2 - 4\partial u \partial t - 4u\right) \tag{14}$$

Using the decomposition series for the linear term $u(t,x)$ gives

$$p^2_{n\geq 0x}\left(u_n\right) = p - 2 + pe^{2x} +_{n\geq 0\ x} \left(\partial^2 u_n \partial x^2 - 4\partial u_n \partial t - 4u\right) \tag{15}$$

We deduce the following Laplace-Adomian algorithm

$$\begin{cases} p_x^2\left(u_0\right) = p - 2 + pe^{2x} \\ p_x^2\left(u_{n+1}\right) =_x \left(Ru_n\right); n \geq 0 \end{cases} \tag{16}$$

We obtain

$$\begin{cases} u_0\left(t,x\right) =_x^{-1} \left[1p^2\left(p-2+pe^{2x}\right)\right] \\ u_{n+1}\left(t,x\right) =_x^{-1} \left[1p^2{}_x\left(Ru_n\right)\right]; n \geq 0 \end{cases}$$

**Determinate** $u_n(t,x)$, **for** $n \geq 0$

$u_0(t,x) =_t^{-1} \left[1p^2\left(p-2+pe^{2x}\right)\right]$

$\Rightarrow u_0(t,x) =_t^{-1} \left[1p\left(1+e^{2x}\right) - 2p^2\right]$

$\Rightarrow u_0(t,x) = e^{2x} - 2t + 1$

$u_1(t,x) =_t^{-1} \left[1p^2{}_t\left(R(u_0)\right)\right]$

$\Rightarrow u_1(t,x) =_t^{-1} \left[1p^2\left[_t\left(8t+4\right)\right]\right]$

$\Rightarrow u_1(t,x) =_t^{-1} \left(8p^4 + 4p^3\right) = 8t^3 3! + 4t2!$

$\Rightarrow u_1(t,x) = (2t)^3 3! + (2t)^2 2!$

$u_2(t,x) =_t^{-1} \left[1p^2{}_t\left(R(u_1)\right)\right]$

$\Rightarrow u_2(t,x) =_t^{-1} \left[1p^2\left[_t\left(-16t - 24t^2 - 32t^3 3!\right)\right]\right]$

$\Rightarrow u_2(t,x) =_t^{-1} \left(-161p^4 - 481p^5 - 321p^6\right)$

$\Rightarrow u_2(t,x) = -2(2t)^3 3! - 3(2t)^4 4! - (2t)^5 5!$

$u_3(t,x) =_t^{-1} \left[1p^2{}_t\left(R(u_2)\right)\right]$

$\Rightarrow u_3(t,x) =_t^{-1} \left[1p^2\left[_t\left(2^6t^2 2! + 2^8t^3 3! + 10 \times 2^5t^4 4! + 2^7t^5 5!\right)\right]\right]$

$\Rightarrow u_3(t,x) =_t^{-1} \left(2^6 1p^5 + 2^8 1p^6 + 5 \times 2^6 1p^7 + 2^7 1p^8\right)$

$\Rightarrow u_3(t,x) = 4(2t)^4 4! + 8(2t)^5 5! + 5(2t)^6 6! + (2t)^7 7!$

$u_4(t,x) =_t^{-1} \left[1p^2{}_t\left(R(u_3)\right)\right]$

$\Rightarrow u_4(t,x) =_t^{-1} \left[1p^2\left[_t\left(-\frac{32}{315}t^7 - \frac{112}{45}t^6 - \frac{96}{5}t^5 - \frac{160}{3}t^4 - \frac{128}{3}t^3\right)\right]\right]$

$\Rightarrow u_4(t,x) =_t^{-1} \left[1p^2\left[_t\left(-2^8t^3 3! - 5 \times 2^8t^4 4! - 9 \times 2^8t^5 5! - 7 \times 2^8t^6 6! - 2^9t^7 7!\right)\right]\right]$

$\Rightarrow u_4(t,x) =_t^{-1} \left(-2^8 1p^6 - 5 \times 2^8 1p^7 - 9 \times 2^8 1p^8 - 7 \times 2^8 1p^9 - 2^9 1p^{10}\right)$

$\Rightarrow u_4(t,x) = -8(2t)^5 5! - 20(2t)^6 6! - 18(2t)^7 7! - 7(2t)^8 8! - (2t)^9 9!$

Step by step, we deduce

$$_{n\geq 0}u_n\,(t,x) = e^{2x} +_{n\geq 0} (-2t)^k n!$$

Then

$$u(t,x) =_{n\geq 0} u_n(t,x) = e^{2x} + e^{-2t}$$

The exact solution of model is

$$u(t,x) = e^{2x} + e^{-2t}$$

## 5 Conclusion

Laplace's Adomian numerical method allowed us to solve some linear partial differential equations by modelling the standard telegraph equation. It is therefore a very powerful numerical analysis tool to solve this type of problem, this method accelerates convergence to the solution . Our study was limited to the linear models of telegraph non-homogeneous reaction, a study of these models in non-homogeneous cases would be an important contribution to the understanding of these models.

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