Original Research Article

# A New Type of Bivariate-Gamma Probability Function

## Abstract

This paper develops a new bivariate-gamma distribution (BGD) and explores its distributional propeties such as marginal probability distributions, moments, product moment and covariance.
**Keywords:** Bivariate Γ-distribution, marginal distribution, covariance.

## 1 Introduction

The bivariate gamma distribution is a powerful probability function that can be suitably applied in epidemiology and medical fields [1], noise theory, modeling of rainfall at two nearby rain gauges, rain-making experiments of two areas with strong correlation coefficient, the dependence between annual stream-flow of rivers and a real precipitation [4], wind gust data [7], dependence between rainfall and runoff [5], reliability theory, strength (fracture, fatigue) of different kind of materials, renewal processes and stochastic routing problems [2], and in every skewed data [8]. If $(P_i, Q_i)|i = 1, 2, \cdots, n$ is a random sample from a bivariate normal distribution with 0 means, then the bivariate random variable $(X, Y)$, where $X = \frac{1}{n}\sum_{i=1}^{n} P_i^2$ and $Y = \frac{1}{n}\sum_{i=1}^{n} Q_i^2$, has bivariate gamma distribution. This can be advantageously proved using characteristic function, $\Phi(X, Y) = \iint e^{it(x+y)} f(x, y)dxdy$.

**Definition 1.1 (Bivariate gamma distribution)** *A continuous bivariate random variable $(P, Q)$ is said to have the bivariate gamma distribution if its joint probability density function is of the form*

$$f(p,q) = \begin{cases} \dfrac{(pq)^{\frac{1}{2}(\alpha-1)}}{(1-\theta)\Gamma(\alpha)(\theta)^{\dfrac{(\alpha-1)}{2}}} e^{-\dfrac{p+q}{1-\theta}} I_{\alpha-1}\left(\dfrac{2\sqrt{\theta\, pq}}{1-\theta}\right), & \textit{if } 0 \le p, q < \infty \\ 0, & \textit{otherwise.} \end{cases} \tag{1}$$

*where*

$$I_k(t) = \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}t\right)^{k+2r}}{r!\Gamma(k+r+1)} \tag{2}$$

*is the modified Bessel function of the first kind of order $k$ and $\theta \in [0, 1)$ and $\alpha > 0$ are parameters. Explicitly, $f(p, q)$ is given by*

$$f(p,q) = \begin{cases} \dfrac{1}{\theta^{\alpha-1}\Gamma(\alpha)} e^{-\dfrac{p+q}{1-\theta}} \sum_{k=0}^{\infty} \dfrac{(\theta\, pq)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}}; & \textit{for } 0 \le p, q < \infty \\ 0; & \textit{otherwise.} \end{cases} \tag{3}$$

The main difficulty in dealing with bivariate gamma is that for many problems, no unique bivariate gamma density function exists. The definition given above is due to Kibble as cited in [9]. In the same period, Cherian developed a bivariate gamma distribution whose probability density function is given by

$$f(p,q)\begin{cases}\frac{e^{-(p+q)}}{\prod_{i=1}^{3}\Gamma(\alpha_i)}\int_0^{\min\{p,q\}}\frac{r^{\alpha_3}(p-r)^{\alpha_1}(q-r)^{\alpha_2}}{z(p-z)(q-z)}e^r\,dr; & \text{if } 0<p,q<\infty\\ 0 & \text{elsewhere,}\end{cases} \tag{4}$$

where $\alpha_1,\ \alpha_2,\ \alpha_3 \in (0,\infty)$ are parameters. Mckay constructed alternative bivariate gamma distribution whose probability density function is of the form

$$f(p,q)\begin{cases}\frac{\theta^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)}p^{\alpha-1}(q-p)^{\beta-1}e^{-\theta q}; & \text{if } 0<p<q<0\\ 0; & \text{elsewhere}\end{cases} \tag{5}$$

where $\theta,\alpha,\beta\in(0,\infty)$ are constants. Equations (4) and (5) are all cited in [9]. [3] have studied a density function of the form,

$$f(x;\alpha,\beta,\delta,\alpha,\delta)=\frac{\beta\alpha^{\frac{m}{\beta}+1}}{\nu_X(0)}x^{\beta+m-1}e^{-\delta x^{\beta}}2F_1\left(\lambda,b;c-\frac{\alpha\,x^{\beta}}{n}\right)I_{\mathbb{R}}+(x), \tag{6}$$

where $\beta\geq 0$, $m+\beta>0$, $\delta$ and $\alpha$ are positive numbers, and $\nu_X(0)$ is as defined below. The $h^{th}$ moment of this distribution is

$$\mu'_X(h)=\frac{\nu_X(h)}{\nu_X(0)}, \tag{7}$$

where

$$\nu_X(h)=\alpha^{\frac{m}{\beta}+1}\Gamma(c)\left[\begin{array}{l}\frac{\delta^{-\frac{m+h-b\beta+\beta}{\beta}}\Gamma\left(\frac{m+h-b\beta+\beta}{\beta}\right)\Gamma(\lambda-b)\alpha^{-b}}{\Gamma(c-b)\Gamma(\lambda)}\times_2F_2\left(b,b-c+1;b-\frac{m+h}{\beta},b-\lambda+1;\frac{\delta}{\alpha}\right)+\\ \left(\frac{\Gamma\left(\frac{m+h+\beta}{\beta}\right)\Gamma\left(-\frac{m+h-b\beta+\beta}{\beta}\right)\Gamma\left(-\frac{m+h+\beta-\beta\lambda}{\beta}\right)(\alpha)^{-\frac{m+h+\beta}{\beta}}}{\Gamma\left(-\frac{m+h-c\beta+\beta}{\beta}\right)\Gamma(\lambda)\Gamma(b)}\times_2F_2(K)\right)+\\ \frac{\delta^{-\frac{m+h+\beta-\beta\lambda}{\beta}}\Gamma(b-\lambda)\Gamma\left(\frac{m+h+\beta-\beta\lambda}{\beta}\right)\alpha^{-\lambda}\Gamma(b)\Gamma(c-\lambda)}{\times}{}_2F_2\left(\lambda,-c+\lambda+1;-b+\lambda+1,\lambda-\frac{m+h}{\beta};\frac{\delta}{\alpha}\right)\end{array}\right] \tag{8}$$

such that $K=\frac{m+h}{\beta}+1,-c+\frac{m+h}{\beta}+2;-b+\frac{m+h}{\beta}+2,-\lambda+\frac{m+h}{\beta}+2;\frac{\delta}{\alpha}$ for $h=0,1,\cdots,$ whenever $h+m+\beta>0$. A multivariate case of bivariate gamma distribution was developed by [2], [4].

From (3), results show that the marginal distributions of $P$ and $Q$, $g(p)=\dfrac{1}{\Gamma(\alpha)}p^{\alpha-1};p>0$ and $g(q)=\dfrac{1}{\Gamma(\alpha)}q^{\alpha-1};q>0$ are univariate gamma with parameter $\alpha$ (and $\theta=1$). This study modifies and reparameterizes Kibble bivariate gamma density function with the aim of making $\theta\in\mathbb{R}$ (other than fixed value $\theta=1$) in the marginal pdf's and other accompanies probability properties.

# 2 Preliminaries

The study states the following lemmas without proof since they have been proved in many elementary Calculus textbooks.

**Lemma 2.1** *From Calculus,*

$$e^t=\sum_{i=0}^{\infty}\frac{t^i}{i!}. \tag{9}$$

*The infinite series on RHS of (9) converges $\forall\, t \in \mathbb{R}$. Differentiating both sides of (9) wrt $t$ and then multiplying the resulting expression by $t$ gives*

$$te^t = \sum_{i=0}^{\infty} i\frac{t^i}{i!} \tag{10}$$

*Differentiating (10) again and multiply the resulting expression by $t$ to get*

$$te^t + t^2e^t = \sum_{i=0}^{\infty} i^2\frac{t^i}{i!} \tag{11}$$

# 3 The Developed Bivariate Gamma Distribution (BGD)

The study begins by letting $p = \theta\, x$ and $q = \theta\, y$ with Jacobian of the transformation as

$$J = \begin{vmatrix} \dfrac{dp}{dx} & \dfrac{dp}{dy} \\ \dfrac{dq}{dx} & \dfrac{dq}{dy} \end{vmatrix} = \begin{vmatrix} \theta & 0 \\ 0 & \theta \end{vmatrix} = \theta^2. \tag{12}$$

Suppose there exists, a one to one transformation that maps set $A = \{(p, q) : 0 \leq\ p \leq \infty,\ \ 0 \leq q \leq \infty\}$ onto $B = \{(x, y) : 0 \leq\ x \leq \infty,\ \ 0 \leq\ y \leq \infty\}$. Hence, from (3), the joint pdf of $X$ and $Y$, say $f(x, y) = f\,(p = \theta\, x, q = \theta\, y)\,|\,J|$, is

$$f(x,y) = \begin{cases} \dfrac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\dfrac{\theta}{1-\theta}(x+y)} \sum\limits_{k=0}^{\infty} \dfrac{(\theta^3 xy)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}}; & \text{for } 0 \leq\ x, y < \infty \\ 0; & \text{otherwise.} \end{cases} \tag{13}$$

where $\theta \in (0, \infty)$ and $\alpha > 0$ are real constants (parameters).

**Theorem 3.1** *The developed BGD in (13) is a proper pdf.*

**Proof:** This is to show that $\iint\ f(x, y)dxdy = 1$. Following from (13),

$$\begin{aligned}
\int_0^{\infty}\int_0^{\infty} f(x,y)dxdy &= \int_0^{\infty}\int_0^{\infty} \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{\frac{-\theta}{1-\theta}(x+y)} \sum_{k=0}^{\infty} \frac{(\theta^3 xy)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} dydx \\
&= \int_0^{\infty} \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}x} \sum_{k=0}^{\infty} \frac{(\theta^3 x)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \int_0^{\infty} y^{\alpha+k-1}e^{-\frac{\theta}{1-\theta}y} dydx \\
&= \int_0^{\infty} \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}x} \sum_{k=0}^{\infty} \frac{(\theta^3 x)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \Gamma(\alpha+k)\left(\frac{1-\theta}{\theta}\right)^{\alpha+k} dx \\
&= \int_0^{\infty} \frac{1}{\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}x} \sum_{k=0}^{\infty} \frac{\theta^{\alpha+2k}}{k!(1-\theta)^k} x^{\alpha+k-1} dx \\
&= \int_0^{\infty} \frac{\theta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{\theta}{1-\theta}x} \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\theta^2 x}{1-\theta}\right)^k dx
\end{aligned}$$

Using (9), $\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\theta^2 x}{1-\theta}\right)^k = e^{\dfrac{\theta^2 x}{1-\theta}}$,

$$
\begin{aligned}
\int_0^\infty \int_0^\infty f(x,y)dxdy &= \int_0^\infty \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{\theta}{1-\theta}x} e^{\frac{\theta^2}{1-\theta}x} dx \\
&= \frac{\theta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-\theta\, x} dx \\
&= \frac{\theta^\alpha}{\Gamma(\alpha)} \left(\frac{\Gamma(\alpha)}{\theta^\alpha}\right) \\
&= 1.
\end{aligned}
$$

**Theorem 3.2** *The marginal distribution of $X$ is univariate gamma with parameter $\alpha$ and $\theta$. That is,*

$$
f_1(x) = \int f(x,y)dy = \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\theta\, x} \sim GAM(x;\alpha,\theta). \tag{14}
$$

**Proof:** This requires integrating (13) wrt $y$. That is,

$$
\begin{aligned}
f_1(x) = \int f(x,y)dy &= \int_0^\infty \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}(x+y)} \sum_{k=0}^{\infty} \frac{(\theta^3 xy)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} dy \\
&= \sum_{k=0}^{\infty} \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}x} \frac{(\theta^3 x)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \int_0^\infty y^{\alpha+k-1} e^{-\frac{\theta}{1-\theta}y} dy \\
&= \sum_{k=0}^{\infty} \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}x} \frac{(\theta^3 x)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \left(\frac{1-\theta}{\theta}\right)^{\alpha+k} \Gamma(\alpha+k) \\
&= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}x} \frac{\theta^{\alpha+2k}}{k!(1-\theta)^k} x^{\alpha+k-1} \\
&= \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{\theta}{1-\theta}x} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\theta^2\, x}{1-\theta}\right)^k \\
&= \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{\theta}{1-\theta}x} e^{\frac{\theta^2}{1-\theta}x} \\
&= \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\theta\, x} \sim GAM(x;\alpha,\theta)
\end{aligned}
$$

Similarly, as shown below, the marginal distribution of $Y$ is also univariate gamma.

$$
\begin{aligned}
f_2(y) = \int f(x,y)dx &= \sum_{k=0}^{\infty} \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}y} \frac{(\theta^3 y)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \int_0^\infty x^{\alpha+k-1} e^{-\frac{\theta}{1-\theta}x} dx \\
&= \sum_{k=0}^{\infty} \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}y} \frac{(\theta^3 y)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \left(\frac{1-\theta}{\theta}\right)^{\alpha+k} \Gamma(\alpha+k) \\
&= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}y} \frac{\theta^{\alpha+2k}}{k!(1-\theta)^k} y^{\alpha+k-1} \\
&= \frac{\theta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\frac{\theta}{1-\theta}y} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\theta^2 y}{1-\theta}\right)^k \\
&= \frac{\theta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\frac{\theta}{1-\theta}y} e^{\frac{\theta^2\, y}{1-\theta}} \\
&= \frac{\theta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\theta\, y} \sim GAM(y;\alpha,\theta)
\end{aligned}
$$

From the two marginals,

$$E(X) = \int_0^\infty x f_1(x) dx = \int_0^\infty x \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\theta\, x} dx = \frac{\theta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\theta^{\alpha+1}} = \frac{\alpha}{\theta} \tag{15}$$

$$E(Y) = \int_0^\infty y f_2(y) dy = \int_0^\infty y \frac{\theta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\theta\, y} dy = \frac{\theta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\theta^{\alpha+1}} = \frac{\alpha}{\theta} \tag{16}$$

and

$$Var(X) = \int_0^\infty x^2 \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\theta\, x} dx - \left[\int_0^\infty x \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\theta\, x} dx\right]^2 = \frac{\theta^\alpha \alpha(\alpha+1)}{\theta^{\alpha+2}} - \left[\frac{\alpha}{\theta}\right]^2 = \frac{\alpha}{\theta^2} \tag{17}$$

$$Var(Y) = \int_0^\infty y^2 \frac{\theta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\theta\, y} dy - \left[\int_0^\infty y \frac{\theta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\theta\, y} dy\right]^2 = \frac{\theta^\alpha \alpha(\alpha+1)}{\theta^{\alpha+2}} - \left[\frac{\alpha}{\theta}\right]^2 = \frac{\alpha}{\theta^2} \tag{18}$$

The joint expectation is

$$\begin{aligned}
E(XY) &= \iint xy f(x,y) dxdy = \int_0^\infty \int_0^\infty xy \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}(x+y)} \sum_{k=0}^\infty \frac{(\theta^3 xy)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} dxdy \\
&= \int_0^\infty y \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}y} \sum_{k=0}^\infty \frac{(\theta^3 y)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \int_0^\infty x^{\alpha+k} e^{-\frac{\theta}{1-\theta}x} dxdy \\
&= \int_0^\infty y \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}y} \sum_{k=0}^\infty \frac{(\theta^3 y)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \Gamma(\alpha+k+1) \left(\frac{1-\theta}{\theta}\right)^{\alpha+k+1} dy \\
&= \int_0^\infty \frac{1}{\Gamma(\alpha)} \sum_{k=0}^\infty \frac{\theta^{\alpha+2k-1}}{k!(1-\theta)^{k-1}} e^{-\frac{\theta}{1-\theta}y} y^{\alpha+k}(\alpha+k) dy \\
&= \frac{\alpha}{\Gamma(\alpha)} \int_0^\infty \sum_{k=0}^\infty \frac{\theta^{\alpha+2k-1}}{k!(1-\theta)^{k-1}} e^{-\frac{\theta}{1-\theta}y} y^{\alpha+k} dy + \frac{1}{\Gamma(\alpha)} \int_0^\infty \sum_{k=0}^\infty \frac{k\theta^{\alpha+2k-1}}{k!(1-\theta)^{k-1}} e^{-\frac{\theta}{1-\theta}y} y^{\alpha+k} dy \\
&= \frac{\alpha(1-\theta)\theta^{\alpha-1}}{\Gamma(\alpha)} \int_0^\infty y^\alpha\, e^{-\frac{\theta}{1-\theta}y} \sum_{k=0}^\infty \frac{1}{k!} \left(\frac{\theta^2 y}{1-\theta}\right)^k dy + \frac{(1-\theta)\theta^{\alpha-1}}{\Gamma(\alpha)} \int_0^\infty y^\alpha e^{\frac{-\theta}{1-\theta}y} \sum_{k=1}^\infty \frac{k}{k!} \left(\frac{\theta^2 y}{1-\theta}\right)^k dy
\end{aligned}$$

Using (9) and (10), $\sum_{k=0}^\infty \frac{1}{k!}\left(\frac{\theta^2 y}{1-\theta}\right)^k = e^{\frac{\theta^2}{1-\theta}y}$ and $\sum_{k=0}^\infty \frac{k}{k!}\left(\frac{\theta^2 y}{1-\theta}\right)^k = \frac{\theta^2 y}{1-\theta} e^{\frac{\theta^2}{1-\theta}y}$. So,

$$\begin{aligned}
E(XY) &= \frac{\alpha(1-\theta)\theta^{\alpha-1}}{\Gamma(\alpha)} \int_0^\infty y^\alpha e^{-\theta\, y} dy + \frac{\theta^{\alpha+1}}{\Gamma(\alpha)} \int_0^\infty y^{\alpha+1} e^{-\theta\, y} dy \\
&= \frac{\alpha(1-\theta)\theta^{\alpha-1}}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\theta^{\alpha+1}} + \frac{\theta^{\alpha+1}}{\Gamma(\alpha)} \frac{\Gamma(\alpha+2)}{\theta^{\alpha+2}} \\
&= \frac{\alpha^2(1-\theta)}{\theta^2} + \frac{\alpha(\alpha+1)}{\theta}
\end{aligned} \tag{19}$$

By definition, $Cov(X,Y) = E(XY) - E(X)E(Y)$. Hence,

$$Cov(X,Y) = \frac{\alpha^2(1-\theta)}{\theta^2} + \frac{\alpha(\alpha+1)}{\theta} - \left(\frac{\alpha}{\theta}\right)\left(\frac{\alpha}{\theta}\right) = \frac{\alpha}{\theta} \tag{20}$$

**Theorem 3.3** *Let the random variable* $(X,Y) \sim K(\alpha,\theta)$*, where* $0 < \alpha < \infty$ *and* $0 \le \theta < 1$*.*

*Then*

$$E(Y/x) = \frac{\alpha(1-\theta)}{\theta} + \theta\, x \tag{21}$$
$$E(X/y) = \frac{\alpha(1-\theta)}{\theta} + \theta\, y \tag{22}$$
$$Var(Y/x) = \frac{(1-\theta)}{\theta^2}\left[2\theta^2 x + \alpha(1-\theta)\right] \tag{23}$$
$$Var(X/y) = \frac{(1-\theta)}{\theta^2}\left[2\theta^2 y + \alpha(1-\theta)\right] \tag{24}$$

**Proof:** First, the conditional probability density function $Y$ given $X = x$, is given by

$$\begin{aligned} f(y/x) = \frac{f(x,y)}{f_1(x)} &= \frac{1}{\theta^{\alpha-3}\Gamma(\alpha)} e^{-\frac{\theta}{1-\theta}(x+y)} \sum_{k=0}^{\infty} \frac{(\theta^3 xy)^{\alpha+k-1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \times \frac{\Gamma(\alpha)}{\theta^{\alpha}\, x^{\alpha-1} e^{-\theta\, x}} \\ &= e^{-\frac{\theta}{1-\theta}x+\theta\, x} \sum_{k=0}^{\infty} \frac{\theta^{3\alpha+3k-3-\alpha+3-\alpha} x^{\alpha+k-1-\alpha+1}}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} y^{\alpha+k-1} e^{-\frac{\theta}{1-\theta}y} \\ &= e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{\theta^{\alpha+3k} x^k}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} y^{\alpha+k-1} e^{-\frac{\theta}{1-\theta}y} \end{aligned}$$

Then, the conditional mean is

$$\begin{aligned} E(Y|\,X=x) &= \int_0^{\infty} y f(y/x) dy = \int_0^{\infty} y e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{\theta^{\alpha+3k} x^k}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} y^{\alpha+k-1} e^{-\frac{\theta}{1-\theta}y} dy \\ &= e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{\theta^{\alpha+3k} x^k}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \int_0^{\infty} y^{\alpha+k} e^{-\frac{\theta}{1-\theta}y} dy \\ &= e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{\theta^{\alpha+3k} x^k}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \frac{\Gamma(\alpha+k+1)}{\left(\frac{\theta}{1-\theta}\right)^{\alpha+k+1}} \\ &= e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{\theta^{2k-1} x^k}{k!(1-\theta)^{k-1}} (\alpha+k) \\ &= \frac{1-\theta}{\theta} e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\theta^2 x}{1-\theta}\right)^k (\alpha+k) \\ &= \frac{1-\theta}{\theta} e^{-\frac{\theta^2}{1-\theta}x} \left[\alpha \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\theta^2 x}{1-\theta}\right)^k + \frac{\theta^2 x}{1-\theta} \sum_{k-1=0}^{\infty} \frac{1}{(k-1)!} \left(\frac{\theta^2 x}{1-\theta}\right)^{k-1}\right] \end{aligned}$$

From (9), $\sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\theta^2 x}{1-\theta}\right)^k = \sum_{k-1=0}^{\infty} \frac{1}{(k-1)!} \left(\frac{\theta^2 x}{1-\theta}\right)^{k-1} = e^{-\dfrac{\theta^2}{1-\theta}x}$. Therefore, the conditional mean, $E(Y|\,X=x)$, is finally presented as

$$E(Y|\,X=x) = \frac{1-\theta}{\theta} e^{-\frac{\theta^2}{1-\theta}x} \left[\alpha\, e^{\frac{\theta^2}{1-\theta}x} + \frac{\theta^2 x}{1-\theta} e^{\frac{\theta^2}{1-\theta}x}\right] = \frac{\alpha(1-\theta)}{\theta} + \theta\, x. \tag{25}$$

In the same manner,

$$
\begin{aligned}
E(Y^2|\, X=x) &= \int_0^\infty y^2 f(y/x)dy = \int_0^\infty y^2 e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{\theta^{\alpha+3k} x^k}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} y^{\alpha+k-1} e^{-\frac{\theta}{1-\theta}y} dy \\
&= e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{\theta^{\alpha+3k} x^k}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \int_0^\infty y^{\alpha+k+1} e^{-\frac{\theta}{1-\theta}y} dy \\
&= e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{\theta^{\alpha+3k} x^k}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \frac{\Gamma(\alpha+k+2)}{\left(\frac{\theta}{1-\theta}\right)^{\alpha+k+2}} \\
&= e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{\theta^{\alpha+3k} x^k}{k!\Gamma(\alpha+k)(1-\theta)^{\alpha+2k}} \frac{(\alpha+k)(\alpha+k+1)\Gamma(\alpha+k)(1-\theta)^{\alpha+k+1}}{\theta^{\alpha+k+2}} \\
&= e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{\theta^{2k-2} x^k}{k!(1-\theta)^{k-2}} (\alpha+k)(\alpha+k+1)
\end{aligned}
$$

which gives

$$
\begin{aligned}
E(Y^2|\, X=x) &= \frac{(1-\theta)^2}{\theta^2} e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\theta^2 x}{1-\theta}\right)^k (\alpha+k)[(\alpha+1)+k] \\
&= \frac{(1-\theta)^2}{\theta^2} e^{-\frac{\theta^2}{1-\theta}x} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\theta^2 x}{1-\theta}\right)^k [\alpha(\alpha+1)+(2\alpha+1)k+k^2]
\end{aligned} \tag{26}
$$

From (9), (10) and (11), $\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\theta^2 x}{1-\theta}\right)^k$, $\sum_{k=0}^{\infty}\frac{k}{k!}\left(\frac{\theta^2 x}{1-\theta}\right)^k$ and $\sum_{k=0}^{\infty}\frac{k^2}{k!}\left(\frac{\theta^2 x}{1-\theta}\right)^k$ are $e^{\dfrac{\theta^2 x}{1-\theta}}$, $\dfrac{\theta^2 x}{1-\theta}e^{\dfrac{\theta^2 x}{1-\theta}}$ and $\dfrac{\theta^2 x}{1-\theta}e^{\dfrac{\theta^2}{1-\theta}x} + \left(\dfrac{\theta^2 x}{1-\theta}\right)^2 e^{\dfrac{\theta^2 x}{1-\theta}}$, respectively. Hence, (26) gives

$$
\begin{aligned}
E(Y^2|\, X=x) &= \frac{(1-\theta)^2}{\theta^2} e^{-\frac{\theta^2}{1-\theta}x} \left[\alpha(\alpha+1)e^{\frac{\theta^2}{1-\theta}x} + (2\alpha+1)\frac{\theta^2 x}{1-\theta} e^{\frac{\theta^2}{1-\theta}x} + \left(\frac{\theta^2 x}{1-\theta}\right) e^{\frac{\theta^2}{1-\theta}x} + \left(\frac{\theta^2 x}{1-\theta}\right)^2 e^{\frac{\theta^2}{1-\theta}x}\right] \\
&= \frac{(1-\theta)^2}{\theta^2} \left[\alpha(\alpha+1) + (2\alpha+1)\frac{\theta^2 x}{1-\theta} + \frac{\theta^2 x}{1-\theta} + \left(\frac{\theta^2 x}{1-\theta}\right)^2\right] \\
&= \frac{\alpha(\alpha+1)(1-\theta)^2}{\theta^2} + 2(\alpha+1)(1-\theta)x + \theta^2 x^2
\end{aligned} \tag{27}
$$

The conditional variance of $Y$ given $X=x$ is

$$
\begin{aligned}
Var(Y|\, x) &= E(Y^2|\, X=x) - [E(Y|\, X=x)]^2 \\
&= \frac{\alpha(\alpha+1)(1-\theta)^2}{\theta^2} + 2(\alpha+1)(1-\theta)x + \theta^2 x^2 - \left[\frac{\alpha(1-\theta)}{\theta} + \theta\, x\right]^2 \\
&= \frac{\alpha(1-\theta)^2}{\theta^2} + 2(1-\theta)x \\
&= \frac{(1-\theta)}{\theta^2}\left[2\theta^2 x + \alpha(1-\theta)\right].
\end{aligned} \tag{28}
$$

The developed density function (13) is symmetric, that is, $f(x,y) = f(y,x)$. As a result, $E(X|\, y)$ and $Var(X|\, y)$ can be obtained by interchanging $x$ with $y$ in (25) and (28), respectively. So,

$$
E(X/y) = \frac{\alpha(1-\theta)}{\theta} + \theta\, y \tag{29}
$$

$$
Var(X/y) = \frac{(1-\theta)}{\theta^2}\left[2\theta^2 y + (1-\theta)\right] \tag{30}
$$

This completes the proof.

Theory establishes that the univariate exponential distribution is a special case of the univariate gamma distribution. Similarly, the bivariate exponential distribution is a special case of bivariate gamma distribution [6]. Taking the index parameter to be unity in the developed bivariate gamma density function presented in (13), we obtain its corresponding bivariate exponential distribution as

$$f(x,y) = \begin{cases} \theta^2 e^{-\frac{\theta}{1-\theta}(x+y)} \sum\limits_{k=0}^{\infty} \frac{(\theta^3 xy)^k}{k!\Gamma(k+1)(1-\theta)^{2k+1}}; & \text{for } 0 \leq x, y < \infty, \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

where $\theta \in (0, \infty)$ is a parameter. In addition, each of the properties of bivariate gamma distribution is same with bivariate exponential distribution with $\alpha = 1$. The equivalent bivariate exponential probability density function corresponding to bivariate gamma distribution of Kibble is

$$f(p,q) = \begin{cases} e^{-\left(\frac{p+q}{1-\theta}\right)} \sum\limits_{k=0}^{\infty} \frac{(\theta\, pq)^k}{k!\Gamma(k+1)(1-\theta)^{2k+1}}; & \text{if } 0 < p, q < \infty \\ 0; & \text{otherwise} \end{cases} \tag{32}$$

where $\theta \in (0, 1)$ is a parameter. The bivariate exponential distribution corresponding to the Cherian bivariate distribution is

$$f(p,q) = \begin{cases} \left[e^{\min\{p,q\}} - 1\right] e^{-(p+q)}; & \text{if } 0 < p, q < \infty \\ 0; & \text{elsewhere} \end{cases} \tag{33}$$

Gumble, cited in [9], constructed a bivariate exponential density function of the form

$$f(p,q) = \begin{cases} [(1+\theta\, p)(1+\theta\, q) - \theta]\, e^{-(p+q+\theta\, pq)}; & \text{if } 0 < p, q < \infty \\ 0; & \text{otherwise} \end{cases} \tag{34}$$

and $\theta > 0$ is a parameter. Marshall and Olkin, as cited in [9], introduced alternative bivariate exponential distribution as

$$f(p,q) = \begin{cases} 1 - e^{(\alpha+\gamma)p} - e^{-(\beta+\gamma)q} + e^{-(\alpha\, p+\beta\, q+\gamma \max\{p,q\})}; & \text{if } p, q > 0 \\ 0; & \text{otherwise} \end{cases} \tag{35}$$

where $\alpha, \beta, \gamma > 0$ are unknown but fixed parameters. (35) satisfies the memoryless property.

# Conclusion

This study developed a new probability function, namely, modified bivariate gamma probability distribution function. The main properties (the marginal densities, moments, product moment and covariance) of the developed probability function are derived. The re-parameterized distribution has the advantage of robustness, simple implementation and opportunity of varying values of $\theta$ for mixed outcome data.

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