

ON PROJECTION PROPERTIES OF MONOTONE INTEGRABLE FUNCTIONS

Abstract

This research formulates an $(i - 1, i)$ - dimensional structure of $\mu_{|f|^p}^{(i-1,i)}$ -vector measure integrable functions for $i = 1, 2, \dots, n$. Fixed point projection properties of a vector measure are applied to determine the measurability of sets in the domain of integrable functions. Measurable sets of the form $\Pi_i A_{i-1}^{(i,i+1)}$ are partitioned into disjoint sets $\Pi_i A_{i-1}^i$ of finite measure. The obtained results demonstrate utility of concepts of vector measure duality, continuity from below of a measure and monotonicity of a vector measure in integrating functions.

Keywords : Projection properties, Measure space, Integrable functions.

1 Introduction

This paper considers a sequence of monotone functions and integrability concepts of integrable functions with respect to $\mu_{|f|^p}^{(i-1,i)}$ -vector measure. The utility of concepts such as vector measure duality, continuity from below and monotonicity of a vector measure are applied in constructing $\mu_{|f|^p}^{(i-1,i)}$ -vector measurable sets with respect to the sigma ring $\rho^{(i-1,i)}$ of subsets of an n -dimensional space X^n where f is an integrable function

with respect to a measure $\mu^{(i-1,i)}$ defined on $\rho^{(i-1,i)}$.

2 Research Methodology

This study involves partitioning of measurable sets into disjoint sets. The research further applies projection properties of vector measure duality with values in a Hilbert space.

3 Basic concepts

Definition 1(p -Integrable function)

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space where $\mu^{(i-1,i)}$ is a measure defined on a sigma ring $\rho^{(i-1,i)}$ of subsets of $X \times X$. Then for $\Pi_i A_{i-1}^i \in \rho^{(i-1,i)}$ there exists a function f defined on $X \times X$ such that $\mu_{|f|^p}^{(i-1,i)}(\Pi_i A_{i-1}^i) \in Z$ where Z is a Hilbert space and $\Pi_i A_{i-1}^i$ is the product of A_i for $i = 1, 2, \dots, n$. The function f defined on $X \times X$ is said to be p -integrable with respect to $\mu^{(i-1,i)}$ if

$$\langle \mu_{|f|^p}^{(i-1,i)}(\Pi_i A_{i-1}^i), z' \rangle < \infty$$

where z' is an element in Z' , the dual space of Z .

Definition 2 (Vector measure)

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space. If $L_P(\mu^{(i-1,i)})$ is the function space of p -integrable functions with respect to $\mu^{(i-1,i)}$, $\Pi_{i=1} A_{i-1}^i \in \rho^{(i-1,i)}$, $f \in L_P(\mu^{(i-1,i)})$ and $\mu_{|f|^p}^{(i-1,i)}(\Pi_{i=1} A_{i-1}^i) \in Z$ where Z is a Hilbert space, then the set function $\mu_{|f|^p}^{(i-1,i)} : \rho^{(i-1,i)} \rightarrow Z$ is called a vector measure.

Definition 3 (Norm of p -integrable functions)

The set $L_P(\mu^{(i-1,i)})$ of p -integrable functions with respect to $\mu^{(i-1,i)}$ defines an order continuous Hilbert function space whose norm is given by

$$\|f\|_p = \sup(\langle \mu_{|f|^p}^{(i-1,i)}(\Pi_{i=1} A_{i-1}^i), z' \rangle)^{1/p}$$

where $\Pi_{i=1} A_{i-1}^i \in \rho^{(i-1,i)}$, $f \in L_P(\mu^{(i-1,i)})$ and $z' \in Z'$.

Definition 4 (k_{i+1} -Projection product measure)

Let $\mu_{i-1}^{(i,i+1)}$ represent the product measure $\mu_{i-1} \times \mu_i \times \mu_{i+1}$ defined on a sigma ring $\rho_{i-1}^{(i,i+1)}$ of subsets of an $i + 1$ -dimensional space for $i = 1, 2, ..n$. For a fixed positive integer k_{i+1} , the set function μ_{i-1}^i where $i = 1, 2, ..n$ is called the projection product measure and is denoted by

$$proj_{k_{i+1}}(\mu_{i-1}^{(i,i+1)})$$

Definition 5 ($(\mu^{i-1,i})_{|f|^p}$ -Measurable set)

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space. If $\Pi_i A_{i-1}^i$ is a measurable set with respect to $\rho^{(i-1,i)}$, then

$$\mu^{(i-1,i)}(\Pi_i A_{i-1}^i) = \mu_{i-1}(A_{i-1}) \times \mu_i(A_i) \text{ for } i = 1, 2, \dots, n$$

If $f \in L_P(\mu_{i-1}^i)$ then for a fixed positive integer k_{i+1} , the set $\Pi_i A_{i-1}^i$ is said to be $(\mu^{i-1,i})_{|f|^p}$ -measurable if

$$\langle \mu_{|f|^p}^{(i-1,i)}(\Pi_i A_{i-1}^i), z' \rangle \text{ is finite for } i = 1, 2, \dots, n$$

Definition 6 (k_{i+1} -Projection of a measurable set)

Let $\Pi_{i=1} A_{i-1}^{(i,i+1)}$ be a measurable set with respect to $\rho^{(i-1,i,i+1)}$. Then the k_{i+1} -projection $\Pi_{i=1} A_{i-1}^i$ of $\Pi_{i=1} A_{i-1}^{(i,i+1)}$ is denoted by $proj_{k_{i+1}}(\Pi_{i=1} A_{i-1}^{(i,i+1)})$ where k_{i+1} is a fixed positive integer.

Proposition 1

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space and $(f_n)_{n=1}^\infty$ be a monotonically decreasing sequence of p -integrable functions with respect to $\mu^{(i-1,i)}$. If $f_n \downarrow 0$ for each n and $((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0)$ for all (x_{i-1}, x_i) , then $\langle \mu_{|f|^p}^{(i-1,i)}((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0), z' \rangle^{1/p}$ is monotonically decreasing to zero for $i = 1, 2, \dots, n$

Proof

Let $proj_{k_{i+1}}(\Pi_{i=1} E_{n_{i-1}}^{(i,i+1)}) = \Pi_{i=1} E_{n_{i-1}}^i$ such that

$$\Pi_{i=1} E_{n_{i-1}}^i = ((x_{i-1}, x_i) : f_n(x_{i-1}, x_i) > \epsilon)$$

where $\epsilon > 0$ and $f_{n+1} \leq f_n$ for all n . It follows that

$$\Pi_{i=1} E_{ni-1}^i \subset ((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0)$$

As a consequence of $f_n(x) \downarrow 0$ and $((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0)$, it follows that $\Pi_{i=1} E_{ni-1}^i \downarrow 0$ for all n (Lech [2])

$$\text{Let } \Pi_{i=1} E_{i-1}^i = ((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \geq f_n(x_{i-1}, x_i)).$$

$$\text{If } (x_{i-1}, x_i) \in \Pi_{i=1} E_{i-1}^i, \text{ then } (f_n \cap f_1)(x_{i-1}, x_i) = f_n(x_{i-1}, x_i)$$

Therefore

$$((x_{i-1}, x_i) : f_n(x_{i-1}, x_i) \neq 0) \subset ((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0)$$

For each set $((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0)$, we have

$$\chi_{((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0)} f_n = f_n \text{ for } i = 1, 2, \dots, n$$

Applying the results on integrable functions (Sanchez [9] and okada [3]) and vector duality functions (Campo *et. al.* [1]), we obtain

$$\begin{aligned} &< \mu_{|f|^p}^{(i-1, i)}((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0), z' >^{1 \setminus p} \\ &= < \mu_{|f|^p}^{(i-1, i)}((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0) \cap (\Pi_{i=1} E_{ni-1}^i)^c, z' >^{1 \setminus p} \\ &\quad + < \mu_{|f|^p}^{(i-1, i)}(\Pi_{i=1} E_{ni-1}^i), z' >^{1 \setminus p} \end{aligned} \quad (*)$$

where $(\Pi_{i=1} E_{ni-1}^i)^c$ represents the complement of $\Pi_{i=1} E_{ni-1}^i$ in the set

$$((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0)$$

Given that $f_n(x_{i-1}, x_i) \leq \epsilon$ on $((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0) \setminus \Pi_{i=1} E_{ni-1}^i$, it follows that

$$\begin{aligned} &< \mu_{|f|^p}^{(i-1, i)}((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0) \cap (\Pi_{i=1} E_{ni-1}^i)^c, z' >^{1 \setminus p} \\ &\leq \epsilon < \mu_{|f|^p}^{(i-1, i)}(((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0) \cap (\Pi_{i=1} E_{ni-1}^i)^c), z' > \end{aligned}$$

$$\leq \epsilon < \mu_{i-1}^i((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0), z' >$$

Let $M = \sup (|f_n(x_{i-1}, x_i)| \mid \forall(x_{i-1}, x_i))$. Then

$$< \mu_{|f|^p}^{(i-1,i)}(\Pi_{i=1} E_{ni-1}^i), z' >^{1/p} \leq M < \mu^{(i-1,i)}(\Pi_{i=1} E_{ni-1}^i), z' > \text{ for all } n$$

Therefore, equation (*) becomes

$$\begin{aligned} < \mu_{|f|^p}^{(i-1,i)}((x_{i-1}, x_i) : f_1(x_{i-1}, x_i) \neq 0), z' >^{1/p} \\ &\leq \epsilon < \mu^{(i-1,i)}(x_{i-1}, x_i : f_1(x_{i-1}, x_i) \neq 0), z' > \\ &+ M < \mu^{(i-1,i)}(\Pi_{i=1} E_{ni-1}^i), z' > \end{aligned}$$

Since ϵ is arbitrary and $< \mu^{(i-1,i)}(\Pi_{i=1} E_{ni-1}^i), z' > \downarrow 0$ for each n , it follows that

$$< \mu_{|f|^p}^{(i-1,i)}((x_{i-1}, x_i : f_1(x_{i-1}, x_i) \neq 0), z' >^{1/p} \downarrow 0 \text{ for } i = 1, 2, \dots, n$$

Proposition 2

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space, f and g be positive p -integrable functions with respect to $\mu^{(i-1,i)}$.

If $\Pi_{i=1} E_{i-1}^i = (x_{i-1}, x_i : g(x_{i-1}, x_i) \geq f(x_{i-1}, x_i))$, then

$$\|f\|_p \leq \|g\|_p$$

Proof

Let $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ be monotonically increasing p -integrable functions such that $\chi_{\Pi_{i=1} A_{i-1}^i} g_n \uparrow \chi_{\Pi_{i=1} A_{i-1}^i} g$ and $\chi_{\Pi_{i=1} A_{i-1}^i} f_n \uparrow \chi_{\Pi_{i=1} A_{i-1}^i} f$ for each n and for every measurable set $\Pi_{i=1} A_{i-1}^i$ of finite measure.

Let $< \mu_{|g_n|^p}^{(i-1,i)}(\Pi_{i=1} A_{i-1}^i), z' >^{1/p} \leq M$ for each n and $M > 0$.

If $(\Pi_{i=1} A_{i-1}^i) \cap ((x_{i-1}, x_i) : h_n(x_{i-1}, x_i) \neq 0)$

$$= (\Pi_{i=1} A_{i-1}^i) \cap ((x_{i-1}, x_i) : (f_n \cap g_n)(x_{i-1}, x_i) \neq 0), \text{ then}$$

$(\Pi_{i=1} A_{i-1}^i) \cap ((x_{i-1}, x_i) : h_n(x_{i-1}, x_i) \neq 0)$ is a subset of

$$(\Pi_{i=1}A_{i-1}^i) \cap ((x_{i-1}, x_i) : g_n(x_{i-1}, x_i) \neq 0)$$

Therefore

$$\langle \mu_{|h_n|^p}^{(i-1,i)}(\Pi_{i=1}A_{i-1}^i), z' \rangle^{1/p} \leq M$$

If $(x_{i-1}, x_i) \in \Pi_{i=1}E_{i-1}^i$, then

$$(f \cap g)(x_{i-1}, x_i) = f(x_{i-1}, x_i)$$

It follows that

$\Pi_{i=1}A_{i-1}^i \cap (x \in X : h_n(x) \neq 0)$ is monotonically increasing to

$$\begin{aligned} (\Pi_{i=1}A_{i-1}^i) \cap ((x_{i-1}, x_i) : (f \cap g)(x_{i-1}, x_i) \neq 0) \\ = (\Pi_{i=1}A_{i-1}^i) \cap ((x_{i-1}, x_i) : f(x_{i-1}, x_i) \neq 0) \end{aligned}$$

Therefore

$$\begin{aligned} \langle \mu_{|f|^p}^{(i-1,i)}(\Pi_{i=1}A_{i-1}^i), z' \rangle^{1/p} = \text{LUB} \langle \mu_{|h_n|^p}^{(i-1,i)}(\Pi_{i=1}A_{i-1}^i), z' \rangle^{1/p} \\ \leq \text{LUB} \langle \mu_{|g_n|^p}^{(i-1,i)}(\Pi_{i=1}A_{i-1}^i), z' \rangle^{1/p} = \langle \mu_{|g|^p}^{(i-1,i)}(\Pi_{i=1}A_{i-1}^i), z' \rangle^{1/p} \end{aligned}$$

Taking the supremum on both sides of the inequality, (Sanchez [9]) we

obtain

$$\| f \|_p \leq \| g \|_p$$

Proposition 3

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space and $(f_n)_{n=1}^\infty$ be a sequence of positive bounded p -integrable functions with respect to $\mu^{(i-1,i)}$ such that $f_n \uparrow f$ for each n . If $\Pi_i E_{i-1}^i = ((x_{i-1}, x_i) : f((x_{i-1}, x_i)) > \epsilon)$, then

$$\langle \mu^{(i-1,i)}(\Pi_i E_{i-1}^i), z' \rangle \text{ is bounded.}$$

Proof

Since $f_n \uparrow f$ for each n (by hypothesis), it follows that $f = \text{LUB} f_n$ and

$$f = (f_n)_{n=1}^\infty$$

Let $\Pi_{i=1} E_{ni-1}^i = ((x_{i-1}, x_i) : f_n(x_{i-1}, x_i) > \epsilon)$ such that

$$\langle \mu_{|f_n|^p}^{(i-1,i)}(\Pi_i E_{ni-1}^i), z' \rangle^{1/p} \leq M \text{ for all } n \text{ and } M > 0$$

It follows that $\Pi_{i=1} E_{ni-1}^i \uparrow \Pi_{i=1} E_{1-1}^i$ for each n

Since $f_n(x_{i-1}, x_i) > \epsilon$ for each (x_{i-1}, x_i) , it follows that

$$\epsilon < \mu^{(i-1,i)}(\Pi_{i=1} E_{ni-1}^i), z' \rangle \leq \mu_{|f_n|^p}^{(i-1,i)}(\Pi_{i=1} E_{ni-1}^i), z' \rangle^{1/p} \leq M$$

Let $(\Pi_{i=1} F_{ni-1}^i)_{n=1}^\infty$ be a sequence of mutually disjoint sets such that

$$\Pi_{i=1} E_{i-1}^i = \bigcup_{n=1}^\infty \Pi_{i=1} F_{ni-1}^i$$

On application of the results in (Rodriguez [8] and Otanga [7]) and

by finiteness of a vector measure (Otanga [4] and Yaogan [10]), we obtain

$$\begin{aligned} \langle \mu^{(i-1,i)}(\Pi_{i=1} E_{i-1}^i), z' \rangle &= \sum_{k=1}^\infty \langle \mu(\Pi_{i=1} F_{ki-1}^i), z' \rangle \\ &= LUB_n \sum_{k=1}^n \langle \mu^{(i-1,i)}(\Pi_{i=1} F_{ki-1}^i), z' \rangle \\ &= LUB_n \langle \mu^{(i-1,i)}(\bigcup_{k=1}^n \Pi_{i=1} F_{ki-1}^i), z' \rangle \\ &= LUB_n \langle \mu^{(i-1,i)}(\Pi_{i=1} E_{ni-1}^i), z' \rangle \leq M \end{aligned}$$

Proposition 4

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space, f be a p -integrable function with respect to $\mu^{(i-1,i)}$ and $(\Pi_{i=1} E_{ni-1}^i)_{n=1}^\infty$ be a sequence of measurable sets such that

$$\Pi_{i=1} E_{ni-1}^i = ((x_{i-1}, x_i) : |f(x_{i-1}, x_i)| \geq 1/n) \text{ for each } n.$$

If $\Pi_{i=1} E_{ni-1}^i$ is a $\mu_{|f|^p}^{(i-1,i)}$ - null set for each n , then

$$\langle \mu^{(i-1,i)}((x_{i-1}, x_i) : f(x) \neq 0), z' \rangle = 0$$

Proof

Consider the measurable sets $((x_{i-1}, x_i) : f(x_{i-1}, x_i) \neq 0)$ and $\Pi_{i=1} E_{ni-1}^i = ((x_{i-1}, x_i) : |f(x_{i-1}, x_i)| \geq 1/n)$

such that $((x_{i-1}, x_i) : f(x_{i-1}, x_i) \neq 0) = LUB_n \Pi_{i=1} E_{ni-1}^i$

It follows that

$$\Pi_{i=1} E_{ni-1}^i \uparrow ((x_{i-1}, x_i) : f(x_{i-1}, x_i) \neq 0)$$

Let $G_{k_i} \cap G_{k_j} = \emptyset$ for $k_i \neq k_j$ where $k_i, k_j = 1, 2, \dots$ and

$$((x_{i-1}, x_i) : f(x_{i-1}, x_i) \neq 0) = \bigcup_{k=1}^{\infty} \Pi_{i=1} G_{ki-1}^i$$

By the property of countable additivity of a vector measure (Otanga *et. al.* [5]), we obtain

$$\begin{aligned} < \mu^{(i-1,i)}((x_{i-1}, x_i) : f(x_{i-1}, x_i) \neq 0), z' > \\ &= \sum_{k=1}^{\infty} < \mu^{(i-1,i)}(\Pi_{i=1} G_{ki-1}^i), z' > \\ &= LUB_n \sum_{k=1}^n < \mu^{(i-1,i)}(\Pi_{i=1} G_{ki-1}^i), z' > \\ &= LUB_n < \mu^{(i-1,i)}(\bigcup_{k=1}^n \Pi_{i=1} G_{ki-1}^i), z' > \\ &= LUB_n < \mu^{(i-1,i)}(\Pi_{i=1} E_{ni-1}^i), z' > \end{aligned}$$

Since $1/n \leq |f(x_{i-1}, x_i)|$ on $\Pi_{i=1} E_{ni-1}^i$ and $\Pi_{i=1} E_{ni-1}^i$ is a $\mu_{|f|^p}^{(i-1,i)}$ - null set for each n (by hypothesis), then

$$1/n < \mu^{(i-1,i)}(\Pi_{i=1} E_{ni-1}^i), z' > \leq < \mu^{(i-1,i)} |f|^p(\Pi_{i=1} E_{ni-1}^i), z' >^{1/p} = 0$$

Therefore $< \mu^{(i-1,i)}((x_{i-1}, x_i) : f(x) \neq 0), z' > = 0$

Proposition 5

Let $(X \times X, \rho^{(i-1,i)}, \mu^{(i-1,i)})$ be a measure space, f be a p -integrable function with respect to $\mu^{(i-1,i)}$ and $((x_{i-1}, x_i) : f(x_{i-1}, x_i) \neq 0)$ be a

$\mu_{|f|^p}^{(i-1,i)}$ - null set, then $f = 0$ on the complement of set $((x_{i-1}, x_i) : f(x_{i-1}, x_i) \neq 0)$

Proof

Let $\Pi_{i=1}G_{i-1}^i = ((x_{i-1}, x_i) : f(x_{i-1}, x_i) \neq 0)$,

$\Pi_{i=1}E_{i-1}^i = ((x_{i-1}, x_i) : f(x_{i-1}, x_i) > 0)$ and

$\Pi_{i=1}F_{i-1}^i = ((x_{i-1}, x_i) : f(x_{i-1}, x_i) < 0)$ be measurable sets with respect

to $\rho^{(i-1,i)}$. Since $((x_{i-1}, x_i) : f(x_{i-1}, x_i) \neq 0)$ is a $\mu_{|f|^p}^{(i-1,i)}$ - null set

(by hypothesis), then $\langle \mu_{|f|^p}^{(i-1,i)}(\Pi_{i=1}G_{i-1}^i), z' \rangle^{1/p} = 0$. Since $f(x) > 0$ for

each $(x_{i-1}, x_i) \in \Pi_{i=1}E_{i-1}^i$, then $\langle \mu_{|f|^p}^{(i-1,i)}(\Pi_{i=1}E_{i-1}^i), z' \rangle^{1/p} = 0$ and

$f(x_{i-1}, x_i) < 0$ for each $(x_{i-1}, x_i) \in \Pi_{i=1}F_{i-1}^i$ implies

that $\langle \mu_{|f|^p}^{(i-1,i)}(\Pi_{i=1}F_{i-1}^i), z' \rangle^{1/p} = 0$.

It follows that

$\Pi_{i=1}G_{i-1}^i = ((x_{i-1}, x_i) : f(x_{i-1}, x_i) > 0) \cup ((x_{i-1}, x_i) : f(x_{i-1}, x_i) < 0)$ is

a $\mu_{|f|^p}^{(i-1,i)}$ - null set

Therefore

$f = 0$ on the complement of $\Pi_{i=1}G_{i-1}^i = ((x_{i-1}, x_i) : f(x_{i-1}, x_i) \neq 0)$

Corollary

Let $(f_n)_{n=1}^\infty$ be a sequence of monotonically increasing p -integrable functions such that $\langle \mu_{|f_n|^p}^{(i-1,i)}(\Pi_{i=1}E_{i-1}^i), z' \rangle^{1/p}$ is bounded for each n .

Let $\Pi_{i=1}E_{ni-1}^i$ be monotonically increasing to $\Pi_{i=1}E_{i-1}^i$ where $\mu_{|f_n|^p}^{(i-1,i)}(\Pi_{i=1}E_{ni-1}^i) < \infty$ for all n and $\Pi_{i=1}E_{i-1}^i = \bigcap_{n=1}^\infty \Pi_{i=1}E_{ni-1}^i$.

If $\Pi_{i=1}E_{ni-1}^i = ((x_{i-1}, x_i) : f_n(x_{i-1}, x_i) \geq M)$ for $M > 0$, then

$\Pi_{i=1}E_{i-1}^i$ is a $\langle \mu_{|f_n|^p}^{(i-1,i)}(\cdot), z' \rangle^{1/p} - \mu$ - null set

Proof

Since $\langle \mu_{|f_n|^p}^{(i-1,i)}(\Pi_{i=1} E_{i-1}^i), z' \rangle^{1/p}$ is bounded for each n , then

$$\langle \mu_{|f_n|^p}^{(i-1,i)}(\Pi_{i=1} E_{i-1}^i), z' \rangle^{1/p} \leq \beta \text{ for } \beta > 0.$$

From the hypothesis, $M \leq f_n(x_{i-1}, x_i)$ on $\Pi_{i=1} E_{ni-1}^i$. Therefore,

$$M < \mu^{(i-1,i)}(\Pi_{i=1} E_{ni-1}^i), z' \rangle \leq \langle \mu_{|f_n|^p}^{(i-1,i)}(\Pi_{i=1} E_{ni-1}^i), z' \rangle^{1/p}.$$

Since $\Pi_{i=1} E_{ni-1}^i$ is monotonically increasing to $\Pi_{i=1} E_{i-1}^i$, it follows that

$$\Pi_{i=1} E_{ni-1}^i \uparrow \Pi_{i=1} E_{i-1}^i \text{ (Otanga and Oduor [6])}$$

Therefore,

$$M < \mu^{(i-1,i)}(\Pi_{i=1} E_{ni-1}^i), z' \rangle \leq \langle \mu_{|f_n|^p}^{(i-1,i)}(\Pi_{i=1} E_{ni-1}^i), z' \rangle^{1/p} \leq \beta \text{ for } \beta > 0.$$

$$LUB \langle \mu^{(i-1,i)}(\Pi_{i=1} E_{ni-1}^i), z' \rangle = \langle \mu^{(i-1,i)}(\Pi_{i=1} E_{i-1}^i), z' \rangle.$$

Subsequently,

$$\langle \mu^{(i-1,i)}(\Pi_{i=1} E_{ni-1}^i), z' \rangle \leq \beta$$

From $\Pi_{i=1} E_{i-1}^i = \bigcap_{n=1}^{\infty} \Pi_{i=1} E_{ni-1}^i$, we have $\Pi_{i=1} E_{ni-1}^i \downarrow \Pi_{i=1} E_{i-1}^i$.

This implies that $\Pi_{i=1} E_{i-1}^i \subset \Pi_{i=1} E_{ni-1}^i$ for $i = 1, 2, \dots, n$

Hence,

$$\langle \mu^{(i-1,i)}(\Pi_{i=1} E_{i-1}^i), z' \rangle \leq \beta \setminus M$$

Taking $M \rightarrow \infty$, we obtain

$$\langle \mu^{(i-1,i)}(\Pi_{i=1} E_{i-1}^i), z' \rangle = 0$$

3 Conclusion

The results obtained in this paper demonstrate utility of concepts of vector measure duality, continuity from below of a measure and

monotonicity of a vector measure in integrating functions in $L_P(\mu^{(i-1,i)})$ for $0 < p < \infty$

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