Review Article

Asymptotic Theorems for Discrete Markov Chains

Abstract

Let X_n be a discrete time Markov chain with state space S (countably infinite, in general) and initial probability distribution $\mu^{(0)} = (P(X_0 = i_1), P(X_0 = i_2), \cdots,)$. Can we estimate the probabilities $P(X_n = j|X_0 = i)$ and $P(X_n = j)$ for large n? We will discuss this question and give some answers even if there exists periodic states. What is the probability of choosing in random some $k \in \mathbb{N}$ with $k \leq n$ such that $X_k = j$ where $j \in S$? This probability is the average $\frac{1}{n}\sum_{k=1}^{n} \mu_j^{(k)}$ where $\mu_j^{(k)} = P(X_k = j)$. In this note we will study the limit of this average without assuming that the chain is irreducible, using elementary mathematical tools. We will also relate the limiting probabilities with the ergodic type of limits and prove that the computation of the limiting probabilities are a stronger result than that of the ergodic theorem. Finally, we study the limit of the average $\frac{1}{n}\sum_{k=1}^{n} g(X_k)$ where g is a given function for a Markov chain not necessarily irreducible. Finally, we will mention some open problems regarding these limiting probabilities.

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1 Introduction

Let X_n be a discrete time Markov chain with state space S (countably infinite, in general) and initial probability distribution $\mu^{(0)}$, that is $\mu_i^{(0)} = P(X_0 = i)$ where $i \in S$ and let P be the transition matrix of this chain.

It is well known that $P(X_n = j|X_0 = i) = (P^n)_{ij}$ and $P(X_n = j) = \sum_{i \in S} \mu_i^{(0)} (P^n)_{ij}$. In the case where S is finite it is easy to compute the matrix P^n for the discrete time case. In [8] and [9] we have discussed this problem in the case where the transision matrix is finite. We have seen that we can indeed compute the nth power of the matrix, even if it is not diagonizable. In [9] we gave the matlab code for this computation using the minimum polynomial of the transition matrix. We gave also a feasible method to compute the minimum polynomial, which is very useful in the case where the transition matrix is big. However, if the transition matrix is big and not sparse, it is not possible to compute the nth power, even if we know the roots of the minimum or the characteristic polynomial.

If S is infinite the situation is much different and we can not compute the probabilities $(P^n)_{ij}$ by the above method.

In the case where the chain is aperiodic we have that the limiting probabilities are such that

$$\lim_{n\to\infty}(P^n)_{ij} = \begin{cases} \frac{f_{ij}}{m_j}, & \text{when } j \text{ is positive recurrent} \\ 0, & \text{otherwise} \end{cases}$$

where m_j is the mean recurrent time of the state j and $f_{ij} = P(\exists\, n \in \mathbb{N} : X_n = j | X_0 = i)$. More compactly we can write

$$\lim_{n\to\infty}(P^n)_{ij} = \frac{f_{ij}}{m_j}$$

where $m_j = \infty$ when j is not positive recurrent. Therefore for big enough n we can say that $(P^n)_{ij} \simeq \lim_{n\to\infty}(P^n)_{ij}$, that is the limiting probabilities are useful for the estimation of $(P^n)_{ij}$ when we are not able to compute them exactly.

For the periodic case we have the following representation of the limiting probabilities. Denoting by $f_n(i|j)$ the probability

$$f_n(i|j) = \mathbb{P}(X_n = j, X_k \neq j, k = 1, \cdots, n-1 | X_0 = i)$$

we have (see [1]) that

$$\lim_{n\to\infty}(P^{nd(j)+a})_{ij} = \frac{d(j)}{m_j}\sum_{k=0}^{\infty} f_{kd(j)+a}(i|j), \quad a = 0, \cdots, d(j)-1 \tag{1}$$

where $d(j)$ is the period of the state j and m_j is the mean recurrent time of j at the chain X_n. Note that when the period $d(j) = 1$ then the above coincides with the aperiodic case because $\sum_{k=0}^{\infty} f_{kd(j)+a}(i|j) = f_{ij}$ in this situation. The probability $\sum_{k=0}^{\infty} f_{kd(j)+a}(i|j)$ (for $d \geq 2$) is sometimes difficult to compute therefore the above representation of the limiting probabilities can not be used in practice in this case. Below we are going to give another representation of the limiting probabilities which is sometimes easier to compute.

Let us recall the dominated convergence theorem for sequences of numbers.

Theorem 1 *Let a_{nk} with $n, k \in \mathbb{N}$ real numbers and b_k no negative numbers such that $\sum_{k=1}^{\infty} b_k < \infty$ and $|a_{nk}| \leq b_k$. If $\lim_{n\to\infty} a_{nk} = a_k$ then*

$$\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk} = \sum_{k=1}^{\infty} a_k$$

Using the above result we can easily prove the following theorem. Denote by Y_n the chain with transition matrix $Q := P^d$ where $d = lcm\{d_1, d_2, \cdots\} < \infty$. By m_j^Q we denote the mean recurrent time of the state j at the chain Y_n and f_{ij}^Q is the probability the chain Y_n to visit sometime the state j starting from i.

Theorem 2 *Let X_n be a discrete time Markov chain with state space S and transition matrix P. Suppose that $d = lcm\{d_1, d_2, \cdots\} < \infty$ where $d_1, d_2, \cdots$ are the periods of the recurrent states. Then*

$$\lim_{n\to\infty} P^{dn+a} = P^a Q^\infty, \quad a = 0, \cdots, d-1 \tag{2}$$

where $Q_{ij}^\infty = \frac{f_{ij}^Q}{m_j^Q}$.

Proof. Consider the chain Y_n with transition matrix $Q = P^d$. It is easy to see that all the states of this chain are aperiodic. Therefore the following holds

$$Q^\infty := \lim_{n\to\infty} (Q^n)_{ij} = \begin{cases} \frac{f_{ij}^Q}{m_j^Q}, & \text{when } j \text{ is positive recurrent} \\ 0, & \text{otherwise} \end{cases}$$

where m_j^Q is the mean recurrent time of the state j at the chain Y_n and f_{ij}^Q is the probability the chain Y_n to visit sometime the state j starting from i. For $a = 0, \cdots, d-1$, using the dominated convergence theorem, we obtain,

$$\begin{aligned} \lim_{n\to\infty} (P^{dn+a})_{ij} &= \lim_{n\to\infty} (P^a \cdot Q^n)_{ij} \\ &= \lim_{n\to\infty} \sum_{k\in S} (P^a)_{ik} (Q^n)_{kj} \\ &= (P^a \cdot Q^\infty)_{ij} \end{aligned}$$

since $\sum_{k\in S} (P^a)_{ik} (Q^n)_{kj} \le \sum_{k\in S} (P^a)_{ik} = 1$. □

In a similar fashion we have the following theorem concerning the limiting probabilities $\lim_{n\to\infty} (P^n)_{ij}$ for specific i, j.

Theorem 3 *Let X_n be a discrete time Markov chain with state space S and transition matrix P. Then*

$$\lim_{n\to\infty} \left(P^{d(j)n+a}\right)_{ij} = (P^a \cdot Q^\infty)_{ij}, \quad a = 0, \cdots, d(j) - 1 \tag{3}$$

where $d(j)$ is the period of the state j if it is recurrent, m_j is the mean recurrent time of the state j at the chain X_n and

$$Q^\infty := \lim_{n\to\infty} \left(P^{d(j)n}\right)_{ij} = \frac{d(j) f_{ij}^Q}{m_j}$$

Proof. Here we construct the chain Y_n with transition matrix $Q = P^{d(j)}$ where $d(j)$ is the period of the state j. The state j is aperiodic in this chain so

$$(Q^\infty)_{ij} := \lim_{n\to\infty} (P^{d(j)n})_{ij} = \begin{cases} \frac{f_{ij}^Q}{m_j^Q}, & \text{when } j \text{ is positive recurrent} \\ 0, & \text{otherwise} \end{cases}$$

where m_j^Q is the mean recurrent time of the state j at the chain Y_n and f_{ij}^Q as before. Note that it is easy to see that $m_j = d(j) m_j^Q$ where m_j is the mean recurrent time of the state j at the chain X_n. Therefore it holds that

$$(Q^\infty)_{ij} := \lim_{n\to\infty} (P^{d(j)n})_{ij} = \begin{cases} \frac{d(j) f_{ij}^Q}{m_j}, & \text{when } j \text{ is positive recurrent} \\ 0, & \text{otherwise} \end{cases}$$

Using again the dominated convergence theorem we get the desired result. □

Remark 1 *The above result is very useful in the case where i is transient and j is recurrent. If both i, j are recurrent then we can have immediately the limiting probabilities. For example, if the i, j belong to different recurrent classes then $(P^n)_{ij} = 0$ for all $n \in \mathbb{N}$. If the i, j belong to the same recurrent class then $(P^{nd(j)+a})_{ij} \to \frac{d(j)}{m_j}$ when i belong to the cyclically moving subclass C_r and j belong to the cyclically moving subclass C_{r+a}. Moreover, $(P^{nd(j)+a})_{ij} = 0$ for all $n \in \mathbb{N}$ when $i \in C_r$ and $j \in C_{r+b}$ with $b \neq a$.* □

Concerning the computation of the limiting probabilities, the following result seems to be new.

Corollary 1 *At the above setting it holds that*

$$\lim_{n\to\infty} \left(P^{nd(j)+a}\right)_{ij} = \left(P^a \cdot \lim_{n\to\infty} P^{d(j)n}\right)_{ij} = \frac{d(j)}{m_j} \sum_{k=0}^{\infty} f_{kd(j)+a}(i|j), \quad a = 0, \cdots, d(j)-1$$

and therefore we obtain the equalities

$$\sum_{k=0}^{\infty} f_{kd(j)+a}(i|j) = \sum_{k\in S} (P^a)_{ik} f^Q_{kj}, \quad a = 0, \cdots, d(j)-1 \tag{4}$$

$$f_{ij} = \sum_{k\in S} W_{ik} f^Q_{kj} \tag{5}$$

where $W_{ik} = \sum_{a=0}^{d(j)-1} (P^a)_{ik}$.

Proof. We will prove only the last equality. Since

$$\begin{aligned}
\sum_{a=0}^{d(j)-1} \sum_{k=0}^{\infty} f_{kd(j)+a}(i|j) &= \sum_{k=0}^{\infty} \sum_{a=0}^{d(j)-1} f_{kd(j)+a}(i|j) \text{ (absolutely convergence series)} \\
&= \sum_{n=1}^{\infty} f_n(i|j) \text{ (setting } f_0(i|j) = 0) \\
&= f_{ij}
\end{aligned}$$

we have that

$$f_{ij} = \sum_{a=0}^{d(j)-1} \sum_{k\in S} (P^a)_{ik} f^Q_{kj} = \sum_{k\in S} W_{ik} f^Q_{kj}$$

□

The above results concerning the limiting probabilities $\lim_{n\to\infty}(P^n)_{ij}$ when i is transient and j is recurrent and periodic, seems to be new. In many cases is much easier to compute these limiting probabilities with the above method than the suggested method in [4], for example.

Remark 2 *Using the system (or the difference equation)*

$$f_{ij} = \sum_{a=0}^{d(j)-1} \sum_{k\in S} (P^a)_{ik} f^Q_{kj} = \sum_{k\in S} W_{ik} f^Q_{kj}, \quad i \in S$$

one can compute the probabilities f_{ij}^Q knowing the f_{ij} without computing the matrix $Q = P^d$. However we should compute first the matrix W but if the period of j equals 2 this remark may be practically useful. $\square$

By means of Remark 2, there is an open question regarding the probabilities f_{ij}^Q. Can we compute these probabilities given the probabilities f_{ij}? In a finite Markov chain, one can find the inverse of the matrix W in order to compute the probabilities f_{ij}^Q given the probabilities f_{ij}. In the infinite case however, the system in Remark 2 is a difference equation, therefore the above method does not give us a result. How can we solve, in general, this difference equation in order to compute the desired probabilities? Denoting by $G^Q(x)$ the probability generating function of the sequence f_{nj}^Q and by $G(x)$ the probability generating function of the sequence f_{nj} can somehow relate these probability generating functions using the relation

$$f_{ij} = \sum_{k\in S} W_{ik} f_{kj}^Q$$

or, even better, solve for $G^Q(x)$ in terms of $G(x)$? If this is the case, how can we compute the probability generating function $G(x)$?

We will study now the limit of the average

$$\frac{1}{n}\sum_{k=1}^{n} \mu_j^{(k)}$$

This quantity gives the probability of choosing in random an integer k with $k \leq n$ such that $X_k = j$. Note that, for any $i, j \in S$, we have

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} \mu_j^{(k)} &= \frac{1}{n}\sum_{k=1}^{n} (\mu^{(0)} \cdot P^k)_j \\
&= \frac{1}{n}\sum_{k=1}^{n}\sum_{i\in S} \mu_i^{(0)} P_{ij}^k \\
&= \sum_{i\in S} \mu_i^{(0)} \frac{1}{n}\sum_{k=1}^{n} P_{ij}^k \qquad (6)
\end{aligned}$$

Therefore, one can study the desired limit by studying the limit of the average $\frac{1}{n}\sum_{k=1}^{n} P_{ij}^k$. To do so one can use the limit theorems for P_{ij}^n (see for example [2]) and the well known fact that if $a_n \to a$ then $\frac{1}{n}\sum_{k=1}^{n} a_k \to a$. However, here we will give a different proof without using the limit theorems and without assuming that the chain is irreducible. Moreover, we will study the behavior of the limit $\frac{1}{n}\sum_{k=1}^{n} g(X_k)$ for a given function g, using elementary mathematical tools. For more on this topic one can see [3], [5], [6], [7], [10], [13], [14], [15] and [16].

2 Ergodic Theorems

Let X_n be a Markov chain with (countably infinite in general) state space S. We will prove the following well known result using elementary mathematical tools.

Theorem 4 *It holds that, for any $i, j \in S$,*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mu_j^{(k)} = \begin{cases} \dfrac{1}{m_j}\displaystyle\sum_{i\in S} \mu_i^{(0)} f_{ij}, & \text{when } j \text{ is positive recurrent} \\ 0, & \text{otherwise} \end{cases}$$

and

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} P_{ij}^k}{n} = \begin{cases} \frac{f_{ij}}{m_j}, & \text{when } j \text{ is positive recurrent} \\ 0, & \text{otherwise} \end{cases}$$

where $f_{ij} = P(\exists\, n \in \mathbb{N} : X_n = j | X_0 = i)$.

Proof. We know (see [2]) that when j is transient or null recurrent $\lim_{n\to\infty} P_{ij}^n = 0$. Therefore $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P_{ij}^n = 0$ and using (6) the result follows. Next we suppose that j is positive recurrent.

We are going now to prove the first assertion of the theorem, dividing the proof into three steps.

Step 1 At this step we will see that crucial role play the quantity $\mathbb{E}\left(\frac{M_j(n)}{n}\right)$.

Let the random variables

$$N_j^k = \begin{cases} 1, & \text{when } X_k = j \\ 0, & \text{otherwise} \end{cases}$$

and $M_j(n) = \sum_{k=1}^{n} N_j^k$. Because

$$\mathbb{E}\left(\frac{M_j(n)}{n}\right) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}N_j^k = \frac{1}{n}\sum_{k=1}^{n} P(X_k = j) = \frac{1}{n}\sum_{k=1}^{n} \mu_j^{(k)} \tag{7}$$

we will study the quantity $\mathbb{E}\left(\frac{M_j(n)}{n}\right)$.

Step 2 At this step we will prove the following assertion

$$P\left(\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = \frac{1}{m_j}\}\middle| A_i\right) = 1$$

Let the event $A_i = \{\exists\, n \in \mathbb{N} : X_n = j\} \cap \{X_0 = i\}$ where $i \in S$. Because $P(A_i) = P(\exists\, n \in \mathbb{N} : X_n = j | X_0 = i) \cdot \mu_i^{(0)}$ we see that $P(A_i) = f_{ij} \cdot \mu_i^{(0)}$ where $f_{ij} = P(\exists\, n \in \mathbb{N} : X_n = j | X_0 = i)$.

We will work under the probability measure $P_{A_i}(\cdot) = P(\cdot | A_i)$ while the corresponding expected value will be denoted by $\mathbb{E}_{A_i}$.

We define the following sequence of random variables,

$$n_1(\omega) = \begin{cases} \min\{n \in \mathbb{N} : X_n(\omega) = j\}, & \text{when } \omega \in A_i \\ \infty, & \text{otherwise} \end{cases}$$

$$n_2(\omega) = \begin{cases} \min\{n > n_1 : X_n(\omega) = j\}, & \text{when } \omega \in A_i \\ \infty, & \text{otherwise} \end{cases}$$

$$\vdots$$

$$n_k(\omega) = \begin{cases} \min\{n > n_{k-1} : X_n(\omega) = j\}, & \text{when } \omega \in A_i \\ \infty, & \text{otherwise} \end{cases}$$

We define also

$$Z_m = \begin{cases} n_{m+1} - n_m, & \text{when } \omega \in A_i \\ 0, & \text{otherwise} \end{cases}$$

for $m \geq 1$ which gives us the number of transitions needed to return back to j. Note that the sequence $Z_1, Z_2, \cdots$, is an independent and identically distributed sequence of random variables. The mean recurrent time m_j is such that $m_j = \mathbb{E}_{A_i}(Z_k)$ for every $k \geq 1$. Next we define the random variable $S_l = Z_1 + \cdots + Z_l$ with $S_0 = 0$. Note that

$$S_l + n_1 = n_{l+1} \quad \text{for every } l \geq 0 \tag{8}$$

Using the strong law of large numbers we have that

$$P_{A_i}\left(\{\omega \in \Omega : \lim_{n\to\infty} \frac{S_n}{n} = m_j\}\right) = 1 \tag{9}$$

Note that $M_j(n) \to \infty$ as $n \to \infty$ for almost all $\omega \in \Omega$ when j is recurrent and its easy to see that $n_{M_j(k)} \leq k$ for every $k \geq 1$.

Using (8) we see that the following inequality hold

$$S_{M_j(n)-1} + n_1 \leq n \leq S_{M_j(n)} + n_1, \quad n \geq 1, \quad \text{for every } \omega \in A_i$$

Dividing the previous inequality by $M_j(n) > 0$ for $n > n_1$ we get, noting that $M_j(n) > 1$ for $n \geq n_2$,

$$\frac{S_{M_j(n)-1} + n_1}{M_j(n) - 1} \frac{M_j(n) - 1}{M_j(n)} \leq \frac{n}{M_j(n)} \leq \frac{S_{M_j(n)}}{M_j(n)}, \quad n \geq n_2, \quad \text{for every } \omega \in A_i$$

Using (9) we have that $\frac{S_{M_j(n)-1}}{M_j(n)-1} \to m_j$, $\frac{n_1}{M_j(n)-1} \to 0$ and $\frac{S_{M_j(n)}}{M_j(n)} \to m_j$ with probability 1, therefore we deduce that

$$P_{A_i}\left(\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = \frac{1}{m_j}\}\right) = 1 \tag{10}$$

Step 3 Next we will study the limit of the quantity

$$\frac{\mathbb{E}_{A_i}(M_j(n))}{n}$$

Using the dominated convergence theorem it follows that

$$\begin{aligned}
\lim_{n\to\infty}\frac{\mathbb{E}_{A_i}(M_j(n))}{n} &= \mathbb{E}_{A_i}\left(\lim_{n\to\infty}\frac{M_j(n)}{n}\right)\\
&= \mathbb{E}_{A_i}\left(\frac{1}{m_j}\right)\\
&= \frac{1}{m_j}
\end{aligned}$$

But, since

$$\mathbb{E}_{A_i}\left(\frac{M_j(n)}{n}\right) = \frac{\mathbb{E}\left(\frac{M_j(n)}{n}\mathbb{I}_{A_i}\right)}{P(A_i)}$$

it follows that

$$\lim_{n\to\infty}\mathbb{E}\left(\frac{M_j(n)}{n}\mathbb{I}_{A_i}\right) = \frac{P(A_i)}{m_j} = \mu_i^{(0)}\frac{f_{ij}}{m_j} \tag{11}$$

Because

$$\mathbb{E}\left(\frac{M_j(n)}{n}\right) = \sum_{i\in S}\mathbb{E}\left(\frac{M_j(n)}{n}\mathbb{I}_{A_i}\right)$$

we obtain, using (11)

$$\begin{aligned}
\lim_{n\to\infty}\mathbb{E}\left(\frac{M_j(n)}{n}\right) &= \lim_{n\to\infty}\sum_{i\in S}\mathbb{E}\left(\frac{M_j(n)}{n}\mathbb{I}_{A_i}\right)\\
&= \sum_{i\in S}\lim_{n\to\infty}\mathbb{E}\left(\frac{M_j(n)}{n}\mathbb{I}_{A_i}\right)\\
&= \sum_{i\in S}\mu_i^{(0)}\frac{f_{ij}}{m_j}\\
&= \frac{1}{m_j}\sum_{i\in S}\mu_i^{(0)}f_{ij}
\end{aligned}$$

where we have used the dominated convergence theorem to get the second equality above. Therefore, we have proved the first assertion of the theorem.

Next, we are going to prove the second assertion of the theorem. If $m_{ij}(n) = \mathbb{E}(M_j(n)|X_0 = i)$ then we have

$$\begin{aligned}
m_{ij}(n) &= \mathbb{E}\left(M_j(n)|X_0 = i\right)\\
&= \mathbb{E}\left(\sum_{k=1}^{n}N_j^k|X_0 = i\right)\\
&= \sum_{k=1}^{n}\mathbb{E}(N_j^k|X_0 = i)\\
&= \sum_{k=1}^{n}P_{ij}^k
\end{aligned}$$

Denoting by $A = \{\exists\, k \in \mathbb{N} : X_k = j\}$, we have

$$\begin{aligned}
\mathbb{E}\left(\frac{M_j(n)}{n}|X_0 = i\right) &= \mathbb{E}\left(\frac{M_j(n)}{n}\mathbb{I}_A|X_0 = i\right) + \mathbb{E}\left(\frac{M_j(n)}{n}\mathbb{I}_{A^c}|X_0 = i\right) \\
&= \mathbb{E}\left(\frac{M_j(n)}{n}\mathbb{I}_A|X_0 = i\right) \\
&= \frac{\mathbb{E}\left(\frac{M_j(n)}{n}\mathbb{I}_{A_i}\right)}{\mu_i^{(0)}}
\end{aligned}$$

because $M_j(n)\mathbb{I}_{A^c} = 0$. That means that

$$\lim_{n\to\infty} \frac{m_{ij}(n)}{n} = \lim_{n\to\infty} \mathbb{E}\left(\frac{M_j(n)}{n}|X_0 = i\right) = \frac{f_{ij}}{m_j}$$

Therefore

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} P_{ij}^k}{n} = \begin{cases} \frac{f_{ij}}{m_j}, & \text{when } j \text{ is positive recurrent} \\ 0, & \text{otherwise} \end{cases}$$

The second assertion of the theorem has been proved also. □

Note that in the case where the chain is aperiodic it holds that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} P_{ij}^k}{n} = \lim_{n\to\infty} (P^n)_{ij} =: (P^\infty)_{ij}$$

We can relate the limiting probabilities with the above ergodic type limits.

Theorem 5 *Let X_n a discrete time Markov chain with transition matrix P. Then it holds that*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} P_{ij}^k}{n} = \frac{1}{d}\sum_{a=0}^{d-1}(P^a \cdot Q^\infty)_{ij} = \frac{1}{d}\sum_{a=0}^{d-1}\lim_{n\to\infty}(P^{nd+a})_{ij} \tag{12}$$

where $Q^\infty := \lim_{n\to\infty} P^{nd}$ and d is the period of the state j.

Proof. Indeed, by theorem 4, we know that the limit $\lim_{n\to\infty} \frac{\sum_{k=1}^{\infty} P_{ij}^k}{n}$ exists and is finite. Therefore it holds that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{dn} P_{ij}^k}{dn} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} P_{ij}^k}{n}$$

and rearranging the terms we have that

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sum_{k=1}^{dn} P_{ij}^k}{dn} &= \lim_{n\to\infty}\left(\frac{P_{ij} + \cdots + P_{ij}^{d-1}}{dn} - \frac{P_{ij}^{dn+1} + \cdots + P_{ij}^{dn+d-1}}{dn}\right) \\
&\quad + \lim_{n\to\infty}\left(\frac{1}{d}\frac{\sum_{k=1}^{n} P_{ij}^{dk}}{n} + \frac{1}{d}\frac{\sum_{k=1}^{n} P_{ij}^{dk+1}}{n} + \cdots + \frac{1}{d}\frac{\sum_{k=1}^{n} P_{ij}^{dk+d-1}}{n}\right) \\
&= \lim_{n\to\infty}\left(\frac{1}{d}\frac{\sum_{k=1}^{n} P_{ij}^{dk}}{n} + \frac{1}{d}\frac{\sum_{k=1}^{n} P_{ij}^{dk+1}}{n} + \cdots + \frac{1}{d}\frac{\sum_{k=1}^{n} P_{ij}^{dk+d-1}}{n}\right)
\end{aligned}$$

But it holds that

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{d}\frac{\sum_{k=1}^{n}P_{ij}^{dk+a}}{n} &= \lim_{n\to\infty}\frac{1}{d}\frac{\sum_{k=1}^{n}(P^a\cdot P^{dk})_{ij}}{n}\\
&= \lim_{n\to\infty}\frac{1}{d}\frac{\sum_{k=1}^{n}\sum_{l\in S}P_{il}^{a}P_{lj}^{dk}}{n}\\
&= \lim_{n\to\infty}\frac{1}{d}\frac{\sum_{l\in S}P_{il}^{a}\sum_{k=1}^{n}P_{lj}^{dk}}{n}\\
&= \frac{1}{d}\sum_{l\in S}P_{il}^{a}\lim_{n\to\infty}\frac{\sum_{k=1}^{n}P_{lj}^{dk}}{n}\\
&= \frac{1}{d}\sum_{l\in S}P_{il}^{a}Q_{lj}^{\infty}\\
&= \frac{1}{d}(P^a\cdot Q^{\infty})_{ij}
\end{aligned}$$

where we have used again the dominated convergence theorem. □

Therefore the result of theorem 3 is stronger than that of theorem 4 because the result of the ergodic theorem is just the average of the limits of the subsequences of $(P^n)_{ij}$.

Example 1 *Let X_n a discrete time Markov chain with transition matrix*

$$P=\begin{bmatrix} 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 1/3 & 2/3 & 0 & 0 & 0\\ 1/10 & 1/5 & 3/10 & 1/5 & 1/5\\ 3/10 & 1/5 & 0 & 1/5 & 3/10 \end{bmatrix}$$

and state space $S=\{1,2,3,4,5\}$. We see that the states $1,2,3$ are periodic with period 2 while the states $4,5$ are transient. We want to estimate the probabilities $\mathbb{P}(X_n=2|X_0=4)$, $\mathbb{P}(X_n=2|X_0=1)$ and $\mathbb{P}(X_n=3|X_0=1)$ for large n. We will work at the chain Y_n with transition matrix

$$Q=P^2=\begin{bmatrix} 1/3 & 2/3 & 0 & 0 & 0\\ 1/3 & 2/3 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ \frac{9}{50} & \frac{7}{25} & \frac{9}{25} & \frac{2}{25} & 1/10\\ \frac{11}{100} & 1/10 & \frac{14}{25} & 1/10 & \frac{13}{100} \end{bmatrix}$$

and the same state space. At this chain, every state is aperiodic therefore we have

$$\lim_{n\to\infty}(Q^n)_{ij}=\frac{f_{ij}^{Q}}{m_j^{Q}}$$

It is easy to see that $m_1^Q = 3$, $m_2^Q = 3/2$ *and* $m_3^Q = 1$ *and finally*

$$Q^\infty = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{27}{152} & \frac{27}{76} & \frac{71}{152} & 0 & 0 \\ \frac{23}{228} & \frac{23}{114} & \frac{53}{76} & 0 & 0 \end{bmatrix}$$

That means that

$$\lim_{n\to\infty} P^{2n} \quad = \quad Q^\infty = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{27}{152} & \frac{27}{76} & \frac{71}{152} & 0 & 0 \\ \frac{23}{228} & \frac{23}{114} & \frac{53}{76} & 0 & 0 \end{bmatrix}$$

$$\lim_{n\to\infty} P^{2n+1} \quad = \quad PQ^\infty = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 & 0 \\ \frac{71}{456} & \frac{71}{228} & \frac{81}{152} & 0 & 0 \\ \frac{53}{228} & \frac{53}{114} & \frac{23}{76} & 0 & 0 \end{bmatrix}$$

Therefore, an estimate of $\mathbb{P}(X_n = 2|X_0 = 4)$ *for large* n *depends on the actual* n*. If* $n = 2k$ *then we can say that this probability is almost* $27/76$ *while if* $n = 2k+1$ *then this probability is almost* $71/228$*. Using the ergodic theorem we will get* $\frac{27/76+71/228}{2} = 1/3$ *as a result, that is the average of the limits of the two subsequences. For the probability* $\mathbb{P}(X_n = 2|X_0 = 1)$ *for large* n *we see that is equal to* $2/3$ *for* $n = 2k$ *and* 0 *for* $n = 2k+1$ *while the ergodic theorem gives* $1/3$ *as a result. For the probability* $\mathbb{P}(X_n = 3|X_0 = 1)$ *for large* n *we have that is equal to* 0 *for* $n = 2k$ *and* 1 *for* $n = 2k+1$ *while the ergodic theorem gives* $1/2$ *as a result.* □

Next, we will give some well known results using elementary mathematical tools.

Proposition 1 *It holds that, when* j *is positive recurrent,*

$$\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = \frac{1}{m_j}\} \cup \{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = 0\} = \Omega \setminus E$$

with $P(E) = 0$*. More precisely, it holds that*

$$P\left(\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = \frac{1}{m_j}\}\right) = \sum_{i\in S} \mu_i^{(0)} \cdot f_{ij}$$

and

$$P\left(\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = 0\}\right) = \sum_{i\in S} \mu_i^{(0)} \cdot (1 - f_{ij})$$

If j is null recurrent or transient, then

$$P\left(\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = 0\}\right) = 1$$

Proof.

· Assume that j is positive recurrent. Denoting by $B = \{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = \frac{1}{m_j}\}$ we can write

$$B = \bigcup_{i\in S} \{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = \frac{1}{m_j}\} \cap \{X_0 = i\} = \bigcup_{i\in S} B_i$$

and therefore $P(B) = \sum_{i\in S} P(B_i)$.

But

$$B_i = B_i \cap \{\exists\, k \in \mathbb{N} : X_k = j\} \bigcup B_i \cap \{\nexists\, k \in \mathbb{N} : X_k = j\}$$

so $P(B_i) = P(B_i \cap \{\exists\, k \in \mathbb{N} : X_k = j\}) + P(B_i \cap \{\nexists\, k \in \mathbb{N} : X_k = j\})$. Recalling (10) we can write that

$$P(B_i \cap \{\exists\, k \in \mathbb{N} : X_k = j\}) = P_{A_i}\left(\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = \frac{1}{m_j}\}\right) \cdot P(A_i) = \mu_i^{(0)} \cdot f_{ij}$$

Moreover

$$P(B_i \cap \{\nexists\, k \in \mathbb{N} : X_k = j\}) = 0$$

since in this event $M_j(n) = 0$. Therefore $P(B_i) = \mu_i^{(0)} \cdot f_{ij}$ and thus

$$P(B) = \sum_{i\in S} \mu_i^{(0)} \cdot f_{ij}$$

Denote now $\Gamma_i = \{\nexists\, k \in \mathbb{N} : X_k = j\} \cap \{X_0 = i\}$. Then

$$P_{\Gamma_i}\left(\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = 0\}\right) = 1$$

where $P_{\Gamma_i}(\cdot) = P(\cdot|\Gamma_i)$. Thus

$$P\left(\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = 0\} \cap \Gamma_i\right) = P(\Gamma_i) = \mu_i^{(0)}(1 - f_{ij})$$

That means that

$$P\left(\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = 0\} \cap \Gamma\right) = \sum_{i\in S} \mu_i^{(0)}(1 - f_{ij})$$

where $\Gamma = \{\nexists\, k \in \mathbb{N} : X_k = j\}$. Thus

$$P\left(\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = 0\}\right) \geq \sum_{i\in S} \mu_i^{(0)}(1 - f_{ij})$$

The events

$$\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = \frac{1}{m_j}\} \quad \text{and} \quad \{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = 0\}$$

are disjoint, therefore

$$\begin{aligned} 1 &\leq P\left(\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = \frac{1}{m_j}\}\right) + P\left(\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = 0\}\right) \\ &= P\left(\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = \frac{1}{m_j}\} \cup \{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = 0\}\right) \\ &\leq 1 \end{aligned}$$

Therefore

$$\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = \frac{1}{m_j}\} \cup \{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = 0\} = \Omega \setminus E$$

with $P(E) = 0$ and

$$P\left(\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = 0\}\right) = \sum_{i\in S} \mu_i^{(0)}(1 - f_{ij})$$

· Assume now that j is null recurrent and let the sequence of random variables

$$Z_m = \begin{cases} n_{m+1} - n_m, & \text{when } \omega \in A_i \\ 0, & \text{otherwise} \end{cases}$$

for $m \geq 1$. Because j is null recurrent we have that $\mathbb{E}(Z_m) = \infty$ for every $m \geq 1$. We define now the sequence $Z_m^R = Z_m \mathbb{I}_{\{Z_m < R\}}$ for $R > 0$ for which it holds that $\mathbb{E}(Z_m^R) < \infty$ for every $m \geq 1$. Moreover, $\mathbb{E}(Z_1^R) = \mathbb{E}(Z_m^R)$ for every $m \geq 1$. This sequence is again an independent and identical distributed sequence of random variables. Therefore we can use the strong law of large numbers to get

$$P_{A_i}\left(\omega \in \Omega : \lim_{n\to\infty} \frac{S_n^R}{n} = \mathbb{E}_{A_i}(Z_1^R)\right) = 1$$

where $S_n^R = Z_1^R + Z_2^R + \cdots + Z_n^R \leq S_n = Z_1 + \cdots + Z_n$ and A_i, P_{A_i} is as before. Therefore it holds that

$$S^R_{M_j(n)-1} + n_1 \leq S_{M_j(n)-1} + n_1 \leq n$$

So

$$\frac{S^R_{M_j(n)-1} + n_1}{M_j(n) - 1} \frac{M_j(n) - 1}{M_j(n)} \leq \frac{n}{M_j(n)}, \quad n \geq n_2, \quad \text{for every } \omega \in A_i$$

Letting $n \to \infty$ we get that

$$0 \leq \limsup_{n\to\infty} \frac{M_j(n)}{n} \leq \frac{1}{\mathbb{E}(Z_m^R)}, \quad \text{almost surely}, \quad \text{for every } R > 0$$

under the probability measure P_{A_i}. Note that Z_m^R is an increasing sequence in R and that $Z_m^R \to Z_m$ as $R \to \infty$ almost surely. Therefore $\mathbb{E}_{A_i}(Z_m^R) \to \mathbb{E}_{A_i}(Z_m) = \infty$ using the monotone convergence theorem. That means that

$$\lim_{n\to\infty} \frac{M_j(n)}{n} = 0 \quad \text{almost surely}$$

under the probability measure P_{A_i}, i.e.

$$P_{A_i}\left(\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = 0\}\right) = 1 \tag{13}$$

Let now the event $\{\omega \in \Omega : \limsup_{n\to\infty} \frac{M_j(n)}{n} \geq \varepsilon\}$ where $\varepsilon > 0$. Noting that

$$P\left(\{\omega \in \Omega : \limsup_{n\to\infty} \frac{M_j(n)}{n} \geq \varepsilon\} \cap A^c\right) = 0$$

where $A = \{\exists\, l \in \mathbb{N} : X_l = j\}$ and

$$\begin{aligned}
P\left(\{\omega \in \Omega : \limsup_{n\to\infty} \frac{M_j(n)}{n} \geq \varepsilon\} \cap A\right) &= \sum_{i\in S} P\left(\{\omega \in \Omega : \limsup_{n\to\infty} \frac{M_j(n)}{n} \geq \varepsilon\} \cap A_i\right) \\
&= \sum_{i\in S} \underbrace{P_{A_i}\left(\{\omega \in \Omega : \limsup_{n\to\infty} \frac{M_j(n)}{n} \geq \varepsilon\}\right)}_{=0,\ \text{see (13)}} P(A_i) \\
&= 0
\end{aligned}$$

we obtain

$$P\left(\{\omega \in \Omega : \limsup_{n\to\infty} \frac{M_j(n)}{n} \geq \varepsilon\}\right) = 0$$

Because $\frac{M_j(n)}{n} \geq 0$ it follows the desired result.

· Finally we assume that j is transient. It is well known that $P(M_j < \infty | X_0 = i) = 1$ for every state i, where $M_j = \lim_{n\to\infty} M_j(n)$. Therefore

$$P(M_j < \infty) = \sum_{i\in S} P(M_j < \infty | X_0 = i) \cdot P(X_0 = i) = \sum_{i\in S} \mu_i^{(0)} = 1$$

Moreover

$$\Omega = \left(\bigcup_{N=0}^{\infty} B_N\right) \cup B_\infty$$

where $B_N = \{M_j = N\}$ and $B_\infty = \{M_j = \infty\}$. Thus

$$\sum_{N=0}^{\infty} P(B_N) = 1$$

since $P(B_\infty) = 0$.

Therefore we can write

$$\begin{aligned} & \{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = 0\} \\ = & \left(\bigcup_{N=0}^{\infty} \{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = 0\} \cap B_N \right) \cup \left(\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = 0\} \cap B_\infty \right) \end{aligned}$$

Thus

$$P\left(\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = 0\} \right) = \sum_{N=0}^{\infty} P\left(\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = 0\} \cap B_N \right)$$

since $P\left(\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = 0\} \cap B_\infty \right) \leq P(B_\infty) = 0$. But

$$\begin{aligned} & P\left(\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = 0\} \cap B_N \right) \\ = & P\left(\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = 0\} | B_N \right) P(B_N) \\ = & P(B_N) \end{aligned}$$

since it holds that $P\left(\{\omega \in \Omega : \lim_{n\to\infty} \frac{M_j(n)}{n} = 0\} | B_N \right) = 1$. Since $\sum_{N=0}^{\infty} P(B_N) = 1$ we obtain the desired result. □

Corollary 2 *If $g : S \to \mathbb{R}$ is such that*

$$\sum_{i\in S} |g(i)| < \infty$$

then it holds that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E} g(X_k) = \sum_{j\in C} \frac{g(j)}{m_j} \sum_{i\in S} \mu_i^{(0)} f_{ij}$$

where $C \subseteq S$ is the subset of S of positive recurrent states.

Proof. Note that $g(X_k) = \sum_{j\in S} g(j) \mathbb{I}_{\{X_k = j\}}$. Therefore

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E} g(X_k) &= \frac{1}{n} \sum_{k=1}^{n} \sum_{j\in S} g(j) \mathbb{E} \mathbb{I}_{\{X_k = j\}} \\ &= \sum_{j\in S} g(j) \frac{1}{n} \sum_{k=1}^{n} \mathbb{E} \mathbb{I}_{\{X_k = j\}} \\ &= \sum_{j\in S} g(j) \mathbb{E} \left(\frac{M_j(n)}{n} \right) \end{aligned}$$

We have interchange the sums $\sum_{k=1}^{n} \sum_{j\in S}$ because the series is absolutely convergent since $\sum_{i\in S} |g(i)| < \infty$.

So

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}g(X_k) = \lim_{n\to\infty}\sum_{j\in S}g(j)\mathbb{E}\left(\frac{M_j(n)}{n}\right) = \sum_{j\in C}\frac{g(j)}{m_j}\sum_{i\in S}\mu_i^{(0)}f_{ij}$$

We have used the dominated convergence theorem to interchange the limit with the sum in the second equality above. □

Corollary 3 *Given a function $g : S \to \mathbb{R}$ such that*

$$\sum_{i\in S}|g(i)| < \infty$$

it holds that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}g(X_k) = \sum_{j\in C}\frac{g(j)}{m_j}\mathbb{I}_{A^j} \qquad \textit{almost surely}$$

where $A^j = \{\omega \in \Omega : \exists\, l \in \mathbb{N} : X_l = j\}$ with $P(A^j) = \sum_{i\in S}\mu_i^{(0)} \cdot f_{ij}$. In particular, when the chain is irreducible it holds that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}g(X_k) = \begin{cases} \displaystyle\sum_{j\in S}g(j)\pi_j, & \textit{when is positive recurrent} \\ 0, & \textit{otherwise} \end{cases}$$

where $\pi = (\pi_1, \pi_2, \cdots,)$ is the unique stationary distribution (if it exists).

Proof. Note that

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n}g(X_k) &= \frac{1}{n}\sum_{k=1}^{n}\sum_{j\in S}g(j)\mathbb{I}_{\{X_k=j\}} \\
&= \sum_{j\in S}g(j)\frac{M_j(n)}{n} \\
&= \sum_{j\in C}g(j)\frac{M_j(n)}{n}\mathbb{I}_{A^j} + \sum_{j\in NR}g(j)\frac{M_j(n)}{n} + \sum_{j\in T}g(j)\frac{M_j(n)}{n}
\end{aligned}$$

where $C \subseteq S$ is the subset of positive recurrent states of S, $NR \subseteq S$ is the subset of null recurrent states of S, $T \subseteq S$ is the subset of transient states of S and $A^j = \{\omega \in \Omega : \exists\, l \in \mathbb{N} : X_l = j\}$. The condition on g, i.e. $\sum_{i\in S}|g(i)| < \infty$ is needed in order to interchange the sums to get the second equation above.

Note that $A^j = \bigcup_{i\in S}\{\exists\, l \in \mathbb{N} : X_l = j\} \cap \{X_0 = i\}$ and therefore $P(A^j) = \sum_{i\in S}\mu_i^{(0)} \cdot f_{ij}$. Finally, using proposition 1, we obtain the result,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}g(X_k) = \sum_{j\in C}\frac{g(j)}{m_j}\mathbb{I}_{A^j}, \qquad \text{almost surely}$$

In the case where the chain is irreducible (i.e. $f_{ij} = 1$ for every $i, j \in S$ and thus $P(A^j) = 1$ for every $j \in S$) it is easy to obtain the desired result. □

3 Summary

In this review article, we are interested on discrete time Markov chains. A basic problem in the study of discrete Markov chains is the computation of the probabilities

$$P(X_n = j|X_0 = i) \text{ and } P(X_n = j)$$

To compute these probabilities one have to compute the nth power of the transition matrix. In [8] and [9] we have discussed this problem in the case where the transision matrix is finite. We have seen that we can indeed compute the nth power of the matrix, even if it is not diagonizable. In [9] we gave the matlab code for this computation using the minimum polynomial of the transition matrix. We gave also a feasible method to compute the minimum polynomial, which is very useful in the case where the transition matrix is big. However, if the transition matrix is big and not sparse, it is not possible to compute the nth power, even we know the roots of the minimum or the characteristic polynomial. Therefore, in order to compute the above probabilities, we have to compute first the limiting probabilities and then, approximately, we deduce that the desired probabilities are almost equal to the limiting ones.

In order to compute the limiting probabilities, a problem arise when there are some periodic states. In this case, the ergodic theorems are useful because they give us a partial answer to our problem. So, in this review article we gave the proofs of some ergodic theorems for discrete Markov chains using elementary mathematical tools. There are some results concerning the limiting probabilities in the general case (see for example [4]) but they are not practical useful in most cases. Therefore, we are interested to find a more practical way to compute these limiting probabilities.

We are interested in particular on the case where some of the states are periodic. In this case we have show that there is a better way to compute the limiting probability $\lim_{n\to\infty}(P^n)_i$ where i is transient and j is recurrent and periodic, than the way already suggested (see for example [4]). In the way suggested by the existing literature one has to compute the probability $\sum_{k=0}^{\infty} f_{kd(j)+a}(i|j)$ where

$$f_{ij} = P(\exists\, n \in \mathbb{N} : X_n = j|X_0 = i)$$

But in most cases, this is not possible. Our suggestion is, first to compute the limiting probabilities $\lim_{n\to\infty}(P^{nd})_{ij}$. Then, in order to compute the limiting probabilities $\lim_{n\to\infty}(P^{nd+a})_{ij}$, we just have to compute the $\left(P^a \cdot \lim_{n\to\infty}(P^{nd})\right)$. This way is much easier than the existing one and the proof relies on the use of the dominated convergence theorem. We also relate the limiting probabilities with the ergodic results and find out that the first result is stronger than the second.

By means of Remark 2, there is an open question regarding the probabilities f_{ij}^Q. Can we compute these probabilities given the probabilities f_{ij}? In a finite Markov chain, one can find the inverse of the matrix W in order to compute the probabilities f_{ij}^Q given the probabilities f_{ij}. In the infinite case however, the system in Remark 2 is a difference equation, therefore the above method does not give us a result. How can we solve, in general, this difference equation in order to compute the desired probabilities? Denoting by $G^Q(x)$ the probability generating function of the sequence f_{nj}^Q and by $G(x)$ the probability generating function of the sequence f_{nj} can somehow relate these probability generating functions using the relation

$$f_{ij} = \sum_{k\in S} W_{ik} f_{kj}^Q$$

or, even better, solve for $G^Q(x)$ in terms of $G(x)$? If this is the case how can we compute the probability generating function $G(x)$?

References

[1] R. Ash, *Basic Probability Theory*, Wiley, 1970.

[2] R. Ash - C. Doleans-Dade, *Probability and Measure Theory*, Elsevier, 2000.

[3] P. Bremaud, *Markov Chains, Gibbs Fields, Monte Carlo Simulation and Queues*, Springer, 1999.

[4] K. Chung, *Markov Chains*. Springer, 1967.

[5] R. Durrett, *Probabiblity: Theory and Examples*, Cambridge University Press, 2010.

[6] Randal Douc - Eric Moulines Pierre Priouret - Philippe Soulier, *Markov Chains*, Springer, 2018.

[7] G. Grimmett - D. Stirzaker, *Probability and Random Processes*, Oxford University Press, 2001.

[8] N. Halidias, *On the Computation of the Minimum Polynomial and Applications*, Asian Research Journal of Mathematics, Volume 18, Issue 11, Page 301-319, 2022.

[9] N. Halidias, *Computing the minimum polynomial, the function and the Drazin inverse of a matrix with Matlab*, ResearhGate.

[10] P. Hoel - S. Port - C. Stone, *Introduction to Stochastic Processes*, Houghton Mifflin Company, 1972.

[11] I. Gikhman - A. Skorokhod, *Introduction to the Theory of Random Processes*, 1969.

[12] S. Karlin - H. Taylor, *A First Course in Stochastic Processes*, Academic Press, 1975.

[13] J. Kemeny - J. Snell, *Finite Markov Chains*, Springer, 1976.

[14] J. Norris, *Markov Chains*, Cambridge University Press, 1997.

[15] D. Revuz, *Markov Chains*, North Holland, 1984.

[16] D. Stroock, *An Introduction to Markov Processes*, Springer, 1999.