Relationship Among Some Selected Probability Distributions

Abstract
Choosing the right probability distribution for a particular phenomenon of interest sometimes becomes an issue for researchers. Relationship among probability distributions can be explored by transformation of random variables, combination of multiple variables and by approximations. This study reviewed some of these commonly used probability distributions and their properties and highlighted their basic differences and links. The distributions considered were both discrete and continuous in nature. While algebraic processes were developed to establish the links among the selected probability distributions, a schematic relationship was utilized to foster better understanding of the relationships among the selected distributions.

Keywords: Probability distributions, Random variables, Moments, Relationship and Approximations

1.0 Introduction

The probability of an event is the possibility of that event to occur subject to pre-event factors. Probability takes its root from the concept of relative frequency of an observed event. It's a way of inferring what might h if certain events have occurred. An event is almost sure if it takes probability 1 while the event that will not h with certainty takes value 0. Hence, for a number of events which assume probability values between 0 and 1, the sum of all the probabilities is 1. The description of the concepts of probability by mathematical functions and graphs is termed probability distribution. There are different distributions that characterized life phenomena; knowledge of such is handy in statistical inference and other modelling problems that involve uncertainty in the parameters of interest (Nwosu et al, 2016).

The concept of distribution is an important aspect of data science that provides foundation for inferential statistics. Reviewing relationships between different probability distributions is handy because finding the probability of a certain distribution sometimes depends on the table of another distribution. Also, parameter estimations and simulation techniques rely heavily on good knowledge of probability distributions as limiting form or summations, product or sometimes integrations of one or more probability distributions lead to another distribution (Viti, Terzi, & Bertolaccini, 2015).

The common distributions are interwoven and somehow confusing especially for non-statistical experts (Wroughton & Cole, 2013). Thus, there is always a problem in the selection of the right probability distribution to use in a particular scenario. When the variables in some distributions change, the distributions are reduced to some other simple distributions. The relationships that exist among distributions can also be said to be specific, general, special, identical or functional in nature (Wen, 2022; Gao, 2018; Mao et al, 2011).

2.0 Probability Distributions

2.1 Bernoulli Distribution

If an experiment is performed with two possible outcomes (success or failure), a random variable that takes value 1 in case of success and 0 in case of failure constitutes a Bernoulli random variable(Oyeka, 2013). Let X be a discrete random variable which supports $R_X = \{0,1\}$. The random variable X has a Bernoulli distribution with parameter $p \in (0,1)$ if its probability mass function is

$$p(x) = \begin{cases} p & \text{if } x = 1 \\ 1-p & \text{if } x = 0 \\ 0 & \text{if } x \notin R_X \end{cases} \quad (1)$$

Mean(x)= p and Variance (x) = p(1-p)

2.2 Binomial Distributions

Let X be a discrete random variable. Let $n \in$N, p $\in (0,1)$ and the support of X be $R_x = \{0,1,\ldots n\}$. The random variable X has a Binomial distribution with parameters n and p if its probability mass function is given by (2).

$$p(x) = \binom{k}{x} p^x (1-p)^{k-x} \quad x = 0,1,2,\ldots,k; 0 \leq p \leq 1 \quad (2)$$

$Mean(x) = kp$ and $Var(x) = kp(1-p)$.

The Binomial distribution brings out Bernoulli, Hypergeometric and Negative Binomial distributions (Banik & Kibria, 2009; Wackerly, Mendenhall, & Scheaffer, 2008). When the number of trials, $x = 1$ in (1), the trial becomes a Bernoulli trial. When the Binomial experiment is done without replacement, we have Hypergeometric distribution. However, when the number of trials in Binomial trials is random until a desired number of success or failure is achieved gives rise to Negative Binomial Distribution.

2.3 Hypergeometric Distribution

The Hypergeometric distribution involves sampling from a finite population without replacement. Hence, the independent assumption of the Binomial distribution is relaxed in the Hypergeometric distribution. However, when sample size is small relative to the population size, the random variable can be approximated to Binomial distribution provided other Binomial conditions are met. Consider a box with b blue balls and N-b white balls, if m blue balls are chosen at random from b blue balls and n-m white balls from the white balls, then the random variable m is said to be Hypergeometric distributed with probability mass function given in (2).

$$f(m; b, n, N) = \frac{\binom{b}{m}\binom{N-b}{n-m}}{\binom{N}{n}} \ \forall \ m\epsilon\{\max(0, n-(N-b)), 1, \ldots, \min(b,n)\}, b \leq N, n \leq N \quad (3)$$

Where m is the number of observed success from the population size N. The mathematical expectations based on the first two moments about the origin are:

$\text{Mean(m)} = np$; $\text{Var(m)} = \frac{(N-n)nb}{(N-1)N}\left[\frac{N-b}{N}\right] = np(1-p)\frac{(N-n)}{(N-1)}$; where $\text{p} = \frac{b}{N}$

If $\frac{b}{N} = p$, then the mean of the Hypergeometric distribution is the same with that of the Binomial distribution and the variance is only different by a factor $\frac{(N-n)}{(N-1)}$. If $n = 1$, the Hypergeometric distribution becomes Binomial distribution. The Binomial distribution is preferred when the sample size n is small in relation to population size N, otherwise the Hypergeometric distribution should be applied.

2.4 Negative Binomial Distribution

The Negative Binomial distribution (NBD) is applicable where there are discrete dichotomous outcomes until the desired number of successes is achieved. A coins can be tossed repeatedly until m number of heads appears. Hence, the number of trials in NBD is not fixed as in Binomial distribution. If the number of trials denoted by k result in m success, then, the probability mass function for NBD is:

$$f(k; m, p) = \binom{k-1}{m-1} p^m (1-p)^{y-m} \; \forall \; k = m, m+1, \ldots \quad (4)$$

$Mean(k) = \frac{m}{p}$ and $Var(k) = \frac{m(1-p)}{p^2}$

2.5 Geometric Distribution

With the probability function of a negative binomial distribution as given in equation (4), the probability function becomes $P(m) = pq^m$ so that the probabilities of the random variables K for $m = 0,1,2, \ldots$ are in geometric progression with common ratio $1-p$. Thus, a random variable K becomes a geometric distribution if it assumes only the non-negative values and its probability function

$$P(m) = \begin{cases} pq^m, m = 0,1,2,\ldots \; 0 < p < 1 \\ 0, \quad \text{otherwise} \end{cases} \quad (5)$$

$Mean = \left(\frac{1}{p}\right)$ and Var $= \left(\frac{q}{p^2}\right)$

Equation (5) also arises from Bernoulli trials in which the outcome of any trial is either a success or a failure and the probability of success of any trial is p and lack of memory (Bhuyan, 2010).

The Geometric distribution is a special case of NBD in which only a success is required from k number of trials. Hence, when $m = 1$ in NBD, it becomes a Geometric distribution. The maximum number of trials in NBD and Geometric distributions is unknown prior to the experiment. The random variable in geometric distribution is the number of trials until the first success in a repeated Bernoulli experiments.

2.6 Poisson Distribution

The Poisson distribution is widely applied in queue systems where counts of a rare and independent event are measured within a given interval of time, length, area or volume. The longer the period or region of observations, the more the chance to observe more than one object. Examples of events that could be described by the Poisson distribution includes: number of misprints in a typed page, distribution of bacteria on some surface, number of arrivals in a particular polling unit and number of disabled customers arriving a particular bank in a given interval of time can said to be Poisson distributed. If the arrival rate is denoted by a parameter δ and the number of customer denoted by m, then, the probability mass function is:

$$f(m: \delta) = \frac{1}{m!} \delta^m e^{-\delta} \; \forall \; m = 0,1,2, \ldots, \infty. \quad (6)$$

$Mean(m) = Var(m) = \delta$ and $e = 2.71828$

The limiting form of a binomial distribution as the number of trials is large approaches a Poisson distribution with $p = \frac{\delta}{n}$.

2.7 Normal Distribution

The Poisson and other discrete distributions discussed so far present finite number of values within a given interval, there are other set of distributions that allow infinite number of values within a given interval. These are continuous probability distributions. The most important of such is the Normal distribution and all other statistical distributions that revolve around it. This is well captured in central limit theorem. For example, a sample from a Poisson distribution, the sample distribution tends to normality as sample size approaches infinity. This is demonstrated with randomly generated sample of size 1000 from Poisson probability distribution (Figure 1). Hence, the limiting form of discrete and other known distribution gives the Normal distribution.

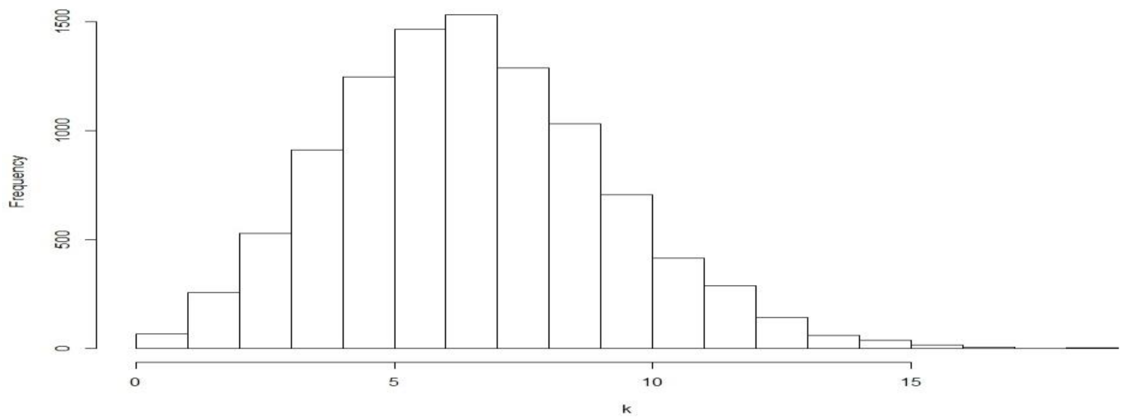

Figure 1. Histogram plot from randomly generated Poisson data

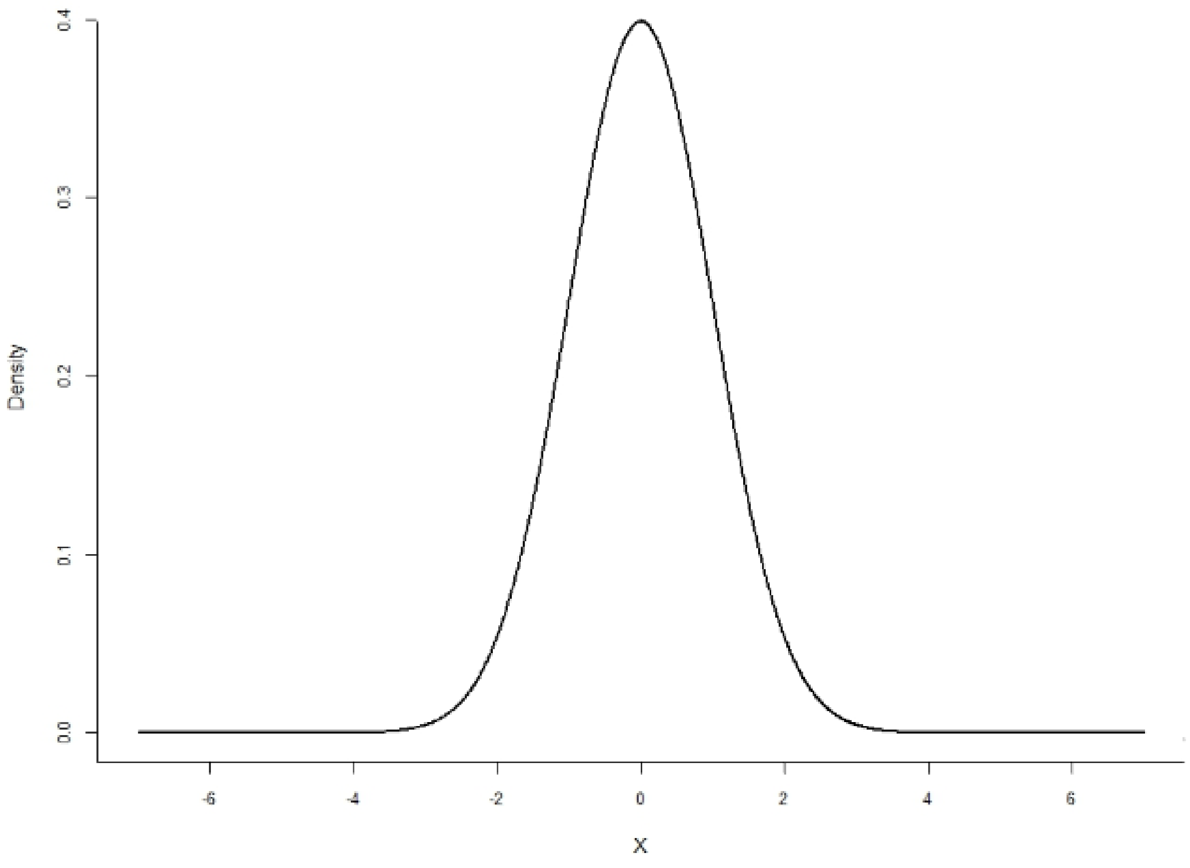

Figure 2. Graphical Description of a Standard Normal Distribution from Number Sequences

The Normal distribution is characterized by two parameters, the mean and the variance. The mean represents the central value while the standard deviation measures how disperse the values are from the mean. As presented in Figure 2, the Normal distribution has the following characteristics:

(i) The area under the curve equals 1
(ii) It is bell shape curve
(iii) It is symmetric around the mean
(iv) The Normal mean, median and mode coincides at the centre of the curve
(v) The probability distribution has two parameters: the mean and the standard deviation.

If a random variable m is distributed as a Normal distribution, the probability density function of m is:

$$f(m;\mu,\delta) = (\frac{1}{2\pi\delta})^{0.5} exp\frac{-(m-\mu)^2}{2\delta^2} \quad \forall -\infty < m < \infty;\ \mu,\delta > 0 \qquad (7)$$

If $\mu = 0$ and $\delta = 1$, it gives a standard Normal distribution.

2.8 Exponential Family of Distributions

There are details of exponential family of probability distributions documented in literature (Ross, 2007), however, only few of these studies established the links between them . In this section, relationships between commonly used probability distributions from exponential family are revisited.

An Exponential distribution is a special case of Gamma family known with modelling the time interval between two successive Poisson events. The Gamma distribution is useful in finding the joint probability distribution of hydrological events (frequency analysis for storms, rainfall for a given time interval). The Gamma distribution also gives probability distribution on the amount of time (waiting time) required for certain number of events or occurrences in Poisson processes. Repeating Exponential distribution with parameter λ for a number of times say n give rise to Gamma with parameters $n\ and\ \lambda$ in (8) while if the shape parameter of Gamma distribution takes only positive integer values, the distribution is described as Erlang distribution which is useful in queue processes to predict waiting times. When n equals 1 in (8), it gives rise to equation (9). Also, when the scale parameter of Gamma(n,λ) is $\frac{1}{2}$ or 2 and shape parameter is $\frac{n}{2}$, then the resulting distribution is the Chi-squared distribution with degrees of freedom equals its mean value in (10).

Likewise, a Beta distribution is related to a Uniform distribution. If the two shape parameters in Beta is unity, then the resulting distribution is the Uniform distribution. Both the Beta and the Uniform distributions serve as prior to commonly used distributions like Binomial in Bayesian analysis. Similarly, the Poisson-Gamma mix distribution produces NBD which is useful in Bayesian and uncertainty analysis. Details relationships of commonly used univariate distributions are display in the schematic diagram (Figure 3). Also, list of commonly encountered continuous probability distributions and their basic properties are displayed in Table 1. Some details of other related probability distributions are available in the literature (Horálek, 2013; Ouarda, Charron, & Chebana, 2016; Rohatgi & Saleh, 2001, Oguntade & Oladimeji, 2023).

$$f(m) = \frac{\lambda^n m^{n-1} exp^{-m\lambda}}{\Gamma(n)}\ ;\ m > 0 \qquad (8)$$

From (8), when n =1 and n = $\frac{n}{2}$, $\lambda = 2$, we have (9) and (10) respectively.

$$f(m)=\lambda exp^{-m\lambda};\ m > 0 \quad (9)$$

$$f(m) = \frac{2^{\frac{n}{2}} m^{\frac{n}{2}-1} exp^{\frac{-m}{2}}}{\Gamma\left(\frac{n}{2}\right)};\ \ m > 0 \quad (10)$$

2.9 Chi-square Distribution

A random variable X has a Chi-square distribution if it can be written as a sum of squares: $X = Y_1^2 + Y_1^2 + \cdots + Y_1^2,\ Y_1, Y_{2,\ldots} Y_n$ are n mutually independent standard normal random variables. Let X be an absolutely continuous random variable. Let its support be the set of positive real numbers: $R_X = [0, \infty)$. If $n \in N,\ then\ X$ has a Chi- square distribution with n degrees of freedom if probability density function (pdf) is

$$f(x) = \begin{cases} cx^{\frac{n}{2}-1}\exp(-\frac{1}{2}x), \text{ if } x \in R_X \\ 0, \quad\quad \text{if } R \notin R_X, \end{cases} \quad (11)$$

where c is a constant: $c = \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}\ and\ \Gamma()$ is the Gamma function

in essence, Chi-square variate implies the square of a standard normal variate. Thus, if U is a normal variate with mean $\mu\ and\ variance\ \sigma^2$ then $X = \frac{U-\mu}{\sigma} \sim N(0,1)\ and\ X^2 = (\frac{U-\mu}{\sigma})^2$ is a Chi-square with 1 degree of freedom. In general, if $U_1, U_2, \ldots U_n$ are independent normal variables with means $\mu_1, \mu_2, \ldots \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \ldots \sigma_n^2$ respectively, then $\chi = \Sigma(\frac{U_i-\mu_i}{\sigma_i})^2$ is a Chi-square variate with n degree of freedom

2.10 Gamma Distribution

Gamma distribution is a generalization of the Chi-square distribution. If a random variable Z has a Chi square distribution with n degrees of freedom and k is a strictly positive constant, then the random variable x defined as $X = \frac{h}{n}Z$ has a Gamma distribution with parameters n and h. Let X be an absolutely continuous random variable and its support be the set of positive real numbers: $R_X = [0, \infty)$. Let $n, h \in \mathbb{R}_{++}$ so that the random variable X has a Gamma distribution with parameters $n\ and\ h$ if its pdf is

$$f(x) = \begin{cases} cx^{\frac{n}{2}-1}\exp(-\frac{n}{h}\frac{1}{2}x), \text{ if } x \in R_X \\ 0, \quad\quad \text{if } R \notin R_X \end{cases} \quad (12)$$

where c is a constant: $c = \frac{(\frac{n}{h})^{\frac{n}{2}}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}\ and\ \Gamma()$ is the Gamma function

2.11 F Distribution

A random variable X has F distribution if it can be written as a ratio $X = \frac{X_{1/}n_1}{X_{2/}n_2}$ between a Chi-square random variable with n_1 degrees of freedom and a Chi-square random variable Y_2 independent of Y_1 with n_2 degrees of freedom.

3.0 Relationship among Selected Distributions

3.1 Binomial Relationship with Bernoulli Distribution

The binomial distribution relates closely with Bernoulli distribution. Recall that a random variable has a binomial distribution with parameters n = 1 and p Iff it has a Bernoulli distribution with parameter p. The two distributions are proposed to be equivalent by showing that they have the same probability mass function (pmf).

The expected pmf of a binomial distribution with n = 1, is

$$p(x) = \begin{cases} \binom{1}{x} p^x (1-p)^{1-x}, if\ x \in \{0,1\} \\ 0\ ,\ if\ x \notin \{0,1\} \end{cases} \quad (13)$$

$$p(x) = \binom{1}{0} p^0 (1-p)^{1-0} = \frac{1!}{0!1!}(1-p) = 1-p \quad (14)$$

$$p(1) = \binom{1}{1} p^1 (1-p)^{1-1} = \frac{1!}{1!0!} p = p \quad (15)$$

Thus, equation (13) becomes (1)

3.2 Poisson Distribution Relationship with Exponential Distribution

Let $r_1, r_2, \ldots. r_n$ be the time elapsed for the first occurrence, between the first and second occurrences and between $(n+1)^{th}$ and n^{th} occurrences respectively

Let X be the number of occurrences of the event so that

$X \geq x\ iff\ r_1 + . . . + r_n \ \leq 1.$

This is true if $p(X \geq x) = P(r_1 + .. + r_n \leq 1) \quad \forall\ x \epsilon R_X$ ………………….

This can be verified as follows:

$P(r_1 + \cdots + r_n \leq 1)$

Also, let $z = r_1 + \cdots + r_n$ be the sum of waiting time. The sum of independent exponential random variables with parameter λ is a Gama random variable with parameters $2x$ and $\frac{x}{\lambda}$

The pdf is given as

$$fz(x) = \begin{cases} cx^{x-1} \quad \exp(-\lambda x)\, , if\ x\epsilon\{0, \infty\} \\ 0 \qquad\qquad\quad if\ x \notin \{0, \infty\}, \end{cases}$$

where $c = \frac{\lambda^x}{\Gamma(x)} = \frac{\lambda^x}{(x-1)!}$

Integrating $f_z(x)$, we have

$$p(r_1 + \cdots + r_x \leq 1) = p(z \leq 1)$$

$$= \int_{-\infty}^{1} fz(x)dx$$

$$= c\int_0^1 z^{x-1} \exp(-\lambda z)\, dx.$$

$$= c[\sum_{i=1}^{x-1} \frac{(x-1)!}{(x-i)!} . \frac{1}{\lambda i} \exp(-\lambda) - \frac{(x-1)!}{\lambda^x} . \exp(-\lambda) + \frac{(x-1)!}{\lambda^x}$$

$$= 1 - \sum_{j=0}^{x-1} \frac{\lambda!}{j!} \exp(-\lambda)$$

Computing $P(X \geq x) = 1 - P(X \leq x)$

$$= 1 - P(X \leq x - 1)$$

$$= 1 - \sum_{j=0}^{x-1} P\, x(j)$$

$$= 1 - \sum_{j=0}^{x-1} \frac{\lambda j}{j!} \exp(-\lambda)$$

$$= P(r_1 + \cdots + r_x \leq 1)$$

3.3 Binomial Distribution Relationship with Poisson Distribution

Poisson distribution is a limit case of binomial distribution.

(i) For a binomial, the parameter n is indefinitely large: $n \to \infty$

(ii) The probability of success for each trial is constant and indefinitely small : $P \to 0$

(a) np is finite so that if $np = \theta,\ p = \frac{\theta}{n}, \quad q = 1 - \frac{\theta}{n}$.

Since by definition,

$$b(x, n, p) = \binom{h}{x} p^x q^{n-x} = \binom{n}{x} \left(\frac{P}{1-p}\right)^x (1-p)^n$$

$$= \frac{n(n-1)(n-2)\ldots(n-x+1)}{x!} \frac{\left(\frac{\theta}{n}\right)^x}{(1-\frac{\theta}{n})^x} \left(1 - \frac{\theta}{h}\right)^n$$

$$= \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \ldots \left(1 - \frac{x-1}{n}\right) \theta^x \left(1 - \frac{\theta}{h}\right)^n$$

$$b(x, n, p) = \frac{e^{-\theta}\theta^x}{x!}, \quad x = 0, 1,2, \ldots$$

3.4 Poisson Distribution as a limiting case of the Negative Binomial Distribution

In this case, Negative Binomial Distribution is approximated to a poison distribution. As r tends to infinite and p tends to zero with rp remaining fixed, the negative binomial distribution tends to a poisson distribution. Therefore

$$\lim_{r\to\infty} P(x) = \lim_{r\to\infty} \binom{x + r - 1}{r - 1} p^r q^r$$

$$= \lim_{r\to\infty} \binom{x + r - 1}{r - 1} q^{-r} \left(\frac{p}{q}\right)^x$$

$$\lim_{r\to\infty} \frac{(x+r-1)\ (x+r-2)\ldots\ (r+1)^r}{x!}\ (1+p)^{-r}$$

$$\left(\frac{P}{1+P}\right)^x$$

$$= \lim_{q\to\infty} \left[\frac{1}{x!} \left(1 + \frac{x-1}{r}\right)\left(1 + \frac{x-2}{r}\right) \ldots (1 +). 1. r^x\ (1+p)^{-r}\ \left(\frac{P}{1+P}\right)^x\right]$$

$$= \frac{1}{x!} \lim_{r\to\infty} [\left(1 + p\right)^{-6}\ \left(\frac{rP}{1+P}\right)^x]$$

$$= \frac{\theta^x}{x!} \lim_{r\to y} \left[\left(1 + \frac{\theta}{r}\right)\right]^{-r} \lim_{r\to\infty} \left(1 + \frac{\theta}{r}\right)^{-x}$$

$$\frac{\theta^x}{x!} e^{-\theta}. 1 = e^{-\theta} \frac{\theta^x}{x}!$$

3.5 Approximation of Hypergeometric Distribution to Binomial Distribution

Let the pdf of Hypergeometric distribution be

$$P(X = k) = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}\ ,\ k = 0, 1, 2 \ldots, \min(n, M)$$

As $N \to \infty$ and $\frac{M}{N} = p$,

$$P(X = k|M, n\ N) = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}$$

$$= \frac{M! \quad . \quad (N-M)!}{K!(M-K)!(n-k)!(N-M-n+k)} \qquad \frac{n!(N-n)!}{N!}$$

$$= \frac{(M-1)(M-2)\dots(M-K+1)}{K!}$$

$$\frac{(N-M)(N-M-1)\dots(N-M-n+K+1)}{(n-k)!}$$

$$\frac{n!}{N(N-1)(N-2)\dots(N-n+1)}$$

$$\frac{n!}{k!(n-k)}\frac{M}{N}\left(\frac{M}{N}-\frac{1}{N}\right)\left(\frac{M}{N}-\frac{2}{N}\right)\dots\left(\frac{M}{N}-\frac{K-1}{N}\right).\ .\frac{\left(-\frac{M}{N}\right)\left(1-\frac{M}{N}-\frac{1}{N}\right)\dots(1-\frac{M}{N}-\frac{n-k-1}{N}}{\left(1-\frac{1}{N}\right)\left(1-\frac{1}{N}\right)\dots(1-\frac{n-1}{N}}$$

As limit $N \to \infty$ and setting $\frac{M}{N} = p$, we have

$$\text{Lim}\ P(X = k|M, n, N) = \frac{n1}{k!(n-k)!}\ (1-p)(1-p)\ \ p.p\dots p\ \ \dots.\ \ (1-p)$$

$$= \binom{n}{k} p^k\ \ q^{n-k}$$

3.6 Gamma Distribution with Chi-square Distribution.

The Gamma distribution is a scaled Chi-square distribution. Let X be a Gamma distribution random variable with parameter n and h, then

$$X = \frac{h}{n} z$$

Z has a Chi-square distribution with n degree of freedom

Proof:

Let $fx(x)$ be absolutely continuous variable so that $fx(x) = fz(g-1)(x)\frac{dg^{-1}(x)}{dx}$

$$= fz(\frac{n}{h^2})\frac{n}{h}$$

The pdf of a Chi-square random variable with n degree of freedom becomes

$$f(z) = \begin{cases} kz\frac{n}{2} - 1 \quad \exp\left(-\frac{1}{2}x\right),\ x\epsilon\{0, \infty\} \\ 0, \qquad\qquad\qquad otherwise, \end{cases}$$

where $k = \frac{1}{2^{n/2}\gamma(n/2)}$

Thus,

$$fx(x) = fz\left(\frac{n}{h}x\right)\frac{n}{h}$$

$$= \begin{cases} k(\frac{n}{h})^{n/2} x^{1/2 - 1} \quad \exp\left(-\frac{1}{2}\frac{n}{h}x\right),\ x\epsilon\{0, \infty\} \\ 0, \qquad\qquad\qquad Otherwise \end{cases}$$

3.7 Gamma Distribution Relationship with Normal Distribution

Let a Chi-square random variable z with $n \in N$ degree freedom be written as a sum of squares of n independent normal random variables.

$Y_1 \dots Y_n$ with $\mu = 0, \quad \sigma^2 = 1$

Thus $z = Y_1^2 + \cdots + Y_n^2$

Let a variable X has a Gamma distribution with parameters n and h be written as

$$X = \frac{h}{n} z$$

By chasing the algebra,

$\times = \frac{h}{n} z = \frac{h}{n}(Y_1^2 + \cdots + Y_n^2)$

$$= \left(\sqrt{\frac{h}{n}}\, Y_1\right)^2 + \cdots + \left(\sqrt{\frac{h}{n}} Y_n\right)^2$$

$= Y_1^2 + \cdots + Y_n^2$, where $Y_i = \sqrt{\frac{h}{n}} W_i \quad i = 1,2 \ldots n \sim N(0, \frac{h}{n})$

3.8 F Distribution Relationship with Gamma Distribution

Let the pdf of a $random\ variable$ X be

$fX(x) = \int_0^\infty fX\,|z = z(x) fz(z) dz,$

where

$fx|z = z(x)$ is the pdf of a Gamma $random\ variable$ nth parameters n_1 and $h_1 = \frac{1}{x}$

(a) $fx|z = z(x) = \frac{(\frac{n_1}{h_1})^{\frac{n_1}{2}}}{2^{n_1}\Gamma(\frac{n_1}{2})}\, x^{\frac{n_1}{2}-1} ex\left(-\frac{n_1}{h_1}\frac{1}{2}x\right)$

$$= \frac{(n_1.z)^{\frac{n_1}{2}}}{2^{\frac{n_1}{2}}\Gamma\left(\frac{n_1}{2}\right)^{x^{\frac{n_1}{2}-1}}} \exp\left(-n_2 \frac{1}{2} x\right)$$

(b) $fz(z)$ is the pdf of a Gamma random variable with parameters n_2 and $h_2 = 1$ so that

$$fz(z) = \frac{(n_2)^{\frac{n_2}{2}}}{2^{\frac{n_2}{2}}\,\Gamma(\frac{n_2}{2})} z^{\frac{n_2}{2}-1} \exp\left(-n_2 \frac{1}{2} z\right)$$

This can be established by proving

$fx(x) = \int_0^\infty f \times |z = z(x) fz(z) dz,$

where

$$fX|z = z(x) = \frac{(n_1 z)^{n_1/2}}{2^{\frac{n_1}{2}}\Gamma(\frac{n_1}{2})}\, x^{\frac{n_2}{2}-1} \exp\left(-n_1 z \frac{1}{2} x\right)$$

$$fz(z) = \frac{n_2^{\frac{n_2}{2}}}{2^{\frac{n_2}{2}}\Gamma(\frac{n_2}{2})} x^{\frac{n_2}{2}-1} \exp\left(-n_2 z \frac{1}{2} z\right)$$

$$fx|z = z(x)\ \ fz(z) = \frac{(n_1 z)^{\frac{n_1}{2}}}{2^{\frac{n_1}{2}}\Gamma(\frac{n_1}{2})} x^{\frac{n_1}{2}-1} \exp\left(-n_1 z \frac{1}{2} x\right)$$

$$= \frac{n_2^{\frac{n_2}{2}}}{2^{\frac{n_2}{2}}\Gamma\left(\frac{n_2}{2}\right)} z^{\frac{n_2}{2}-1} \exp\left(-n_2 \frac{1}{2} z\right)$$

$$= \frac{n_1^{\frac{n_2}{2}}\ n_2^{\frac{n_2}{2}}}{2^{\left(\frac{n_1+n_2}{2}\right)}\Gamma\left(\frac{(n_1+n_2)}{2}\right)} x^{\frac{n_1}{2}-1} \frac{1}{c} fz|\times = x(z)$$

where $c = \frac{(n_1+n_2)^{(\frac{n_1+n_2}{2})}}{2^{\left(\frac{n_1+n_2}{2}\right)}\Gamma\left(\frac{(n_1+n_2)}{2}\right)}$

$fz|\times = x(z)$ is the pdf of a random variable having a Gamma distribution with parameters

$h_1 + n_2$ and $\frac{n_{1+}n_2}{n_1 x + n_2}$

Thus, $\int_0^\infty f \times |z = (x) fz(z) dz$

$$=\int_0^\infty \frac{n_1^{\frac{n_1}{2}}\, n_2^{\frac{n_2}{2}}}{2\left(\frac{n_1+n_2}{2}\right)\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_1}{2}\right)}\, x^{\frac{n_1}{2}-1}\,\frac{1}{c} fz|X = x(z)dz$$

$$= \frac{n_1^{\frac{n_1}{2}}\, n_2^{\frac{n_2}{2}}}{2^{\left(\frac{n_1+n_2}{2}\right)}\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\, x^{\frac{n_1}{2}-1}\,\frac{1}{c}\int_0^\infty f\ z|X = x(z)dz$$

$$= \frac{n_1^{\frac{n_1}{2}}\, n_2^{\frac{n_2}{2}}}{2^{\left(\frac{n_1+n_2}{2}\right)}\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\, x^{\frac{n_1}{2}-1}\,\frac{1}{c}$$

$$= \left(\frac{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}{\Gamma\left(\frac{(n_1+n_2)}{2}\right)}\right)^{-1}\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} x^{\frac{n_1}{2}-1}\left(1+\frac{n_1}{n_2}x\right)^{-\left(\frac{n_1+n_2}{2}\right)}$$

$$= \frac{1}{B\left(\frac{n_1}{2},\frac{n_2}{2}\right)}\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} x^{\frac{n_1}{2}-1}\left(1+\frac{n_1}{n_2}x\right)^{-\frac{(n_1+n_2)}{2}} = fX(x)$$

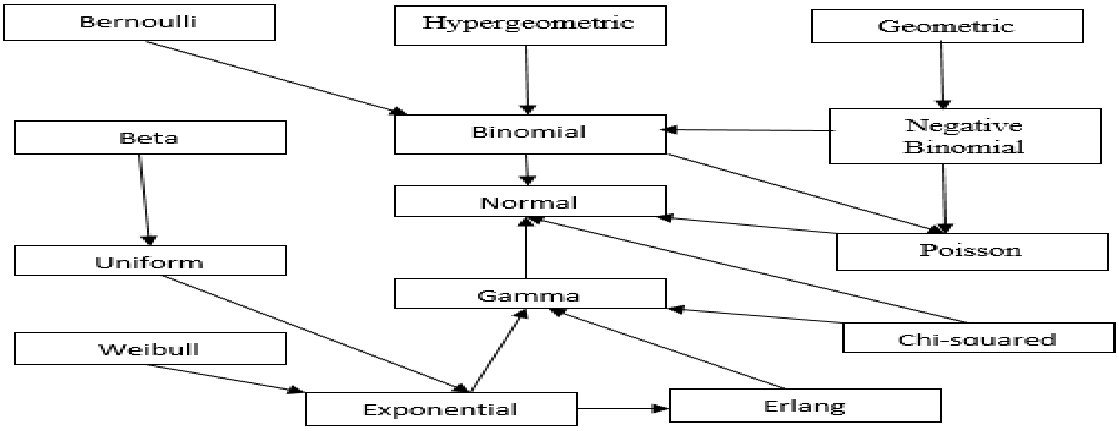

Figure 3. Schematic Relationships of Commonly used Univariate Distributions

Table 1. List of Some Selected Continuous Probability Distributions, Domains and Properties

Probability distribution	Probability density function $f(m)$	Domain	Parameters	Mean	Variance
Beta	$f(m) = \frac{\Gamma(n+\lambda)}{\Gamma(n)\Gamma(\lambda)} m^{n-1}(1-m)^{\lambda-1}$	$0 < m < 1$	2 shape $n > 0$ $\lambda > 0$	$\frac{n}{n+\lambda}$	$\frac{n\lambda}{(n+\lambda)^2(n+\lambda+1)}$
Uniform	$f(m) = \frac{1}{b-a}$	$a < m < b$	$-\infty < a < b < \infty$	$\frac{(a+b)}{2}$	$\frac{(b-a)^2}{12}$
Gamma	$f(m) = \frac{\lambda^n m^{n-1} exp^{-m\lambda}}{\Gamma(n)}$	$m > 0$	1 scale 1 shape $n, \lambda > 0$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$
Exponential	$f(m) = \lambda exp^{-m\lambda}$	$m > 0$	1 scale $\lambda > 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Erlang	$f(m) = \frac{\lambda^n m^{n-1} exp^{-m\lambda}}{(n-1)!}$	$m > 0$	1 shape 1 rate $n, \lambda > 0$ $(n \in \mathrm{N})$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Chi-Squared	$f(m) = \frac{\lambda^{\frac{n}{2}} m^{\frac{n}{2}-1} exp^{\frac{-m}{2}}}{\Gamma\left(\frac{n}{2}\right)}$	$m > 0$	$n > 0$ $(n \in \mathrm{N})$	n	$2n$

4.0 Conclusion

Identification of the characteristics that describe random variables of interest helps to find the right probability distributions for our study. These distributions and their links as highlighted in the review are handy for successful modelling, predictions and general assignment of probability distributions to random variables. Binomial, Hypergeometric, Negative Binomial and Poisson discrete distributions and exponential, Gamma, Chi-square, F continuous distributions are discussed and relationships between them established.

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Details of the AI usage are given below:

1.
2.
3.

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