Global solutions of an initial boundary value problem for the Euler-Poisson-Korteweg System

Abstract

The Euler-Poisson–Korteweg system is a mathematical model arising from hydrodynamics and quantum hydrodynamics. It can be used to describe at interface the flow of capillary flows, such as the liquid–vapor mixture. In this paper, we obtain the global existence of solutions for high-dimensional compressible Euler-Poisson-Korteweg systems with small initial values by the energy method. The study can provide a theoretical basis for the development of efficient numerical solution methods, as well as contribute to the further study of other properties of the solution such as vibrational and bursting properties.

Keywords: Energy estimates; Euler-Poisson-Korteweg System; Global existence
MSC: 35J05; 35B35; 35Q35; 76N10;

1. Introduction

In this paper, we discuss the following high dimensional compressible Euler-Poisson-Korteweg (EPK) system:

$$\begin{aligned} \rho_t + \operatorname{div}(\rho u) &= 0, \\ (\rho u)_t - \Delta u + \beta \rho u + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) &= \rho \nabla (\kappa(\rho) \Delta \rho + \frac{1}{2} \kappa'(\rho) |\nabla \rho|^2) + \rho \nabla \Phi, \\ \Delta \Phi &= \rho - \bar{\rho}, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^{\mathrm{N}}, \end{aligned} \tag{EPK1}$$

where $\rho \in \mathbb{R}$ is the electron density, $u \in \mathbb{R}^N$ is the electron velocity and $\Phi \in \mathbb{R}$ denote the electrostatic potential. p is a given pressure function respective to ρ.

A number of models are used to describe materials of Korteweg type, such as Euler-Korteweg and Euler-Poisson-Korteweg. It has been extensively investigated in references [1, 5, 6, 7, 8, 9, 10]. For example, Hattori and Li investigated the local existence and global existence of solutions for the Euler-Korteweg system in [1, 10]. Donatelli et al [8] studied the existence of global-in-time weak solutions for Euler-Poisson-Korteweg system under difference initial data. Danchin and Desjardins [7] established the existence of smooth solutions for an isothermal model of capillary compressible fluids. However, as we know, global-in-time existence of (EPK1) have not been studied. In this paper, we are going to prove the global existence of solutions for the Euler-Poisson-Korteweg system due to its widespread application.

In the system (EPK1), $\bar{\rho}$ represents the background profile and satisfies that $\bar{\rho}(x) \equiv \bar{\rho}_0$ is a positive constant. Denote by and assume $\kappa : (0, \infty) \to (0, \infty)$ is a smooth function corresponding to the capillary coefficient. In particular, assuming $\kappa > 0$ to be a constant in (EPK1), we can derive the standard equation of an inviscid capillary fluid (see Kotchote [2], [3], Bresch et al [4]). For convenience, let $\kappa = \beta = 1$ and $N = 2$, we ultimately obtain the following system for $[\rho, u, \Phi]$:

$$\begin{aligned} \rho_t + \operatorname{div}(\rho u) &= 0, \\ (\rho u)_t + \rho u + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) &= \rho \nabla \Phi + \Delta u + \rho \nabla \Delta \rho, \\ \Delta \Phi &= \rho - \bar{\rho}, \end{aligned} \tag{EPK2}$$

with the initial data

$$(\rho, u, \Phi)(x, 0) = (\rho_0, u_0, \Phi_0)(x). \tag{1}$$

Assuming p is a soomth function of ρ and satisfies

$$p'(\rho) > 0, \qquad p''(\rho) > 0. \tag{2}$$

Define function $H(\rho)$ by

$$H'(\rho) = h(\rho) = \int_{\bar{\rho}_0}^{\rho} \frac{p'(s)}{s} \mathrm{d}s, \qquad H(\bar{\rho}_0) = 0. \tag{3}$$

Then we have $H'(0) = 0$, $H''(\rho) > 0$. Furthermore, we get

$$H(\rho) \geq \gamma(\rho - \bar{\rho}_0)^2, \tag{4}$$

where γ is a positive constant.

Let $w = (\rho - \bar{\rho}_0, u, \nabla\Phi)$ and H^k represents the usual Sobolev space while $\| \cdot \|_k$ the standrad kth order Sobolev norm in $\mathbb{R}^N$. Denote

$$|||w|||_{0,T}^2 \equiv \sup_{0 \leq t \leq T} (\|w(t)\|_0^2 + \|\nabla \rho(t)\|_0^2) + \int_0^t \|u(s)\|_1^2 + \|\rho(s) - \bar{\rho}_0\|_2^2 \mathrm{d}s,$$

and

$$|||w|||_{k,T}^2 \equiv \sum_{|i| \geq 0}^{k} |||\partial_x^i w|||_{0,T}^2.$$

Denote by $\partial_i f$ the spatial derivative of the i^{th} component for any function f.

Theorem 1.1. *Assume that $p(\rho)$ satisfies (2) and $\bar{\rho}_0 > 0$ is small enough. For the system (EPK2) with the initial data $w_0 = (\rho_0, u_0, \Phi_0)$ satisfies*

$$\|w_0\|_3^2 \le \epsilon, \tag{5}$$

where $0 < \epsilon \ll 1$ and $\rho_0 \ge \delta_0 > 0$. Then there exists a unique classical solution (ρ, u, Φ) in $[0, \infty)$ and satisfies

$$|||w|||_{3,\infty}^2 \le C\|w_0\|_3^2.$$

The proof of Theorem 1.1 is based on the existence of local solutions for the system (EPK2), which can be derived by the similar method in [1]. We can prove Theorem 1.1 by the following theorem:

Theorem 1.2. *Let w be a sufficiently smooth solution in the time interval $[0, T]$ for the system (EPK2) with the initial data $w_0 = (\rho_0, u_0, \Phi_0)$. Assume that $p(\rho)$ satisfies (2). If there exists $\delta \ll 1$ such that w satisfies*

$$\sup_{0 \le t \le T} (\|w\|_3 + \|\nabla\rho\|_3) \le \delta \tag{6}$$

and $0 < \bar{\rho}_0 < \delta$, then we have the estimate

$$|||w|||_{3,T}^2 \le C_\delta(\|w_0\|_3^2 + \|\nabla\rho_0\|_3^2). \tag{7}$$

Remark 1.1. Theorem 1.1 and Theorem 1.2 make sense for $N = 2$. For general N Sobolev space H^3 should be replace by H^k, where $k \ge 2 + \frac{n}{2}$.

Remark 1.2. Given assumptions in Theorem 1.2, we can choose ϵ in Theorem 1.1 so small that

$$C_\delta \epsilon \le \delta.$$

Then for the initial data satisfies (5), the existence of the local existence solutions and the standard continuation argument give the result of Theorem 1.1.

2. Proof of Theorem 1.2

We use induction on k to prove the Theorem 1.2, where $0 \le k \le 3$.

2.1. Estimate for $k = 0$

Multiplying the second equation of (EPK2) by u and integrating over $(0, T) \times \mathbb{R}^N$, we have

$$\begin{aligned}\int_0^t \int_{\mathbf{R}^2} \Big[&(\rho\partial_t u_j + \rho u_i \partial_i u_j) u_j + \nabla_j p(\rho) u_j + \rho u_j u_j \\ &- \rho\nabla_j \Delta\rho u_j - \rho\nabla_j \Phi u_j - \Delta u_j u_j\Big] \mathrm{d}x\mathrm{d}s \\ =:& I_1 + I_2 + I_3 + I_4 + I_5 + I_6 = 0.\end{aligned} \tag{8}$$

Using the first equation in system (EPK2) and integrating by parts, we have

$$I_1 = \frac{1}{2}\int \rho|u|^2 \mathrm{d}x\Big|_0^t \tag{9}$$

According to (3) and (4), we can derive

$$\begin{aligned} I_2 &= \int_0^t \int_{\mathbf{R}^2} p'(\rho)\partial_j \rho u_j = \int_0^t \int_{\mathbf{R}^2} \partial_j h(\rho)\rho u_j = \int_{\mathbf{R}^2} H(\rho)\mathrm{d}x\Big|_0^t \\ &\geq \gamma \int_{\mathbf{R}^2} |\rho - \bar{\rho}_0|^2 \mathrm{d}x\Big|_0^t . \end{aligned} \tag{10}$$

The third term I_3 gives

$$I_3 = \int_0^t \int_{\mathbf{R}^2} \rho|u|^2 \mathrm{d}x\mathrm{d}s. \tag{11}$$

The fourth term I_4 integrating by parts and using the first equation of (EPK2) gives

$$I_4 = -\int_0^t \int_{\mathbf{R}^2} \rho\nabla\Delta\rho \cdot u \mathrm{d}x\mathrm{d}s = \int_0^t \int_{\mathbf{R}^2} \Delta\rho\nabla\cdot(\rho u) = \frac{1}{2}\int_{\mathbf{R}^2} |\nabla\rho|^2 \mathrm{d}x\Big|_0^t . \tag{12}$$

Integrating by parts and using the first and second equation of (EPK2), we obtain

$$\begin{aligned} I_5 &= -\int_0^t \int_{\mathbf{R}^2} \nabla\Phi\cdot(\rho u)\mathrm{d}x\mathrm{d}s = \int_0^t \int_{\mathbf{R}^2} \Phi\nabla\cdot(\rho u)\mathrm{d}x\mathrm{d}s = -\int_0^t \int_{\mathbf{R}^2} \Phi\rho_t \mathrm{d}x\mathrm{d}s \\ &= -\int_0^t \int_{\mathbf{R}^2} \Phi\partial_t\Delta\Phi \mathrm{d}x\mathrm{d}s = \frac{1}{2}\int_{\mathbf{R}^2} |\nabla\Phi|^2 \mathrm{d}x\Big|_0^t . \end{aligned} \tag{13}$$

Integrating by parts, we have

$$I_6 = \int_0^t \int_{\mathbf{R}^2} |\nabla u|^2 \mathrm{d}x\mathrm{d}s. \tag{14}$$

Combining (9)-(14) and using (6), we have

$$\|w(t)\|_0^2 + \|\nabla\rho(t)\|_0^2 + \int_0^t \|u(s)\|_1^2 \mathrm{d}s \leq C_\delta(\|w_0\|_0^2 + \|\nabla\rho_0\|_0^2). \tag{15}$$

Next, mutiplying the second of system (EPK2) by $\nabla\rho$ and integrating over $(0,T)\times\mathbb{R}^N$, we have

$$\begin{aligned} &\int_0^t \int_{\mathbf{R}^2} \Big[(\rho\partial_t u_j + \rho u_i\partial_i u_j)\partial_j\rho + \nabla_j p(\rho)\partial_j\rho + \rho u_j\partial_j\rho \\ &\qquad - \rho\nabla_j\Delta\rho\partial_j\rho - \rho\nabla_j\Phi\partial_j\rho - \Delta u_j\partial_j\rho\Big]\mathrm{d}x\mathrm{d}s \\ &=: J_1 + J_2 + J_3 + J_4 + J_5 + J_6 = 0. \end{aligned} \tag{16}$$

Using the first equation of (EPK2), we have

$$
\begin{aligned}
J_1 =& \int_0^t \int_{\mathbf{R}^2} (\rho\partial_t u_j + \rho u_i \partial_i u_j)\partial_j \rho \mathrm{d}x\mathrm{d}s \\
=& \int_0^t \int_{\mathbf{R}^2} \partial_t u_j \times \frac{1}{2}\partial_j \rho^2 + \rho u_i \partial_i u_j \partial_j \rho \mathrm{d}x\mathrm{d}s \\
=& \int_0^t \rho \partial_j \rho u \mathrm{d}s \Big|_0^t - \int_0^t \int_{\mathbf{R}^2} \partial_j u_j \partial_i(\rho u_i)\rho + \rho u_i \partial_i u_j \partial_j \rho \mathrm{d}x\mathrm{d}s \\
\leq& C_\delta \left(\|\rho(s) - \bar{\rho}_0\|_1^2 + \|u(s)\|_0^2\right)\Big|_0^t + C_\delta \int_0^t \|\nabla u(s)\|_0^2 \mathrm{d}s + C\delta \int_0^t \|\nabla(\rho(s) - \bar{\rho}_0)\|_0^2 \mathrm{d}s \\
\leq& C_\delta(\|w(t)\|_0^2 + \|w_0\|_0^2 + \|\nabla\rho(t)\|_0^2 + \|\nabla\rho_0\|_0^2) \\
&+ C_\delta \int_0^t \|\nabla u(s)\|_0^2 \mathrm{d}s + C\delta \int_0^t \|\nabla(\rho(s) - \bar{\rho}_0)\|_0^2 \mathrm{d}s.
\end{aligned}
\tag{17}
$$

Thanks to $p'(\rho) > 0$, we have

$$
J_2 \geq c_0 \int_0^t \|\nabla(\rho(s) - \bar{\rho}_0)\|_0^2 \mathrm{d}s, \tag{18}
$$

where c_0 is a positive constant. Using the first equation of (EPK2), we have

$$
J_3 = \int_0^t \int_{\mathbf{R}^2} \rho u \cdot \nabla\rho = \frac{1}{2} \|\rho - \bar{\rho}_0\|_0^2\Big|_0^t. \tag{19}
$$

Integrating by parts, we have

$$
\begin{aligned}
J_4 =& -\int_0^t \int_{\mathbf{R}^2} \rho \nabla\Delta\rho \cdot \nabla\rho \\
\geq& c_0 \int_0^t \|\nabla^2(\rho(s) - \bar{\rho}_0)\|_0^2 \mathrm{d}s - C\delta \int_0^t \|\nabla(\rho(s) - \bar{\rho}_0)\|_0^2 \mathrm{d}s.
\end{aligned}
\tag{20}
$$

Integrating by parts and using the third equation of system (EPK2), we get

$$
\begin{aligned}
J_5 =& -\int_0^t \int_{\mathbf{R}^2} \rho \nabla\Phi \cdot \nabla(\rho - \bar{\rho}_0) \mathrm{d}x\mathrm{d}s \\
=& -\int_0^t \int_{\mathbf{R}^2} (\rho - \bar{\rho}_0)\nabla\Phi \cdot \nabla(\rho - \bar{\rho}_0) \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbf{R}^2} \bar{\rho}_0 \nabla\Phi \cdot \nabla(\rho - \bar{\rho}_0) \mathrm{d}x\mathrm{d}s \\
=:& J_{51} + J_{52}.
\end{aligned}
$$

Using the third equation of (EPK1) and integrating by parts, we obtain

$$
\begin{aligned}
J_{52} =& \bar{\rho}_0 \int_0^t \|\rho(s) - \bar{\rho}_0\|_0^2 \mathrm{d}s, \\
J_{51} \geq& c_\delta \int_0^t \|(\rho(s) - \bar{\rho}_0)\|_0^2 \mathrm{d}s.
\end{aligned}
$$

Therefore, we have

$$J_5 \geq C\delta \int_0^t \|(\rho(s) - \bar{\rho}_0)\|_0^2 \mathrm{d}s. \tag{21}$$

Integrating by parts, we obtain

$$J_6 \leq \delta \int_0^t \|\nabla^2(\rho(s) - \bar{\rho}_0)\|_0^2 \mathrm{d}s + C_\delta \int_0^t \|\nabla u\|_0^2 \mathrm{d}s. \tag{22}$$

Combining (17)-(23) and (15), we obtain

$$\begin{aligned} &\|w(t)\|_0^2 + \|\nabla(\rho(t) - \bar{\rho}_0)\|_0^2 + \int_0^t \|u(s)\|_1^2 + \|\rho - \bar{\rho}_0\|_2^2 \mathrm{d}s \\ \leq &C_\delta(\|w_0\|_0^2 + \|\nabla(\rho_0 - \bar{\rho}_0)\|_0^2). \end{aligned} \tag{23}$$

2.2. Estimate of $k \geq 1$

Here we prove (7) by induction on k. Assume (7) makes sense for all $k \leq k-1$, i.e.

$$\begin{aligned} &\|w(t)\|_{k-1}^2 + \|\nabla \rho(t)\|_{k-1}^2 + \int_0^t \|u(s)\|_k^2 + \|\rho - \bar{\rho}_0\|_{k+1}^2 \mathrm{d}s \\ \leq &C_\delta(\|w_0\|_{k-1}^2 + \|\nabla(\rho_0 - \bar{\rho}_0)\|_{k-1}^2). \end{aligned} \tag{24}$$

Denote by ∂^k the operator vector with components consisting of all the differential operators D^α with multi-index $|\alpha| = k$. Denote $[\nabla^k, f]g \equiv \nabla^k(fg) - g\nabla^k f$ for any function f, g. Multiplying the second equation of (EPK2) by ρ^{-1} and applying the operator $\rho\nabla^k$, then multiplying by $\nabla^k u$ and integrating over $(0, T) \times \mathbb{R}^N$, we have

$$\begin{aligned} &\int_0^t \int_{\mathbf{R}^2} \Big[\rho\nabla^k(\partial_t u_j + u_i\partial_i u_j)\nabla^k u_j + \rho\nabla^k\rho^{-1}\nabla_j p(\rho)\nabla^k u_j + \rho\nabla^k u_j \nabla^k u_j \\ &\qquad - \rho\nabla_j\Delta\rho\nabla^k u_j - \rho\nabla^k\nabla_j\Phi\nabla^k u_j - \rho\nabla^k\rho^{-1}\Delta u_j\nabla^k u_j\Big]\mathrm{d}x\mathrm{d}s \\ =: &K_1 + K_2 + K_3 + K_4 + K_5 + K_6 = 0. \end{aligned} \tag{25}$$

Using the first equation in (EPK2), we have

$$\begin{aligned} K_1 = &\frac{1}{2}\int_0^t \rho|\nabla^k u(s)|^2 \mathrm{d}x\Big|_0^t + \int_0^t \int_{\mathbf{R}^2} \rho[\nabla^k, u_i]\partial_i u_j \nabla^k u_j \mathrm{d}x\mathrm{d}s \\ \geq &\delta(\|\nabla^k u(t)\|_0^2) - \|\nabla^k w_0\|_0^2 - C_\delta \int_0^t \|\nabla u(s)\|_{k-1}^2 \mathrm{d}s. \end{aligned} \tag{26}$$

Thanks to (2),(3) and (4) and using the first equation of (EPK2), we have

$$\begin{aligned}
K_2 =& \int_0^t \int_{\mathbf{R}^2} \nabla^k \nabla h(\rho) \cdot \rho \nabla^k u \mathrm{d}x\mathrm{d}s \\
=& \int_0^t \int_{\mathbf{R}^2} \nabla^k \nabla h(\rho) \cdot \nabla^k (\rho u) \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbf{R}^2} \nabla^k \nabla h(\rho) \cdot [\nabla^k, \rho] u \mathrm{d}x\mathrm{d}s \\
=& -\int_0^t \int_{\mathbf{R}^2} \nabla^k h(\rho) \nabla^k \nabla \cdot (\rho u) \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbf{R}^2} \nabla^k \nabla h(\rho) \cdot [\nabla^k, \rho] u \mathrm{d}x\mathrm{d}s \\
=& \int_0^t \int_{\mathbf{R}^2} \nabla^k h(\rho) \partial_t \nabla^k \rho \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbf{R}^2} \nabla^k \nabla h(\rho) \cdot [\nabla^k, \rho] u \mathrm{d}x\mathrm{d}s \\
=& \int_0^t \int_{\mathbf{R}^2} h'(\rho) \nabla^k \rho \partial_t \nabla^k \rho \mathrm{d}x\mathrm{d}s + \int_0^t \int_{\mathbf{R}^2} [\nabla^{k-1}, h'(\rho)] \nabla \rho \partial_t \nabla^k \rho \mathrm{d}x\mathrm{d}s \\
& - \int_0^t \int_{\mathbf{R}^2} \nabla^k \nabla h(\rho) \cdot [\nabla^k, \rho] u \mathrm{d}x\mathrm{d}s \\
=& \int_0^t \int_{\mathbf{R}^2} \partial_t (h'(\rho) \nabla^k \rho \nabla^k \rho) \mathrm{d}x\mathrm{d}s + \int_0^t \int_{\mathbf{R}^2} [\nabla^{k-1}, h'(\rho)] \nabla \rho \partial_t \nabla^k \rho \mathrm{d}x\mathrm{d}s \\
& - \int_0^t \int_{\mathbf{R}^2} \nabla^k \nabla h(\rho) \cdot [\nabla^k, \rho] u dx \mathrm{d}s \\
\geq& C_\delta^{-1} \left\| \nabla^k (\rho(s) - \bar{\rho}_0) \right\|_0^2 \Big|_0^t - C \int_0^t \left\| \nabla(\rho(s) - \bar{\rho}_0) \right\|_{k-1}^2 \mathrm{d}s - \delta \int_0^t (\|\nabla u(s)\|_k^2) \mathrm{d}s.
\end{aligned} \tag{27}$$

The third term K_3 gives

$$K_3 \geq \delta \int_0^t \|\nabla^k u\|_0^2 \mathrm{d}s. \tag{28}$$

Using the first equation of (EPK2) and integrating by parts, we have

$$\begin{aligned}
K_4 =& -\int_0^t \int_{\mathbf{R}^2} \rho \nabla^k \nabla \Delta \rho \cdot \nabla^k u \mathrm{d}x\mathrm{d}s \\
=& -\int_0^t \int_{\mathbf{R}^2} \nabla^k \nabla \Delta \rho \cdot \nabla^k (\rho u) \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbf{R}^2} \nabla^k \nabla \Delta \rho \cdot [\nabla^k, \rho] u \mathrm{d}x\mathrm{d}s \\
=& \int_0^t \int_{\mathbf{R}^2} \nabla^k \Delta \rho \nabla \cdot \nabla^k (\rho u) \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbf{R}^2} \nabla^k \nabla \Delta \rho \cdot [\nabla^k, \rho] u \mathrm{d}x\mathrm{d}s \\
=& \frac{1}{2} \left\| \nabla^{k+1} (\rho(s) - \bar{\rho}_0) \right\|_0^2 \Big|_0^t - \int_0^t \int_{\mathbf{R}^2} \nabla^k \nabla \Delta \rho \cdot [\nabla^k, \rho] u \mathrm{d}x\mathrm{d}s \\
\geq& \frac{1}{2} \left\| \nabla^{k+1} (\rho(s) - \bar{\rho}_0) \right\|_0^2 \Big|_0^t - C_\delta \int_0^t \|\nabla(\rho(s) - \bar{\rho}_0)\|_k^2 + \|u(s)\|_{k-1}^2 \mathrm{d}s.
\end{aligned} \tag{29}$$

Using the first and the third equation of (EPK2), we have

$$K_5 = \int_0^t \int_{\mathbf{R}^2} \nabla^k \Phi \nabla^k \nabla \cdot (\rho u) - \nabla^k \Phi \nabla \cdot ([\nabla^k, \rho] u) \mathrm{d}x \mathrm{d}s$$
$$=: K_{51} + K_{52}.$$

We can deduce that

$$K_{51} = \|\nabla^{k+1} \Phi\|_0^2 \Big|_0^t. \tag{30}$$

Thanks to $k \geq 1$, using the second-order elliptic PDEs regularity theory and integrating by parts, we have

$$K_{52} = \int_0^t \int_{\mathbf{R}^2} \nabla^k \nabla \Phi \cdot ([\nabla^k, \rho] u) \mathrm{d}x \mathrm{d}s \leq -C_\delta \int_0^t \|\rho(s) - \bar{\rho}\|_k^2 + \|u(s)\|_{k-1}^2 \mathrm{d}s. \tag{31}$$

Therefore we have

$$K_5 \geq \|\nabla^{k+1} \Phi\|_0^2 \Big|_0^t - C_\delta \int_0^t \|\rho(s) - \bar{\rho}\|_k^2 + \|u(s)\|_{k-1}^2 \mathrm{d}s. \tag{32}$$

The sixth term gives

$$\begin{aligned} K_6 =& - \int_0^t \int_{\mathbf{R}^2} \rho \nabla^k \rho^{-1} \Delta u \cdot \nabla^k u \mathrm{d}x \mathrm{d}s \\ =& \int_0^t \|\nabla^{k+1} u(s)\|_0^2 \mathrm{d}s + \int_0^t \int_{\mathbf{R}^2} \rho [\nabla^k, \rho^{-1}] \Delta u \cdot \nabla^k u \mathrm{d}x \mathrm{d}s \\ \geq& C \int_0^t \|\nabla^{k+1} u(s)\|_0^2 \mathrm{d}s - C_\delta \int_0^t \|\nabla u(s)\|_{k-1}^2 + \|\nabla(\rho(s) - \bar{\rho}_0)\|_{k-1}^2 \mathrm{d}s. \end{aligned} \tag{33}$$

Combining (26)-(33) and (24), we obtain

$$\|w(t)\|_k^2 + \|\nabla \rho(t)\|_k^2 + \int_0^t \|u(s)\|_{k+1}^2 \mathrm{d}s \leq C_\delta (\|w_0\|_k^2 + \|\nabla \rho_0\|_k^2). \tag{34}$$

Next, multiplying the second equation of (EPK2) by ρ^{-1} and applying the operator $\rho \nabla^k$, then multiplying by $\nabla^k \nabla \rho$ and integrating over $(0, T) \times \mathbb{R}^N$, we have

$$\begin{aligned} &\int_0^t \int_{\mathbf{R}^2} \Big[\rho \nabla^k (\partial_t u_j + u_i \partial_i u_j) \nabla_j \nabla^k \rho + \rho \nabla^k \rho^{-1} \nabla_j p(\rho) \nabla_j \nabla^k \rho + \rho \nabla^k u_j \nabla_j \nabla^k \rho \\ &\qquad - \rho \nabla_j \Delta \rho \nabla_j \nabla^k \rho - \rho \nabla^k \nabla_j \Phi \nabla_j \nabla^k \rho - \rho \nabla^k \rho^{-1} \Delta u_j \nabla_j \nabla^k \rho \Big] \mathrm{d}x \mathrm{d}s \\ =:& L_1 + L_2 + L_3 + L_4 + L_5 + L_6 = 0. \end{aligned} \tag{35}$$

Using a similar method to (17) to estimate the first term L_1, we obtain

$$\begin{aligned} |L_1| \leq& C_\delta (\|\nabla(\rho(s) - \bar{\rho}_0)\|_k^2 + \|\nabla u(s)\|_{k-1}^2 + \|w_0\|_k^2 + \|\rho_0 - \bar{\rho}_0\|_{k+1}^2) \\ &+ C_\delta \int_0^t \|\nabla u(s)\|_k^2 + \|\nabla(\rho(s) - \bar{\rho}_0)\|_k^2 \mathrm{d}s. \end{aligned} \tag{36}$$

Integrating by parts and using (3) and (4), we obtain

$$\begin{aligned} L_2 =& \int_0^t \int_{\mathbf{R}^2} \nabla^k \nabla h(\rho) \cdot \rho \nabla^k \nabla \rho \mathrm{d}x \mathrm{d}s \\ \geq & C\delta \int_0^t \|\nabla^{k+1}(\rho(s) - \bar{\rho}_0)\|_0^2 \mathrm{d}s - C_\delta \int_0^t \|(\rho(s) - \bar{\rho}_0)\|_k^2 \mathrm{d}s. \end{aligned} \tag{37}$$

Obviously we have

$$L_3 \leq C\delta \int_0^t \|\nabla^k u(s)\|_0^2 + \|\nabla^{k+1}(\rho(s) - \bar{\rho}_0)\|_0^2 \mathrm{d}s. \tag{38}$$

Integrating by parts, the fourth term gives

$$\begin{aligned} L_4 =& -\int_0^t \int_{\mathbf{R}^2} \rho \nabla^k \nabla \Delta \rho \cdot \nabla^k \nabla \rho \mathrm{d}x \mathrm{d}s \\ =& \int_0^t \int_{\mathbf{R}^2} \rho (\nabla^k \Delta \rho)^2 \mathrm{d}x \mathrm{d}s + \int_0^t \int_{\mathbf{R}^2} \nabla^k \Delta \rho \nabla \rho \cdot \nabla^k \nabla \rho \mathrm{d}x \mathrm{d}s \\ \geq & C\delta \int_0^t \|\nabla^{k+2}(\rho(s) - \bar{\rho}_0)\|_0^2 \mathrm{d}s - C_\delta \int_0^t \|\nabla^{k+1}(\rho(s) - \bar{\rho}_0)\|_0^2 \mathrm{d}s \end{aligned} \tag{39}$$

The fifth term using similar method on (21) gives

$$\begin{aligned} L_5 =& -\int_0^t \int_{\mathbf{R}^2} \rho \nabla^k \nabla \Phi \nabla^k \nabla \rho \mathrm{d}x \mathrm{d}s \\ =& -\int_0^t \int_{\mathbf{R}^2} (\rho - \bar{\rho}_0) \nabla^k \nabla \Phi \cdot \nabla^k \nabla (\rho - \bar{\rho}_0) - \bar{\rho}_0 \nabla^k \nabla \Phi \cdot \nabla^k \nabla (\rho - \bar{\rho}_0) \mathrm{d}x \mathrm{d}s \\ =:& L_{51} + L_{52}. \end{aligned} \tag{40}$$

Obviously, we have

$$L_{52} = \bar{\rho}_0 \int_0^t \|\nabla^k (\rho(s) - \bar{\rho}_0)\|_0^2 \mathrm{d}s. \tag{41}$$

Thanks to $k \geq 1$, using the third equation of (EPK2), we obtain

$$L_{51} \geq c_\delta \int_0^t \|\nabla^k (\rho(s) - \bar{\rho}_0)\|_0^2 \mathrm{d}s - C_\delta \int_0^t \|\rho(s) - \bar{\rho}_0\|_{k-1}^2 \mathrm{d}s. \tag{42}$$

Therefore, we have

$$L_5 \geq c_\delta \int_0^t \|\nabla^k (\rho(s) - \bar{\rho}_0)\|_0^2 \mathrm{d}s - C_\delta \int_0^t \|\rho(s) - \bar{\rho}_0\|_{k-1}^2 \mathrm{d}s. \tag{43}$$

Obviously, we obtain

$$L_6 \leq \frac{C}{\delta} \int_0^t \|\nabla u(s)\|_k^2 \mathrm{d}s + C\delta \int_0^t \|\nabla^2 (\rho(s) - \bar{\rho}_0)\|_k^2 \mathrm{d}s. \tag{44}$$

Combining (36)-(44), (24) and (34), we obtain our result. This finishes the proof of Theorem 1.2.

3. Conclutsions

The above proofs illustrate that the solution of the equation exists globally and the Sobolev norm of global solutions is controlled by the Sobolev norm of the initial data in the case of small initial values. The study in this article provides a reliable basis for the numerical solution of this equation and other properties of the equation.

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