ON PROJECTION PROPERTIES OF MONOTONE INTEGRABLE FUNCTIONS

Abstract

This research formulates an $(i-1,i)$ - dimensional structure of $\mu_{|f|^p}^{(i-1,i)}$-vector measure integrable functions for $i = 1, 2, ...n$. Fixed point projection properties of a vector measure are appplied to determine the measurability of sets in the domain of integrable functions. Measurable sets of the form $\Pi_i A_{i-1}^{(i,i+1)}$ are partitioned into disjoint sets $\Pi_i A_{i-1}^{i}$ of finite measure.The obtained results demonstrate utility of concepts of vector measure duality, continuity from below of a measure and monotonicity of a vector measure in integrating functions.

Keywords : Projection properties, Measure space, Integrable functions.

1 Introduction

This paper considers a sequence of monotone functions and integrability concepts of integrable functions with respect to $\mu_{|f|^p}^{(i-1,i)}$-vector measure. The utility of concepts such as vector measure duality, continuity from below and monotonicity of a vector measure are applied in constructing $\mu_{|f|^p}^{(i-1,i)}$-vector measurable sets with respect to the sigma ring $\rho^{(i-1,i)}$ of subsets of an n-dimensional spcace X^n where f is an integrable function with respect to a measure $\mu^{(i-1,i)}$ defined on $\rho^{(i-1,i)}$.

This study involves partitioning of measurable sets into disjoint sets. The research further apllies projection properties of vector measure duality with values in a Hilbert space.

2 Preliminaries

Definition 1(p -Integrable Function) (Sanchez [9])

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space where $\mu^{(i-1,i)}$ is a measure defined on a sigma ring $\rho^{(i-1,i)}$ of subsets of $X \times X$. Then for $\Pi_i A_{i-1}^i \in \rho^{(i-1,i)}$ there exists a function f defined on $X \times X$ such that $\mu_{|f|^p}^{(i-1,i)}(\Pi_i A_{i-1}^i) \in Z$ where Z is a Hilbert space and $\Pi_i A_{i-1}^i$ is the product of A_i for $i = 1, 2, ...n$. The function f defined on $X \times X$ is said to be p-integrable with respect to $\mu^{(i-1,i)}$ if

$$< \mu_{|f|^p}^{(i-1,i)}(\Pi_i A_{i-1}^i), z' >< \infty$$

where z' is an element in Z', the dual space of Z.

Definition 2 (Vector Measure)(Otanga [6])

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space. If $L_P(\mu^{(i-1,i)})$ is the function space of p-integrable functions with respect to $\mu^{(i-1,i)}$, $\Pi_{i=1} A_{i-1}^i \in \rho^{(i-1,i)}$, $f \in L_P(\mu^{(i-1,i)})$ and $\mu_{|f|^p}^{(i-1,i)}(\Pi_{i=1} A_{i-1}^i) \in Z$ where Z is a Hilbert space, then the set function $\mu_{|f|^p}^{(i-1,i)} : \rho^{(i-1,i)} \rightarrow Z$ is called a vector measure.

Definition 3 (Norm of p -Integrable Functions)(Sanchez [9])

The set $L_P(\mu^{(i-1,i)})$ of p-integrable functions with respect to $\mu^{(i-1,i)}$ defines an order continuous Hilbert function space whose norm is given by

$$\| f \|_p = sup(< \mu_{|f|^p}^{(i-1,i)}(\Pi_{i=1} A_{i-1}^i), z' >)^{1\backslash p}$$

where $\Pi_{i=1} A_{i-1}^i \in \rho^{(i-1,i)}$, $f \in L_P(\mu^{(i-1,i)})$ and $z' \in Z'$.

Definition 4 (k_{i+1}-Projection Product Measure) (Otanga [5])

Let $\mu_{i-1}^{(i,i+1)}$ represent the product measure $\mu_{i-1} \times \mu_i \times \mu_{i+1}$ defined on a sigma ring $\rho_{i-1}^{(i,i+1)}$ of subsets of an $i+1$-dimensional space for $i = 1, 2, ..n$. For a fixed positive integer k_{i+1}, the set function μ_{i-1}^{i} where $i = 1, 2, ..n$ is called the projection product measure and is denoted by

$proj_{k_{i+1}}(\mu_{i-1}^{(i,i+1)})$

Definition 5 ($(\mu^{i-1,i})_{|f|^p}$-Measurable Set) (Sanchez [9])

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space. If $\Pi_i A_{i-1}^{i}$ is a measurable set with respect to $\rho^{(i-1,i)}$, then

$\mu^{(i-1,i)}(\Pi_i A_{i-1}^{i}) = \mu_{i-1}(A_{i-1}) \times \mu_i(A_i)$ for $i = 1, 2, ...n$

If $f \in L_P(\mu_{i-1}^{i})$ then for a fixed positive integer k_{i+1}, the set $\Pi_i A_{i-1}^{i}$ is said to be $(\mu^{i-1,i})_{|f|^p}$-measurable if

$< \mu_{|f|^p}^{(i-1,i)}(\Pi_i A_{i-1}^{i}), z' >$ is finite for $i = 1, 2, ...n$

Definition 6 (k_{i+1}-Projection of a Measurable Set) (Otanga [6])

Let $\Pi_{i=1} A_{i-1}^{(i,i+1)}$ be a measurable set with respect to $\rho^{(i-1,i,i+1)}$. Then the k_{i+1}-projection $\Pi_{i=1} A_{i-1}^{i}$ of $\Pi_{i=1} A_{i-1}^{(i,i+1)}$ is denoted by $proj_{k_{i+1}}(\Pi_{i=1} A_{i-1}^{(i,i+1)})$ where k_{i+1} is a fixed positive integer.

Definition 7 (Monotone p-Integrable Functions)

According to the results in (Otanga [5] and Sanchez [9]), a sequence $(f_n)_{n=1}^{\infty}$ of p-integrable functions is said to be monotonically increasing if

$\Pi_{i=1} A_{i-1}^{i} \subseteq \Pi_{i=1} B_{i-1}^{i}$ for $i = 1, 2, ...n$ implies that

$< \mu_{|f_n|^p}^{(i-1,i)}(\Pi_{i=1} A_{i-1}^{i}), z' >^{1\backslash p} \leq < \mu_{|f_n|^p}^{(i-1,i)}(\Pi_{i=1} B_{i-1}^{i}), z' >^{1\backslash p}$

Similarly a sequence $(f_n)_{n=1}^{\infty}$ of p-integrable functions is said to be monotonically decreasing if

$\Pi_{i=1}A^i_{i-1} \subseteq \Pi_{i=1}B^i_{i-1}$ for $i = 1, 2, ...n$ implies that

$< \mu^{(i-1,i)}_{|f_n|^p}(\Pi_{i=1}A^i_{i-1}), z' >^{1\backslash p} \geq < \mu^{(i-1,i)}_{|f_n|^p}(\Pi_{i=1}B^i_{i-1}), z' >^{1\backslash p}$

3. Main Results

Proposition 1

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space and $(f_n)_{n=1}^{\infty}$ be a monotonically decreasing sequence of p-integrable functions with respect to $\mu^{(i-1,i)}$. If $f_n \downarrow 0$ for each n and $((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0)$ for all (x_{i-1}, x_i), then $< \mu^{(i-1,i)}_{|f|^p}((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0), z' >)^{1\backslash p}$ is monotonically decreasing to zero for $i = 1, 2, ..., n$

Proof

Let $proj_{k_{i+1}}(\Pi_{i=1}E^{(i,i+1)}_{n i-1}) = \Pi_{i=1}E_{n i-1}{}^i$ such that

$\Pi_{i=1}E_{n i-1}{}^i = ((x_{i-1}, x_i) : f_n(x_{i-1}, x_i)) > \epsilon)$

where $\epsilon > 0$ and $f_{n+1} \leq f_n$ for all n. It follows that

$\Pi_{i=1}E_{n i-1}{}^i \subset ((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0)$

As a consequence of $f_n(x) \downarrow 0$ and $((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0)$, it follows

that $\Pi_{i=1}E_{n i-1}{}^i \downarrow 0$ for all n (Lech [2])

Let $\Pi_{i=1}E^i_{{}_{i-1}} = ((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \geq f_n(x_{i-1}, x_i))$.

If $(x_{i-1}, x_i) \in \Pi_{i=1}E^i_{{}_{i-1}}$, then $(f_n \cap f_1)(x_{i-1}, x_i) = f_n(x_{i-1}, x_i)$

Therefore

$((x_{i-1}, x_i) : f_n(x_{i-1}, x_i) \neq 0) \subset ((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0)$

For each set $((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0)$, we have

$\chi_{((x_{i-1},x_i):f_1(x_{i-1},x_i)\neq 0)}f_n = f_n$ for $i = 1, 2, ...n$

Applying the results on integrable functions (Sanchez [9] and okada [3]) and vector duality functions (Campo *et. al.* [1]), we obtain

$$< \mu_{|f|^p}^{(i-1,i)}((x_{i-1},x_i) : f_1(x_{i-1},x_i) \neq 0), z' >^{1\backslash p}$$

$$= < \mu_{|f|^p}^{(i-1,i)}((x_{i-1},x_i) : f_1(x_{i-1},x_i) \neq 0) \cap (\Pi_{i=1}E_{ni-1}{}^{i})^c, z' >)^{1\backslash p}$$

$$+ < \mu_{|f|^p}^{(i-1,i)}(\Pi_{i=1}E_{ni-1}{}^{i}), z' >)^{1\backslash p} \qquad (*)$$

where $(\Pi_{i=1}E_{ni-1}{}^{i})^c$ represents the complement of $\Pi_{i=1}E_{ni-1}{}^{i}$ in the set

$((x_{i-1},x_i) : f_1(x_{i-1},x_i) \neq 0)$

Given that $f_n(x_{i-1},x_i) \leq \epsilon$ on $((x_{i-1},x_i) : f_1(x_{i-1},x_i) \neq 0) \setminus \Pi_{i=1}E_{ni-1}{}^{i})$, it follows that

$$< \mu_{|f|^p}^{(i-1,i)}((x_{i-1},x_i : f_1(x_{i-1},x_i) \neq 0) \cap (\Pi_{i=1}E_{ni-1}{}^{i})^c, z' >)^{1\backslash p}$$

$$\leq \epsilon < \mu^{(i-1,i)}(((x_{i-1},x_i) : f_1(x_{i-1},x_i) \neq 0) \cap (\Pi_{i=1}E_{ni-1}{}^{i})^c), z' >$$

$$\leq \epsilon < \mu_{i-1}^{i}((x_{i-1},x_i) : f_1(x_{i-1},x_i) \neq 0), z' >$$

Let $M = sup\ (\mid f_n(x_{i-1},x_i) \mid \forall (x_{i-1},x_i)$. Then

$$< \mu_{|f|^p}^{(i-1,i)}(\Pi_{i=1}E_{ni-1}{}^{i}), z' >)^{1\backslash p} \leq M < \mu^{(i-1,i)}(\Pi_{i=1}E_{ni-1}{}^{i}), z' > \text{ for all } n$$

Therefore, equation (*) becomes

$$< \mu_{|f|^p}^{(i-1,i)}((x_{i-1},x_i) : f_1(x_{i-1},x_i) \neq 0), z' >^{1\backslash p}$$

$$\leq \epsilon < \mu^{(i-1,i)}(x_{i-1},x_i : f_1(x_{i-1},x_i) \neq 0), z' >$$

$$+ M < \mu^{(i-1,i)}(\Pi_{i=1}E_{ni-1}{}^{i}), z' >$$

Since ϵ is arbitrary and $< \mu^{(i-1,i)}(\Pi_{i=1}E_{ni-1}{}^{i}), z' >\downarrow 0$ for each n, it follows that

$$< \mu_{|f|^p}^{(i-1,i)}((x_{i-1},x_i : f_1(x_{i-1},x_i) \neq 0), z' >^{1\backslash p}\downarrow 0 \text{ for } i = 1, 2, ...n$$

Proposition 2

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space, f and g be positive p-integrable functions with respect to $\mu^{(i-1,i)}$.
If $\Pi_{i=1}E^i_{i-1} = (x_{i-1}, x_i : g(x_{i-1}, x_i) \geq f(x_{i-1}, x_i))$, then

$\| f \|_p \leq \| g \|_p$

Proof
Let $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ be monotonically increasing p-integrable functions such that $\chi_{\Pi_{i=1}A^i_{i-1}} g_n \uparrow \chi_{\Pi_{i=1}A^i_{i-1}} g$ and $\chi_{\Pi_{i=1}A^i_{i-1}} f_n \uparrow \chi_{\Pi_{i=1}A^i_{i-1}} f$ for each n and for every measurable set $\Pi_{i=1}A^i_{i-1}$ of finite measure.

Let $< \mu^{(i-1,i)}_{|g_n|^p}(\Pi_{i=1}A^i_{i-1}), z' >^{1\backslash p} \leq M$ for each n and $M > 0$.

If $(\Pi_{i=1}A^i_{i-1}) \cap ((x_{i-1}, x_i) : h_n(x_{i-1}, x_i) \neq 0)$

$$= (\Pi_{i=1}A^i_{i-1}) \cap ((x_{i-1}, x_i) : (f_n \cap g_n)(x_{i-1}, x_i) \neq 0), \text{ then}$$

$(\Pi_{i=1}A^i_{i-1}) \cap ((x_{i-1}, x_i) : h_n(x_{i-1}, x_i) \neq 0)$ is a subset of

$(\Pi_{i=1}A^i_{i-1}) \cap ((x_{i-1}, x_i) : g_n(x_{i-1}, x_i) \neq 0)$

Therefore

$< \mu^{(i-1,i)}_{|h_n|^p}(\Pi_{i=1}A^i_{i-1}), z' >^{1\backslash p} \leq M$

If $(x_{i-1}, x_i) \in \Pi_{i=1}E^i_{i-1}$,then

$(f \cap g)(x_{i-1}, x_i) = f(x_{i-1}, x_i)$

It follows that

$\Pi_{i=1}A^i_{i-1} \cap (x \in X : h_n(x) \neq 0)$ is monotonically increasing to

$(\Pi_{i=1}A^i_{i-1}) \cap ((x_{i-1}, x_i) : (f \cap g)(x_{i-1}, x_i) \neq 0)$

$$= (\Pi_{i=1}A^i_{i-1}) \cap ((x_{i-1}, x_i) : f(x_{i-1}, x_i) \neq 0)$$

Therefore

$$< \mu^{(i-1,i)}_{|f|^p}(\Pi_{i=1}A^i_{i-1}), z' >^{1\backslash p} = \text{LUB} < \mu^{(i-1,i)}_{|h_n|^p}(\Pi_{i=1}A^i_{i-1}), z' >^{1\backslash p}$$

$$\leq LUB < \mu^{(i-1,i)}_{|g_n|^p}(\Pi_{i=1}A^i_{i-1}), z' >^{1\backslash p} = < \mu^{(i-1,i)}_{|g|^p}(\Pi_{i=1}A^i_{i-1}), z' >^{1\backslash p}$$

Taking the supremum on both sides of the inequality, (Sanchez [9]) we

obtain

$\| f \|_p \leq \| g \|_p$

Proposition 3

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space and $(f_n)_{n=1}^{\infty}$ be a sequence of positive bounded p-integrable functions with respect to $\mu^{(i-1,i)}$ such that $f_n \uparrow f$ for each n. If $\Pi_i E^i_{i-1} = ((x_{i-1}, x_i) : f((x_{i-1}, x_i)) > \epsilon)$, then

$< \mu^{(i-1,i)}(\Pi_i E^i_{i-1}), z' >$ is bounded.

Proof

Since $f_n \uparrow f$ for each n (by hypothesis), it follows that $f = LUB f_n$ and

$f = (f_n)_{n=1}^{\infty}$

Let $\Pi_{i=1}{E_{ni-1}}^i = ((x_{i-1}, x_i) : f_n(x_{i-1}, x_i) > \epsilon)$ such that

$< \mu^{(i-1,i)}_{|f_n|^p}(\Pi_i {E_{ni-1}}^i), z' >^{1\backslash p} \leq M$ for all n and $M > 0$

It follows that $\Pi_{i=1}{E_{ni-1}}^i \uparrow \Pi_{i=1}E^i_{1-1}$ for each n

Since $f_n(x_{i-1}, x_i) > \epsilon$ for each (x_{i-1}, x_i), it follows that

$$\epsilon < \mu^{(i-1,i)}(\Pi_{i=1}{E_{ni-1}}^i)), z' > \leq < \mu^{(i-1,i)}_{|f_n|^p}(\Pi_{i=1}{E_{ni-1}}^i), z' >^{1\backslash p} \leq M$$

Let $(\Pi_{i=1}{F_{ni-1}}^i)_{n=1}^{\infty}$ be a sequence of mutually disjoint sets such that

$\Pi_{i=1}E^i_{_{i-1}} = \bigcup_{n=1}^{\infty} \Pi_{i=1}{F_{ni-1}}^i$

On application of the results in (Rodriguez [8] and Otanga [7]) and by finiteness of a vector measure (Otanga [4] and Yaogan [10]), we obtain

$$< \mu^{(i-1,i)}(\Pi_{i=1}E^{i}_{i-1}), z' >= \sum_{k=1}^{\infty} < \mu(\Pi_{i=1}F_{ki-1}{}^{i}), z' >$$

$$= LUB_n \sum_{k=1}^{n} < \mu^{(i-1,i)}(\Pi_{i=1}F_{ki-1}{}^{i}), z' >$$

$$= LUB_n < \mu^{(i-1,i)}(\bigcup_{k=1}^{n} \Pi_{i=1}F_{ki-1}{}^{i}), z' >$$

$$= LUB_n < \mu^{(i-1,i)}(\Pi_{i=1}E_{ni-1}{}^{i}), z' > \leq M$$

Proposition 4

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space, f be a p-integrable function with respect to $\mu^{(i-1,i)}$ and $(\Pi_{i=1}E_{ni-1}{}^{i})_{n=1}^{\infty}$ be a sequence of measurable sets such that

$\Pi_{i=1}E_{ni-1}{}^{i} = ((x_{i-1}, x_i) :| f(x_{i-1}, x_i) |\geq 1 \setminus n)$ for each n.

If $\Pi_{i=1}E_{ni-1}{}^{i}$ is a $\mu^{(i-1,i)}_{|f|^p}$ - null set for each n, then

$< \mu^{(i-1,i)}((x_{i-1}, x_i) : f(x) \neq 0), z' >= 0$

Proof
Consider the measurable sets $((x_{i-1}, x_i) : f(x_{i-1}, x_i) \neq 0)$ and $\Pi_{i=1}E_{ni-1}{}^{i} = ((x_{i-1}, x_i) :| f(x_{i-1}, x_i) |\geq 1 \setminus n)$

such that $((x_{i-1}, x_i) : f(x_{i-1}, x_i) \neq 0) = LUB_n\Pi_{i=1}E_{ni-1}{}^{i}$

It follows that

$\Pi_{i=1}E_{ni-1}{}^{i} \uparrow ((x_{i-1}, x_i) : f(x_{i-1}, x_i) \neq 0)$

Let $G_{k_i} \bigcap G_{k_j} = \emptyset$ for $k_i \neq k_j$ where $k_i, k_j = 1, 2, ...$ and

$((x_{i-1}, x_i) : f(x_{i-1}, x_i) \neq 0) = \bigcup_{k=1}^{\infty} \Pi_{i=1}G_{ki-1}{}^{i}$

By the property of countable additivity of a vector measure (Otanga

et. al. [5]), we obtain

$< \mu^{(i-1,i)}((x_{i-1},x_i) : f(x_{i-1},x_i) \neq 0), z' >$

$$= \sum_{k=1}^{\infty} < \mu^{(i-1,i)}(\Pi_{i=1}G_{ki-1}{}^{i}), z' >$$

$$= LUB_n \sum_{k=1}^{n} < \mu^{(i-1,i)}(\Pi_{i=1}G_{ki-1}{}^{i}), z' >$$

$$= LUB_n < \mu^{(i-1,i)}(\bigcup_{k=1}^{n} \Pi_{i=1}G_{ki-1}{}^{i}), z' >$$

$$= LUB_n < \mu^{(i-1,i)}(\Pi_{i=1}E_{ni-1}{}^{i}), z' >$$

Since $1 \setminus n \leq \mid f(x_{i-1},x_i) \mid$ on $\Pi_{i=1}E_{ni-1}{}^{i}$ and $\Pi_{i=1}E_{ni-1}{}^{i}$ is a $\mu_{|f|^p}^{(i-1,i)}$ - null set for each n (by hypothesis), then

$$1 \setminus n < \mu^{(i-1,i)}(\Pi_{i=1}E_{ni-1}{}^{i}), z' > \leq < \mu^{(i-1,i)} \mid f \mid^p (\Pi_{i=1}E_{ni-1}{}^{i}), z' >^{1\setminus p} = 0$$

,

Therefore $< \mu^{(i-1,i)}(x_{i-1},x_i) : f(x) \neq 0), z' >= 0$

Proposition 5

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space, f be a p-integrable function with respect to $\mu^{(i-1,i)}$ and $((x_{i-1},x_i) : f(x_{i-1},x_i) \neq 0)$ be a $\mu_{|f|^p}^{(i-1,i)}$ - null set, then $f = 0$ on the complement of set $((x_{i-1},x_i) : f(x_{i-1},x_i) \neq 0)$

Proof

Let $\Pi_{i=1}G_{i-1}^{i} = ((x_{i-1},x_i) : f(x_{i-1},x_i) \neq 0)$,

$\Pi_{i=1}E_{i-1}^{i} = ((x_{i-1},x_i) : f(x_{i-1},x_i) > 0$ and

$\Pi_{i=1}F_{i-1}^{i} = ((x_{i-1},x_i) : f(x_{i-1},x_i) < 0)$ be measurable sets with respect

to $\rho^{(i-1,i)}$. Since $((x_{i-1},x_i) : f(x_{i-1},x_i) \neq 0)$ is a $\mu_{|f|^p}^{(i-1,i)}$ - null set

(by hypothesis), then $< \mu_{|f|^p}^{(i-1,i)}(\Pi_{i=1}G_{i-1}^{i}), z' >^{1\setminus p} = 0$. Since $f(x) > 0$ for

each $(x_{i-1},x_i) \in \Pi_{i=1}E_{i-1}^{i}$, then $< \mu_{|f|^p}^{(i-1,i)}(\Pi_{i=1}E_{i-1}^{i}), z' >^{1\setminus p} = 0$ and

$f(x_{i-1},x_i) < 0$ for each $(x_{i-1},x_i) \in \Pi_{i=1}F_{i-1}^{i}$ implies that $< \mu_{|f|^p}^{(i-1,i)}(\Pi_{i=1}F_{i-1}^i), z' >^{1\backslash p} = 0$.

It follows that

$\Pi_{i=1}G_{i-1}^i = ((x_{i-1}, x_i) : f(x_{i-1}, x_i) > 0) \cup ((x_{i-1}, x_i) : f(x_{i-1}, x_i) < 0)$ is

a $\mu_{|f|^p}^{(i-1,i)}$ - null set

Therefore

$f = 0$ on the complement of $\Pi_{i=1}G_{i-1}^i = ((x_{i-1}, x_i) : f(x_{i-1}, x_i) \neq 0)$

Corollary 1

Let $(f_n)_{n=1}^{\infty}$ be a sequence of monotonically increasing p-integrable functions such that $< \mu_{|f_n|^p}^{(i-1,i)}(\Pi_{i=1}E_{i-1}^i), z' >^{1\backslash p}$ is bounded for each n. Let $\Pi_{i=1}{E_{ni-1}}^i$ be monotonically increasing to $\Pi_{i=1}E_{i-1}^i$ where $\mu^{(i-1,i)}(\Pi_{i=1}{E_{ni-1}}^i) < \infty$ for all n and $\Pi_{i=1}E_{i-1}^i = \cap_{n=1}^{\infty}\Pi_{i=1}{E_{ni-1}}^i$. If $\Pi_{i=1}{E_{ni-1}}^i = ((x_{i-1}, x_i) : f_n(x_{i-1}, x_i) \geq M)$ for $M > 0$, then $\Pi_{i=1}E_{i-1}^i$ is a $< \mu^{(i-1,i)}(), z' > \mu$ - null set

Proof
Since $< \mu_{|f_n|^p}^{(i-1,i)}(\Pi_{i=1}E_{i-1}^i), z' >^{1\backslash p}$ is bounded for each n, then

$< \mu_{|f_n|^p}^{(i-1,i)}(\Pi_{i=1}E_{i-1}^i), z' >^{1\backslash p} \leq \beta$ for $\beta > 0$.

From the hypothesis, $M \leq f_n(x_{i-1}, x_i)$ on $\Pi_{i=1}{E_{ni-1}}^i$. Therefore,

$M < \mu^{(i-1,i)}(\Pi_{i=1}{E_{ni-1}}^i), z' > \leq < \mu_{|f_n|^p}^{(i-1,i)}(\Pi_{i=1}{E_{ni-1}}^i), z' >^{1\backslash p}$.

Since $\Pi_{i=1}{E_{ni-1}}^i$ is monotonically increasing to $\Pi_{i=1}E_{i-1}^i$, it follows that

$\Pi_{i=1}{E_{ni-1}}^i \uparrow \Pi_{i=1}E_{i-1}^i$ (Otanga and Oduor [6])

Therefore,

$M < \mu^{(i-1,i)}(\Pi_{i=1}{E_{ni-1}}^i), z' > \leq < \mu_{|f_n|^p}^{(i-1,i)}(\Pi_{i=1}{E_{ni-1}}^i), z' >^{1\backslash p} \leq \beta$ for

$\beta > 0$.

$LUB < \mu^{(i-1,i)}(\Pi_{i=1} E_{ni-1}{}^{i}), z' >=< \mu^{(i-1,i)}(\Pi_{i=1} E^{i}_{i-1}), z' >.$

Subsequently,

$< \mu^{(i-1,i)}(\Pi_{i=1} E_{ni-1}{}^{i}), z' > \leq \beta$

From $\Pi_{i=1} E^{i}_{i-1} = \cap_{n=1}^{\infty} \Pi_{i=1} E_{ni-1}{}^{i}$, we have $\Pi_{i=1} E_{ni-1}{}^{i} \downarrow \Pi_{i=1} E^{i}_{i-1}$.

This implies that $\Pi_{i=1} E^{i}_{i-1} \subset \Pi_{i=1} E_{ni-1}{}^{i}$ for $i = 1, 2, ...n$

Hence,

$< \mu^{(i-1,i)}(\Pi_{i=1} E^{i}_{i-1}), z' > \leq \beta \setminus M$

Taking $M \to \infty$, we obtain

$< \mu^{(i-1,i)}(\Pi_{i=1} E^{i}_{i-1}), z' >= 0$

4 Conclusion

The results obtained in this paper demonstrate utility of concepts of vector measure duality, continuity from below of a measure and monotonicity of a vector measure in integrating functions in $L_P(\mu^{(i-1,i)})$ for $0 < p < \infty$

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REFERENCES

[1] Campo, R., Fernandez, A., Ferrrando, I., Mayoral, F. and Naranjo, F. Compactness of multiplication operators on spaces of integrable functions with respect to a vector measure, in vector measure, integration and related topics. *Birkhauser Verlag, Basel*, 2010, *201*, 109 - 113.

[2] Lech, D. and Iwo, L. Topological vector spaces of Brochner measurable functions. *Illinois j. Math. Spring*, 2000, *46*, 287 - 318.

[3] Okada, S., Ricker, W. J. and Sanchez, P. E. A. Optimal

domain and integral extension of operators. *Birkhauser, Basel, Switzerland*, 2008, *180*, 141 - 148.

[4] Otanga, O. L. On the finiteness and sigma finiteness of a cubic measure function. *Inter. Math. Forum, Hikari Ltd*, 2015, *10(3)*, 111 - 114.

[5] Otanga, O. L., Oduor, M. O. and Aywa, S. O. Partition of measurable sets. *Journal of Advances in Mathematics*, 2015, *10 (8)*, 3759 - 3763.

[6] Otanga, O. L. and Oduor, M. O. On Pointwise product vector measure duality. *Journal of Advances in Mathematics*, 2021, *20*, 8 - 18

[7] Otanga, O. L.. On Absolute continuity of non negative functions. *Asian Research Journal of Mathematics*, 2022, *8(11)*, 385 - 392.

[8] Rodriguez, J. On integration of vector functions with respect to vector measures. *Czechoslovak Math. J.*, 2006, *56(3)*, 805 - 825

[9] Sanchez, P. E. A. Vector measure duality and tensor product representation of L_p - spaces of vector measures. *Amer. Math. Soc.*, 2004, *132*, 3319 - 3326

[10] Yaogan, M. Facts about the fourier-stieltjes transform of vector measures on compact groups. *Inter. J. Analy. Appl.*, 2013, 19 - 25.