A Comprehensive Study of k-Circulant Matrices Derived from Generalized Padovan Numbers

Abstract. This paper presents a review of the k-circulant matrices and the generalized Padovan numbers, it further outlines the importance of these numbers and matrices with regard to matrices analysis and number theory. Considering the potential practical applications of k-circulant matrices in combination and numerical analysis, we derive explicit formulas for sum of entries, maximum column and row sum norms, Euclidean norm, spectral norm, eigenvalues and determinant of these matrices. Our research also shows the analytical relationships which exist between the usual structure of k-circulant matrices and generalized Padovan numbers that could be useful for the theoretical and practical researches.

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1. Introduction

Throughout the present section, we review concepts and a few characteristics of the generalized Padovan sequence. The third-order recurrence relations defining a generalized Padovan sequence is given as:

$$\mathcal{V}_n = \mathcal{V}_{n-2} + \mathcal{V}_{n-3} \tag{1.1}$$

with some value of $\mathcal{V}_0 = d_0$, $\mathcal{V}_1 = d_1$, $\mathcal{V}_2 = d_2$ that are not all zero.

It is possible to extend the sequence $\{\mathcal{V}_n\}_{n\geq 0}$ is extended to include negative subscripts through the following definition

$$\mathcal{V}_{-n} = -\mathcal{V}_{-(n-1)} + \mathcal{V}_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Thus, the recurrence relation (1.1) is true for all values of n.

The Binet representation for the generalized Padovan numbers is expressed as follow

$$\mathcal{V}_n = \frac{\beta_1 \alpha_1^n}{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)} + \frac{\beta_2 \alpha_2^n}{(\alpha_2 - \alpha_1)(\alpha_2 - \alpha_3)} + \frac{\beta_3 \alpha_3^n}{(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)}$$

where

$$\begin{aligned} \beta_1 &= \mathcal{V}_2 - (\alpha_2 + \alpha_3)\mathcal{V}_1 + \alpha_2\alpha_3\mathcal{V}_0, \\ \beta_2 &= \mathcal{V}_2 - (\alpha_1 + \alpha_3)\mathcal{V}_1 + \alpha_1\alpha_3\mathcal{V}_0, \\ \beta_3 &= \mathcal{V}_2 - (\alpha_1 + \alpha_2)\mathcal{V}_1 + \alpha_1\alpha_2\mathcal{V}_0. \end{aligned}$$

In this context, α_1, α_2 and α_3 denote the solutions to the cubic equation $x^3 - x - 1 = 0$. Moreover

$$\begin{aligned} \alpha_1 &= \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} = 1.324\,717\,957\,24 \\ \alpha_2 &= \omega\left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \omega^2\left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} \\ \alpha_3 &= \omega^2\left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \omega\left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} \end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= 0, \\ \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 &= -1, \\ \alpha_1\alpha_2\alpha_3 &= 1. \end{aligned}$$

Now, we shall delineate four distinct instances of the sequence $\{\mathcal{V}_n\}$. The Padovan sequence (Cordonnier), denoted as $\{\mathcal{P}_n\}_{n\geq 0}$, the Perrin sequence (Padovan-Lucas), represented as $\{\mathcal{E}_n\}_{n\geq 0}$, the Padovan-Perrin sequence, indicated as $\{\mathcal{S}_n\}_{n\geq 0}$, and the modified Padovan sequence, referred to as $\{\mathcal{A}_n\}_{n\geq 0}$ are given as follows by the corresponding third-order recursive equations:

$$\begin{aligned} \mathcal{P}_{n+3} &= \mathcal{P}_{n+1} + \mathcal{P}_n, \qquad \mathcal{P}_0 = 1, \mathcal{P}_1 = 1, \mathcal{P}_2 = 1, \\ \mathcal{E}_{n+3} &= \mathcal{E}_{n+1} + \mathcal{E}_n, \qquad \mathcal{E}_0 = 3, \mathcal{E}_1 = 0, \mathcal{E}_2 = 2, \\ \mathcal{S}_{n+3} &= \mathcal{S}_{n+1} + \mathcal{S}_n, \qquad \mathcal{S}_0 = 0, \mathcal{S}_1 = 0, \mathcal{S}_2 = 1, \\ \mathcal{A}_{n+3} &= \mathcal{A}_{n+1} + \mathcal{A}_n, \qquad \mathcal{A}_0 = 3, \mathcal{A}_1 = 1, \mathcal{A}_2 = 3. \end{aligned}$$

It is noteworthy that the situation $\mathcal{V}_n = R_n$, $R_0 = 1, R_1 = 0, R_2 = 1$ (or $\mathcal{V}_n = R_n$, $R_0 = 0, R_1 = 1, R_2 = 0$) is referred to in the literature as the Van der Laan sequence.

The sequences $\{\mathcal{P}_n\}_{n\geq 0}$, $\{\mathcal{E}_n\}_{n\geq 0}$, $\{\mathcal{S}_n\}_{n\geq 0}$ and $\{\mathcal{A}_n\}_{n\geq 0}$ can be defined to encompass negative indices by assigning corresponding values to

$$\begin{aligned}
\mathcal{P}_{-n} &= -\mathcal{P}_{-(n-1)} + \mathcal{P}_{-(n-3)} \\
\mathcal{E}_{-n} &= -\mathcal{E}_{-(n-1)} + \mathcal{E}_{-(n-3)} \\
\mathcal{S}_{-n} &= -\mathcal{S}_{-(n-1)} + \mathcal{S}_{-(n-3)} \\
\mathcal{A}_{-n} &= -\mathcal{A}_{-(n-1)} + \mathcal{A}_{-(n-3)}
\end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. It should be noted the fact $\mathcal{P}_n$ and $\mathcal{S}_n$ are two different forms of a sequence from [16]. Since these are really nothing more than left-over variations of the same sequence in [16]. Including A000931, A078027, A096231, A124745, A133034, A134816, A164001, A182097, A228361 and very likely A020720 (although they all have their own distinctive features and are best handled as separate entries in themselves quite often). $\mathcal{E}_n$ is the sequence A001608 in [16] and $\mathcal{A}_n$ is the sequence A276276 in [16].

For additional details about the generalized Padovan numbers, refer to the work by Soykan [25].

The following Theorem yields a sum for the generalized Padovan numbers.

Theorem 1.1. *Assume that x is any nonzero real (or complex) number. For the case where n is greater than or equal to zero, the following expression is established: if the condition $x^3 + x^2 - 1 \neq 0$, is not equal to zero, then the summation of the terms*

$$\sum_{k=0}^{n} x^k \mathcal{V}_k = \frac{\Theta_1(x)}{\Theta(x)}$$

can be represented as:

$\Theta_1(x) = x^{n+3}\mathcal{V}_{n+3} + x^{n+2}\mathcal{V}_{n+2} - (x^2-1)x^{n+1}\mathcal{V}_{n+1} - x^2\mathcal{V}_2 - x\mathcal{V}_1 + (x^2-1)\mathcal{V}_0,$

$\Theta(x) = x^3 + x^2 - 1.$

Proof. In [17, Theorem 2.1. (a)], let $r = 0$, $s = 1, t = 1$ $\square$

The following theorem presents several summation formulas for the generalized Padovan numbers.

Theorem 1.2. *For all non-negative integers n, the subsequent formulas are established:*

- **(a):** $\sum_{i=0}^{n} \mathcal{V}_i = \mathcal{V}_{n+3} + \mathcal{V}_{n+2} - \mathcal{V}_2 - \mathcal{V}_1.$
- **(b):** $\sum_{i=0}^{n} i\mathcal{V}_i = (n-2)\mathcal{V}_{n+3} + (n-3)\mathcal{V}_{n+2} - 2\mathcal{V}_{n+1} + 3\mathcal{V}_2 + 4\mathcal{V}_1 + 2\mathcal{V}_0.$
- **(c):** $\sum_{i=0}^{n} \mathcal{V}_i^2 = -\mathcal{V}_{n+3}^2 - \mathcal{V}_{n+2}^2 - 2\mathcal{V}_{n+1}^2 + 2\mathcal{V}_{n+2}\mathcal{V}_{n+3} + 2\mathcal{V}_{n+1}\mathcal{V}_{n+3} + \mathcal{V}_2^2 + \mathcal{V}_1^2 + 2\mathcal{V}_0^2 - 2\mathcal{V}_0\mathcal{V}_2 - 2\mathcal{V}_1\mathcal{V}_2.$
- **(d):** $\sum_{i=0}^{n} i\mathcal{V}_i^2 = -(n+6)\mathcal{V}_{n+3}^2 - (n+5)\mathcal{V}_{n+2}^2 - 2(n+4)\mathcal{V}_{n+1}^2 + 2(n+4)\mathcal{V}_{n+3}\mathcal{V}_{n+2} + 2(n+5)\mathcal{V}_{n+3}\mathcal{V}_{n+1} - 2\mathcal{V}_{n+1}\mathcal{V}_{n+2} + 5\mathcal{V}_2^2 + 4\mathcal{V}_1^2 + 6\mathcal{V}_0^2 - 6\mathcal{V}_1\mathcal{V}_2 - 8\mathcal{V}_0\mathcal{V}_2 + 2\mathcal{V}_0\mathcal{V}_1.$

Proof.

In order to proof of the theorem, we introduce the following substitutions:

- **(a):** Select $x = 1, r = 0$, $s = 1, t = 1$ in [17, Theorem 2.1. (a)] or select $r = 0$, $s = 1, t = 1$ in [20, Theorem 2.1. (a)].

(b): Choose $x=1, r=0,\ s=1, t=1$ in [22, Theorem 2.1. (a)] or choose $r=0,\ s=1, t=1$ in [24, Theorem 2.1. (a)].

(c): Get $x=1, r=0,\ s=1, t=1$ in [19, Theorem 3.1 (a)]. See also [18, Theorem 2.1].

(d): Select $x=1, r=0,\ s=1, t=1$ in [21, Theorem 2.1. (a)] or select $r=0,\ s=1, t=1$ in [23, Theorem 2.1. (a)]. $\square$

It is to be noted that the theorem can also be stated in the form given above using the recurrence relation $\mathcal{V}_{n+3}=\mathcal{V}_{n+1}+\mathcal{V}_n$.

Theorem 1.3. *For $n\geq 0$, the following relationships hold:*

(a): $\sum_{i=0}^{n}\mathcal{V}_i=\mathcal{V}_{n+2}+\mathcal{V}_{n+1}+\mathcal{V}_n-\mathcal{V}_2-\mathcal{V}_1=\dfrac{\Theta_1}{\Theta}$.

(b): $\sum_{i=0}^{n}i\mathcal{V}_i=(n-3)\mathcal{V}_{n+2}+(n-4)\mathcal{V}_{n+1}+(n-2)\mathcal{V}_n+3\mathcal{V}_2+4\mathcal{V}_1+2\mathcal{V}_0=\dfrac{\Psi_1}{\Psi}$.

(c): $\sum_{i=0}^{n}\mathcal{V}_i^2=-\mathcal{V}_{n+2}^2-\mathcal{V}_{n+1}^2-\mathcal{V}_n^2+2\mathcal{V}_{n+1}\mathcal{V}_{n+2}+2\mathcal{V}_n\mathcal{V}_{n+2}+\mathcal{V}_2^2+\mathcal{V}_1^2+2\mathcal{V}_0^2-2\mathcal{V}_0\mathcal{V}_2-2\mathcal{V}_1\mathcal{V}_2=\dfrac{\Delta_1}{\Delta}$.

(d): $\sum_{i=0}^{n}i\mathcal{V}_i^2=-(n+5)\mathcal{V}_{n+2}^2-(n+4)\mathcal{V}_{n+1}^2-(n+6)\mathcal{V}_n^2+2(n+3)\mathcal{V}_{n+2}\mathcal{V}_{n+1}+2(n+4)\mathcal{V}_{n+2}\mathcal{V}_n-2\mathcal{V}_n\mathcal{V}_{n+1}+5\mathcal{V}_2^2+4\mathcal{V}_1^2+6\mathcal{V}_0^2-6\mathcal{V}_1\mathcal{V}_2-8\mathcal{V}_0\mathcal{V}_2+2\mathcal{V}_0\mathcal{V}_1=\dfrac{\Omega_1}{\Omega}$.

Corollary 1.4. *For $n\geq 0$, if $\mathcal{V}_n=\mathcal{P}_n$ represents the Padovan sequence with $\mathcal{P}_0=1$, $\mathcal{P}_1=1$ and $\mathcal{P}_2=1$, the following formulas hold:*

(a): $\sum_{i=0}^{n}\mathcal{P}_i=\mathcal{P}_{n+2}+\mathcal{P}_{n+1}+\mathcal{P}_n-2$.

(b): $\sum_{i=0}^{n}i\mathcal{P}_i=(n-3)\mathcal{P}_{n+2}+(n-4)\mathcal{P}_{n+1}+(n-2)\mathcal{P}_n+9$.

(c): $\sum_{i=0}^{n}\mathcal{P}_i^2=-\mathcal{P}_{n+2}^2-\mathcal{P}_{n+1}^2-\mathcal{P}_n^2+2\mathcal{P}_{n+1}\mathcal{P}_{n+2}+2\mathcal{P}_n\mathcal{P}_{n+2}$.

(d): $\sum_{i=0}^{n}i\mathcal{P}_i^2=-(n+5)\mathcal{P}_{n+2}^2-(n+4)\mathcal{P}_{n+1}^2-(n+6)\mathcal{P}_n^2+2(n+3)\mathcal{P}_{n+2}\mathcal{P}_{n+1}+2(n+4)\mathcal{P}_{n+2}\mathcal{P}_n-2\mathcal{P}_n\mathcal{P}_{n+1}+3$.

From the previous theorem, we can deduce the following corollary, which provides the sum formulas for the Perrin numbers. Let $\mathcal{V}_n=\mathcal{E}_n$, where $\mathcal{E}_0=3, \mathcal{E}_1=0, \mathcal{E}_2=2$.

Corollary 1.5. *The Perrin numbers, for $n\geq 0$, fulfill the following conditions:*

(a): $\sum_{i=0}^{n}\mathcal{E}_i=\mathcal{E}_{n+2}+\mathcal{E}_{n+1}+\mathcal{E}_n-2$.

(b): $\sum_{i=0}^{n}i\mathcal{E}_i=(n-3)\mathcal{E}_{n+2}+(n-4)\mathcal{E}_{n+1}+(n-2)\mathcal{E}_n+12$.

(c): $\sum_{i=0}^{n}\mathcal{E}_i^2=-\mathcal{E}_{n+2}^2-\mathcal{E}_{n+1}^2-\mathcal{E}_n^2+2\mathcal{E}_{n+1}\mathcal{E}_{n+2}+2\mathcal{E}_n\mathcal{E}_{n+2}+10$.

(d): $\sum_{i=0}^{n}i\mathcal{E}_i^2=-(n+5)\mathcal{E}_{n+2}^2-(n+4)\mathcal{E}_{n+1}^2-(n+6)\mathcal{E}_n^2+2(n+3)\mathcal{E}_{n+2}\mathcal{E}_{n+1}+2(n+4)\mathcal{E}_{n+2}\mathcal{E}_n-2\mathcal{E}_n\mathcal{E}_{n+1}+26$.

From the previous theorem, we can deduce the following corollary, which provides the sum formulas for the Padovan-Perrin numbers. Let $\mathcal{V}_n=\mathcal{S}_n$, where $\mathcal{S}_0=0, \mathcal{S}_1=0, \mathcal{S}_2=1$.

Corollary 1.6. *For all $n\geq 0$, the following properties hold for the Padovan-Perrin numbers:*

(a): $\sum_{i=0}^{n}\mathcal{S}_i=\mathcal{S}_{n+2}+\mathcal{S}_{n+1}+\mathcal{S}_n-1$.

(b): $\sum_{i=0}^{n} i\mathcal{S}_i = (n-3)\mathcal{S}_{n+2} + (n-4)\mathcal{S}_{n+1} + (n-2)\mathcal{S}_n + 3$.

(c): $\sum_{i=0}^{n} \mathcal{S}_i^2 = -\mathcal{S}_{n+2}^2 - \mathcal{S}_{n+1}^2 - \mathcal{S}_n^2 + 2\mathcal{S}_{n+1}\mathcal{S}_{n+2} + 2\mathcal{S}_n\mathcal{S}_{n+2} + 1$.

(d): $\sum_{i=0}^{n} i\mathcal{S}_i^2 = -(n+5)\mathcal{S}_{n+2}^2 - (n+4)\mathcal{S}_{n+1}^2 - (n+6)\mathcal{S}_n^2 + 2(n+3)\mathcal{S}_{n+2}\mathcal{S}_{n+1} + 2(n+4)\mathcal{S}_{n+2}\mathcal{S}_n - 2\mathcal{S}_n\mathcal{S}_{n+1} + 5$.

By substituting $\mathcal{V}_n = \mathcal{A}_n$, where $\mathcal{A}_0 = 3, \mathcal{A}_1 = 1, \mathcal{A}_2 = 3$ into the previous theorem, we obtain the following corollary, which provides the sum formulas for the modified Padovan numbers.

Corollary 1.7. *The modified Padovan numbers, for $n \geq 0$, fulfill the following conditions:*

(a): $\sum_{i=0}^{n} \mathcal{A}_i = \mathcal{A}_{n+2} + \mathcal{A}_{n+1} + \mathcal{A}_n - 4$.

(b): $\sum_{i=0}^{n} i\mathcal{A}_i = (n-3)\mathcal{A}_{n+2} + (n-4)\mathcal{A}_{n+1} + (n-2)\mathcal{A}_n + 19$.

(c): $\sum_{i=0}^{n} \mathcal{A}_i^2 = -\mathcal{A}_{n+2}^2 - \mathcal{A}_{n+1}^2 - \mathcal{A}_n^2 + 2\mathcal{A}_{n+1}\mathcal{A}_{n+2} + 2\mathcal{A}_n\mathcal{A}_{n+2} + 4$.

(d): $\sum_{i=0}^{n} i\mathcal{A}_i^2 = -(n+5)\mathcal{A}_{n+2}^2 - (n+4)\mathcal{A}_{n+1}^2 - (n+6)\mathcal{A}_n^2 + 2(n+3)\mathcal{A}_{n+2}\mathcal{A}_{n+1} + 2(n+4)\mathcal{A}_{n+2}\mathcal{A}_n - 2\mathcal{A}_n\mathcal{A}_{n+1} + 19$.

2. Main Results

During this section, we revisit the concepts of k-circulant matrices and various matrix norms, including the Frobenius norm, spectral norm, and the norms based on the maximum column and row lengths. Consider n as an integer such that $n \geq 2$, and let k be any real or complex number. An $n \times n$ matrix $\mathcal{C}_k = (d_{ij}) \in M_{n\times n}(\mathbb{C})$ is termed a k-circulant matrix if it is structured as follows:

$$\mathcal{C}_k = \begin{pmatrix} d_0 & d_1 & d_2 & \cdots & d_{n-2} & d_{n-1} \\ kd_{n-1} & d_0 & d_1 & \cdots & d_{n-3} & d_{n-2} \\ kd_{n-2} & kd_{n-1} & d_0 & \cdots & d_{n-4} & d_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ kd_2 & kd_3 & kd_4 & \cdots & d_0 & d_1 \\ kd_1 & kd_2 & kd_3 & \cdots & kd_{n-1} & d_0 \end{pmatrix}_{n\times n}.$$

This matrix $\mathcal{C}_k$ can be expressed as $\mathcal{C}_k = \text{Circ}_k(d_0, d_1, ..., d_{n-1})$.

When $k = 1$, the 1-circulant matrix is simply referred to as a circulant matrix and denoted by $\mathcal{C}_k = \text{Circ}(d_0, d_1, ..., d_{n-1})$. The concept of circulant matrices was initially introduced by Davis in [4].

The Frobenius norm (or Euclidean norm) and the spectral norm for a $m \times n$ matrix $\mathcal{A} = (e_{ij})_{m\times n} \in M_{m\times n}(\mathbb{C})$ are formally defined as:

$$\|\mathcal{A}\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |e_{ij}|^2\right)^{1/2} \quad \text{and} \quad \|\mathcal{A}\|_2 = \left(\max_{1\leq i\leq n} |\lambda_i(\mathcal{A}^*\mathcal{A})|\right)^{1/2}$$

where $\lambda_i(\mathcal{A}^*\mathcal{A})$ represents the eigenvalues of the matrix $\mathcal{A}^*\mathcal{A}$ and $\mathcal{A}^*$ denotes the conjugate transpose of matrix $\mathcal{A}$. For any matrix $\mathcal{A} = (e_{ij})_{m\times n} \in M_{n\times n}(\mathbb{C})$, the following bound is always valid (see [30, Theorem

1 and Table 1]):

$$\frac{1}{\sqrt{n}}\left\Vert \mathcal{A}\right\Vert_F \leq \left\Vert \mathcal{A}\right\Vert_2 \leq \left\Vert \mathcal{A}\right\Vert_F . \tag{2.1}$$

It follows that

$$\left\Vert \mathcal{A}\right\Vert_2 \leq \left\Vert \mathcal{A}\right\Vert_F \leq \sqrt{n}\left\Vert \mathcal{A}\right\Vert_2 .$$

In matrix analysis, several other types of norms are frequently employed. For an $n\times n$ matrix $\mathcal{A}=(e_{ij})$, the maximum column sum norm is defined as:

$$\left\Vert \mathcal{A}\right\Vert_1 = \max_{1\leq j\leq n}\sum_{i=1}^{n}\left\vert e_{ij}\right\vert ,$$

whereas the maximum row sum norm is expressed as:

$$\left\Vert \mathcal{A}\right\Vert_\infty = \max_{1\leq i\leq n}\sum_{j=1}^{n}\left\vert e_{ij}\right\vert .$$

In addition, for an $m\times n$ matrix $\mathcal{A}=(e_{ij})$, we define the maximum column length norm and maximum row length norm as:

$$c_1(\mathcal{A}) = \max_{1\leq j\leq n}\left(\sum_{i=1}^{n}\left\vert e_{ij}\right\vert^2\right)^{1/2} \quad \text{and} \quad r_1(\mathcal{A}) = \max_{1\leq i\leq n}\left(\sum_{j=1}^{n}\left\vert e_{ij}\right\vert^2\right)^{1/2} .$$

The following Lemma gives the relation between $\left\Vert .\right\Vert_2$, $c_1(.)$ and $r_1(.)$ norms.

Lemma 2.1. *[8]For two matrices* $\mathcal{A}=(e_{ij})_{m\times n}\in M_{m\times n}(\mathbb{C})$ *and* $\mathcal{B}=(b_{ij})_{m\times n}\in M_{m\times n}(\mathbb{C})$, *the following inequalities hold:*

$$\left\Vert \mathcal{A}\circ\mathcal{B}\right\Vert_2 \leq r_1(\mathcal{A})c_1(\mathcal{B})$$

and

$$\left\Vert \mathcal{A}\circ\mathcal{B}\right\Vert_2 \leq \left\Vert \mathcal{A}\right\Vert_2\left\Vert \mathcal{B}\right\Vert_2$$

and

$$\left\Vert \mathcal{A}\otimes\mathcal{B}\right\Vert_2 = \left\Vert \mathcal{A}\right\Vert_2\left\Vert \mathcal{B}\right\Vert_2 ,$$

where $\mathcal{A}\circ\mathcal{B}$ *denotes the Hadamard product defined by:*

$$\mathcal{A}\circ\mathcal{B} = (e_{ij}b_{ij}),$$

and $\mathcal{A}\otimes\mathcal{B}$ *represents the Kronecker product defined as:*$\mathcal{A}\otimes\mathcal{B}=(e_{ij}\mathcal{B})$.

These relationships provide important bounds and equalities that are useful in matrix analysis and applications, particularly when dealing with products of matrices.

For further insights into matrix norms, one can refer to sources such as [7]. In Table 1 below, we provide a detailed study of the spectral norm, Frobenius norm, maximum row length norm, and maximum column length norm specifically for different types of circulant matrices, including k-circulant, geometric circulant, and semicirculant matrices. These studies are particularly focused on the generalized m-step Fibonacci sequences ($m = 2, 3, 4$) and involve deriving sum formulas for the second powers of numbers in these m-step Fibonacci sequences.

Table 1. Papers on the norms.

Name of sequence	Papers
second order↓	
Fibonacci, Lucas	[5,6,11]
Pell, Pell-Lucas	[1,26]
Jacobsthal, Jacobsthal-Lucas	[12,27,28,29]
third order↓	
Tribonacci, Tribonacci-Lucas	[13,14]
Padovan, Perrin	[3,10,15]
fourth order↓	
Tetranacci, Tetranacci-Lucas	[9]

These papers explore different matrix types and their applications in Fibonacci, Lucas, and other generalized sequences.

For the calculations to be presented, the following two lemmas are necessary.

Lemma 2.2. *[2, Lemma 4]Consider $\mathcal{C}_k = \mathrm{Circ}_k(d_0, d_1, ..., d_{n-1})$ as a $n \times n$ matrix of the k-circulant form. The eigenvalues $\lambda_j(\mathcal{C}_k)$ are given by the following expression:*

$$\lambda_j(\mathcal{C}_k) = \sum_{\mathcal{P}=0}^{n-1} k^{\frac{p}{n}} \omega^{-jp} d_p = \sum_{p=0}^{n-1} \left(k^{\frac{1}{n}} \omega^{-j}\right)^p d_p$$

where $\omega = \exp(2\pi i/n) = e^{\frac{2\pi i}{n}}, j = 0, 1, 2, ..., n-1$. Furthermore, the coefficients d_p can be recovered by the inverse formula:

$$d_p = \frac{1}{n} \sum_{j=0}^{n-1} \left(k^{\frac{1}{n}} \omega^{-j}\right)^{-p} \lambda_j(\mathcal{C}_k), \qquad p = 0, 1, 2, ..., n-1.$$

Lemma 2.3. *[7]Consider a $n \times n$ matrix $\mathcal{A}$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$. The matrix $\mathcal{A}$ is normal if and only if the eigenvalues of the matrix $\mathcal{A}\mathcal{A}^*$ are $|\lambda_1|^2, |\lambda_2|^2, |\lambda_3|^2, ..., |\lambda_n|^2$, where $\mathcal{A}^*$ represents the conjugate transpose of $\mathcal{A}$.*

Next, we introduce the concept of a k-circulant matrix with entries defined by generalized Padovan numbers. Throughout this paper, we will denote the k-circulant matrix with generalized Padovan numbers as $\mathcal{C}_n(\mathcal{V})_k = \mathrm{Circ}_k(\mathcal{V}_0, \mathcal{V}_1, ..., \mathcal{V}_{n-1})$:

Definition 2.4. *A $n\times n$ matrix $\mathcal{C}_n(\mathcal{V})_k$, where the entries are derived from the generalized Padovan numbers, is a k-circulant matrix defined as follows:*

$$\mathcal{C}_n(\mathcal{V})_k=\operatorname{Circ}_k(\mathcal{V}_0,\mathcal{V}_1,...,\mathcal{V}_{n-1})=\begin{pmatrix}\mathcal{V}_0 & \mathcal{V}_1 & \mathcal{V}_2 & \cdots & \mathcal{V}_{n-2} & \mathcal{V}_{n-1}\\ k\mathcal{V}_{n-1} & \mathcal{V}_0 & \mathcal{V}_1 & \cdots & \mathcal{V}_{n-3} & \mathcal{V}_{n-2}\\ k\mathcal{V}_{n-2} & k\mathcal{V}_{n-1} & \mathcal{V}_0 & \cdots & \mathcal{V}_{n-4} & \mathcal{V}_{n-3}\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ k\mathcal{V}_1 & k\mathcal{V}_2 & k\mathcal{V}_3 & \cdots & k\mathcal{V}_{n-1} & \mathcal{V}_0\end{pmatrix}_{n\times n}. \quad (2.2)$$

This matrix is referred to as the generalized Padovan k-circulant matrix. We now explore four particular instances of this matrix, specifically the Padovan k-circulant matrix: $\mathcal{C}_n(\mathcal{P})_k=\operatorname{Circ}_k(\mathcal{P}_0,\mathcal{P}_1,...,\mathcal{P}_{n-1})$, Perrin k-circulant matrix: $\mathcal{C}_n(\mathcal{E})_k=\operatorname{Circ}_k(\mathcal{E}_0,\mathcal{E}_1,...,\mathcal{E}_{n-1})$, Padovan-Perrin k-circulant matrix: $\mathcal{C}_n(\mathcal{S})_k=\operatorname{Circ}_k(\mathcal{S}_0,\mathcal{S}_1,...,\mathcal{S}_{n-1})$ and modified Padovan k-circulant matrix: $\mathcal{C}_n(\mathcal{A})_k=\operatorname{Circ}_k(\mathcal{A}_0,\mathcal{A}_1,...,\mathcal{A}_{n-1})$.

We define $\mathcal{S}(\mathcal{C}_n(\mathcal{V})_k)$ as the sum of the entries of $\mathcal{C}_n(\mathcal{V})_k$.

Lemma 2.5. *The cumulative sum of the entries in $\mathcal{C}_n(\mathcal{V})_k$ is given by:*
$\mathcal{S}(\mathcal{C}_n(\mathcal{V})_k)=(kn-3k+3)\mathcal{V}_{n+2}+(kn-4k+4)\mathcal{V}_{n+1}+2(1-k)\mathcal{V}_n+(3k-n-3)\mathcal{V}_2+(4k-n-4)\mathcal{V}_1+2(k-1)\mathcal{V}_0$

Proof. Based on the definition of $\mathcal{C}_n(\mathcal{V})_k$, and applying Theorem 1.3, we derive the result

$$\begin{aligned}\mathcal{S}(\mathcal{C}_n(\mathcal{V})_k) &= n\mathcal{V}_0+((n-1)+k)\mathcal{V}_1+((n-2)+2k)\mathcal{V}_2+...+(1+(n-1)k)\mathcal{V}_{n-1}\\ &= \sum_{i=0}^{n-1}(n-i)\mathcal{V}_i+k\sum_{i=1}^{n-1}i\mathcal{V}_i\\ &= n\sum_{i=0}^{n-1}\mathcal{V}_i+(k-1)\sum_{i=1}^{n-1}i\mathcal{V}_i\\ &= (kn-3k+3)\mathcal{V}_{n+2}+(kn-4k+4)\mathcal{V}_{n+1}+2(1-k)\mathcal{V}_n\\ &\quad +(3k-n-3)\mathcal{V}_2+(4k-n-4)\mathcal{V}_1+2(k-1)\mathcal{V}_0.\ \square\end{aligned}$$

By substituting $\mathcal{V}_n=\mathcal{P}_n$ with $\mathcal{P}_0=1$, $\mathcal{P}_1=1$, $\mathcal{P}_2=1$, $\mathcal{V}_n=\mathcal{E}_n$ with $\mathcal{E}_0=3$, $\mathcal{E}_1=0$, $\mathcal{E}_2=2$, $\mathcal{V}_n=\mathcal{S}_n$ with $\mathcal{S}_0=0$, $\mathcal{S}_1=0$, $\mathcal{S}_2=1$ and $\mathcal{V}_n=\mathcal{A}_n$ with $\mathcal{A}_0=3$, $\mathcal{A}_1=1$, $\mathcal{A}_2=3$, respectively in the previous Lemma, we can derive the following corollary.

Corollary 2.6. *The following results are obtained:*

(a): *The overall sum of the entries in $\mathcal{C}_n(\mathcal{P})_k$ is given by:*

$$\mathcal{S}(\mathcal{C}_n(\mathcal{P})_k)=(kn-3k+3)\mathcal{P}_{n+2}+(kn-4k+4)\mathcal{P}_{n+1}+2(1-k)\mathcal{P}_n+(9k-2n-9).$$

(b): *The overall sum of the entries in $d_n(\mathcal{E})_k$ is expressed as:*

$$\mathcal{S}(\mathcal{C}_n(\mathcal{E})_k)=(kn-3k+3)\mathcal{E}_{n+2}+(kn-4k+4)\mathcal{E}_{n+1}+2(1-k)\mathcal{E}_n+(12k-2n-12).$$

(c): *The overall sum of the entries in $d_n(\mathcal{S})_k$ is given by:*

$$\mathcal{S}(\mathcal{C}_n(\mathcal{S})_k)=(kn-3k+3)\mathcal{S}_{n+2}+(kn-4k+4)\mathcal{S}_{n+1}+2(1-k)\mathcal{S}_n+(3k-n-3).$$

(d): *The overall sum of the entries in* $d_n(\mathcal{A})_k$ *is expressed as:*

$$\mathcal{S}(\mathcal{C}_n(\mathcal{A})_k) = (kn-3k+3)\mathcal{A}_{n+2} + (kn-4k+4)\mathcal{A}_{n+1} + 2(1-k)\mathcal{A}_n + (19k-4n-19).$$

Considering specific conditions on the generalized Padovan sequence $\mathcal{V}_n$ and the parameter k, we compute the maximum column sum matrix norm $\|\mathcal{C}_n(\mathcal{V})_k\|_1$ and the maximum row sum matrix norm $\|\mathcal{C}_n(\mathcal{V})_k\|_\infty$ of the matrix $\mathcal{C}_n(\mathcal{V})_k = (d_{ij})$.

Theorem 2.7. *Assuming that* $\mathcal{V}_p \geq 0$ *for all* $p \geq 0$*, the following formulas hold: Provided that* $k \geq 1$*, then*

$$\|\mathcal{C}_n(\mathcal{V})_k\|_1 = \|\mathcal{C}_n(\mathcal{V})_k\|_\infty = k\mathcal{V}_{n+2} + k\mathcal{V}_{n+1} - k\mathcal{V}_2 - k\mathcal{V}_1 + (1-k)\mathcal{V}_0$$

and provided that $k < 1$*, then*

$$\|\mathcal{C}_n(\mathcal{V})_k\|_1 = \|\mathcal{C}_n(\mathcal{V})_k\|_\infty = \mathcal{V}_{n+2} + \mathcal{V}_{n+1} - \mathcal{V}_2 - \mathcal{V}_1$$

Proof. Assuming $k \geq 1$, from the definition of the matrix $\mathcal{C}_n(\mathcal{V})_k = (d_{ij})$, and utilizing Theorem 1.3, we can express the following result:

$$\begin{aligned}
\|\mathcal{C}_n(\mathcal{V})_k\|_1 &= \max_{1\leq j\leq n} \sum_{i=1}^{n} |d_{ij}| = \max_{1\leq j\leq n} \{|d_{1j}| + |d_{2j}| + |d_{3j}| + ... + |d_{nj}|\} \\
&= |d_{11}| + |d_{21}| + |d_{31}| + ... + |d_{n1}| \\
&= \mathcal{V}_0 + k\mathcal{V}_{n-1} + k\mathcal{V}_{n-2} + ... + k\mathcal{V}_3 + k\mathcal{V}_2 + k\mathcal{V}_1 \\
&= (\mathcal{V}_0 - k\mathcal{V}_0 - k\mathcal{V}_n) + k\sum_{i=0}^{n} \mathcal{V}_i \\
&= k\mathcal{V}_{n+2} + k\mathcal{V}_{n+1} - k\mathcal{V}_2 - k\mathcal{V}_1 + (1-k)\mathcal{V}_0.
\end{aligned}$$

Similarly, we have

$$\|d_n(\mathcal{V})_k\|_\infty = k\mathcal{V}_{n+2} + k\mathcal{V}_{n+1} - k\mathcal{V}_2 - k\mathcal{V}_1 + (1-k)\mathcal{V}_0.$$

Assuming $k < 1$, from the definition of the matrix $\mathcal{C}_n(\mathcal{V})_k = (d_{ij})$, and utilizing Theorem 1.3, we can express the following result:

$$\begin{aligned}
\|\mathcal{C}_n(\mathcal{V})_k\|_1 &= \max_{1\leq j\leq n} \sum_{i=1}^{n} |d_{ij}| = \max_{1\leq j\leq n} \{|d_{1j}| + |d_{2j}| + |d_{3j}| + ... + |d_{nj}|\} \\
&= |d_{1n}| + |d_{2n}| + |d_{3n}| + ... + |d_{nn}| \\
&= \mathcal{V}_{n-1} + \mathcal{V}_{n-2} + ... + \mathcal{V}_3 + \mathcal{V}_2 + \mathcal{V}_1 + \mathcal{V}_0 \\
&= -\mathcal{V}_n + \sum_{i=0}^{n} \mathcal{V}_i \\
&= \mathcal{V}_{n+2} + \mathcal{V}_{n+1} - \mathcal{V}_2 - \mathcal{V}_1.
\end{aligned}$$

Similarly, we have

$$\|d_n(\mathcal{V})_k\|_\infty = \mathcal{V}_{n+2} + \mathcal{V}_{n+1} - \mathcal{V}_2 - \mathcal{V}_1. \square$$

By setting $\mathcal{V}_n = \mathcal{P}_n$, where $\mathcal{P}_0 = 1, \mathcal{P}_1 = 1, \mathcal{P}_2 = 1$, $\mathcal{V}_n = \mathcal{E}_n$,where $\mathcal{E}_0 = 3, \mathcal{E}_1 = 0, \mathcal{E}_2 = 2$, $\mathcal{V}_n = \mathcal{S}_n$,where $\mathcal{S}_0 = 0, \mathcal{S}_1 = 0, \mathcal{S}_2 = 1$ and $\mathcal{V}_n = \mathcal{A}_n$,where $\mathcal{A}_0 = 3, \mathcal{A}_1 = 1, \mathcal{A}_2 = 3$, respectively, into the previous theorem, we derive the following corollary.

Corollary 2.8. *The derived results are as follows:*

(a): *If $k \geq 1$ then*

$$\|\mathcal{C}_n(\mathcal{P})_k\|_1 = \|\mathcal{C}_n(\mathcal{P})_k\|_\infty = k\mathcal{P}_{n+2} + k\mathcal{P}_{n+1} + (1 - 3k),$$

and if $k < 1$ then

$$\|\mathcal{C}_n(\mathcal{P})_k\|_1 = \|\mathcal{C}_n(\mathcal{P})_k\|_\infty = \mathcal{P}_{n+2} + \mathcal{P}_{n+1} - 2.$$

(b): *If $k \geq 1$ then*

$$\|\mathcal{C}_n(\mathcal{E})_k\|_1 = \|\mathcal{C}_n(\mathcal{E})_k\|_\infty = k\mathcal{E}_{n+2} + k\mathcal{E}_{n+1} + (3 - 5k),$$

and if $k < 1$ then

$$\|\mathcal{C}_n(\mathcal{E})_k\|_1 = \|\mathcal{C}_n(\mathcal{E})_k\|_\infty = \mathcal{E}_{n+2} + \mathcal{E}_{n+1} - 2.$$

(c): *If $k \geq 1$ then*

$$\|\mathcal{C}_n(\mathcal{S})_k\|_1 = \|\mathcal{C}_n(\mathcal{S})_k\|_\infty = k\mathcal{S}_{n+2} + k\mathcal{S}_{n+1} - k,$$

and if $k < 1$ then

$$\|\mathcal{C}_n(\mathcal{S})_k\|_1 = \|\mathcal{C}_n(\mathcal{S})_k\|_\infty = \mathcal{S}_{n+2} + \mathcal{S}_{n+1} - 1.$$

(d): *If $k \geq 1$ then*

$$\|\mathcal{C}_n(\mathcal{A})_k\|_1 = \|\mathcal{C}_n(\mathcal{A})_k\|_\infty = k\mathcal{A}_{n+2} + k\mathcal{A}_{n+1} + (3 - 7k),$$

and if $k < 1$ then

$$\|\mathcal{C}_n(\mathcal{A})_k\|_1 = \|\mathcal{C}_n(\mathcal{A})_k\|_\infty = \mathcal{A}_{n+2} + \mathcal{A}_{n+1} - 4.$$

Let's now establish the Frobenius (Euclidean) norm for the k-circulant matrix $\mathcal{C}_n(\mathcal{V})_k$.

Theorem 2.9. *We express the Frobenius (Euclidean) norm of the k-circulant matrix $\mathcal{C}_n(\mathcal{V})_k$ as follows:*

$$\|\mathcal{C}_n(\mathcal{V})_k\|_F = \sqrt{n\left(\varphi_1(\mathcal{V})\right) + \varphi_2(\mathcal{V})},$$

where

$\varphi_1(\mathcal{V}) = -\mathcal{V}_{n+2}^2 - \mathcal{V}_{n+1}^2 - \mathcal{V}_n^2 - \mathcal{V}_n^2 + 2\mathcal{V}_{n+1}\mathcal{V}_{n+2} + 2\mathcal{V}_n\mathcal{V}_{n+2} + \mathcal{V}_2^2 + \mathcal{V}_1^2 + 2\mathcal{V}_0^2 - 2\mathcal{V}_0\mathcal{V}_2 - 2\mathcal{V}_1\mathcal{V}_2,$

$\varphi_2(\mathcal{V}) = (|k|^2-1)(-(n+5)\mathcal{V}_{n+2}^2-(n+4)\mathcal{V}_{n+1}^2-2(n+3)\mathcal{V}_n^2+2(n+3)\mathcal{V}_{n+2}\mathcal{V}_{n+1}+2(n+4)\mathcal{V}_{n+2}\mathcal{V}_n-2\mathcal{V}_n\mathcal{V}_{n+1}+ 5\mathcal{V}_2^2 + 4\mathcal{V}_1^2 + 6\mathcal{V}_0^2 - 6\mathcal{V}_1\mathcal{V}_2 - 8\mathcal{V}_0\mathcal{V}_2 + 2\mathcal{V}_0\mathcal{V}_1).$

Proof. By applying the definition of the Euclidean norm of a matrix and utilizing Theorem 1.3, we derive the following:

$$\begin{aligned}(\|\mathcal{C}_n(\mathcal{V})_k\|_F)^2 &= \sum_{i=1,j=1}^{n} |d_{ij}|^2 \\ &= \sum_{i=0}^{n-1}(n-i)\mathcal{V}_i^2 + |k|^2\sum_{i=1}^{n-1} i\mathcal{V}_i^2 \\ &= n\sum_{i=0}^{n-1}\mathcal{V}_i^2 + (|k|^2-1)\sum_{i=1}^{n-1} i\mathcal{V}_i^2 \\ &= n\left(\varphi_1(\mathcal{V})\right) + \varphi_2(\mathcal{V}),\end{aligned}$$

where $\varphi_1(\mathcal{V})$ and $\varphi_2(\mathcal{V})$ are defined in the theorem. Thus, we obtain

$$\|\mathcal{C}_n(\mathcal{V})_k\|_F = \sqrt{n\left(\varphi_1(\mathcal{V})\right) + \varphi_2(\mathcal{V})}. \quad \square$$

Note that

$$\varphi_1(\mathcal{V}) = \sum_{i=0}^{n-1}\mathcal{V}_i^2$$

and

$$\varphi_2(\mathcal{V}) = (|k|^2-1)\sum_{i=1}^{n-1} i\mathcal{V}_i^2.$$

By setting $\mathcal{V}_n = \mathcal{P}_n$, where $\mathcal{P}_0 = 1, \mathcal{P}_1 = 1, \mathcal{P}_2 = 1$, $\mathcal{V}_n = \mathcal{E}_n$, where $\mathcal{E}_0 = 3, \mathcal{E}_1 = 0, \mathcal{E}_2 = 2$, $\mathcal{V}_n = \mathcal{S}_n$, where $\mathcal{S}_0 = 0, \mathcal{S}_1 = 0, \mathcal{S}_2 = 1$ and $\mathcal{V}_n = \mathcal{A}_n$, where $\mathcal{A}_0 = 3, \mathcal{A}_1 = 1, \mathcal{A}_2 = 3$, respectively into the previous theorem, we derive the following corollary.

Corollary 2.10. *The derived results are as follows:*

(a): *The Frobenius (Euclidean) norm of k-circulant matrix $\mathcal{C}_n(\mathcal{P})_k$ expressed as:*

$$\|\mathcal{C}_n(\mathcal{P})_k\|_F = \sqrt{n\left(\varphi_1(\mathcal{P})\right) + \varphi_2(\mathcal{P})}$$

where

$\varphi_1(\mathcal{P}) = -\mathcal{P}_{n+2}^2 - \mathcal{P}_{n+1}^2 - \mathcal{P}_n^2 - \mathcal{P}_n^2 + 2\mathcal{P}_{n+1}\mathcal{P}_{n+2} + 2\mathcal{P}_n\mathcal{P}_{n+2}$,

$\varphi_2(\mathcal{P}) = (|k|^2-1)(-(n+5)\mathcal{P}_{n+2}^2 - (n+4)\mathcal{P}_{n+1}^2 - 2(n+3)\mathcal{P}_n^2 + 2(n+3)\mathcal{P}_{n+2}\mathcal{P}_{n+1} + 2(n+4)\mathcal{P}_{n+2}\mathcal{P}_n - 2\mathcal{P}_n\mathcal{P}_{n+1} + 3)$.

(b): *The Frobenius (Euclidean) norm of k-circulant matrix $\mathcal{C}_n(\mathcal{E})_k$ expressed as:*

$$\|\mathcal{C}_n(\mathcal{E})_k\|_F = \sqrt{n\left(\varphi_1(\mathcal{E})\right) + \varphi_2(\mathcal{E})}$$

where

$\varphi_1(\mathcal{E}) = -\mathcal{E}_{n+2}^2 - \mathcal{E}_{n+1}^2 - \mathcal{E}_n^2 - \mathcal{E}_n^2 + 2\mathcal{E}_{n+1}\mathcal{E}_{n+2} + 2\mathcal{E}_n\mathcal{E}_{n+2} + 10$,

$\varphi_2(\mathcal{E}) = (|k|^2-1)(-(n+5)\mathcal{E}_{n+2}^2 - (n+4)\mathcal{E}_{n+1}^2 - 2(n+3)\mathcal{E}_n^2 + 2(n+3)\mathcal{E}_{n+2}\mathcal{E}_{n+1} + 2(n+4)\mathcal{E}_{n+2}\mathcal{E}_n - 2\mathcal{E}_n\mathcal{E}_{n+1} + 26)$.

(c): *The Frobenius (Euclidean) norm of k-circulant matrix* $\mathcal{C}_n(\mathcal{S})_k$ *expressed as:*

$$\|\mathcal{C}_n(\mathcal{S})_k\|_F = \sqrt{n\left(\varphi_1(\mathcal{S})\right) + \varphi_2(\mathcal{S})}$$

where

$\varphi_1(\mathcal{S}) = -\mathcal{S}_{n+2}^2 - \mathcal{S}_{n+1}^2 - \mathcal{S}_n^2 - \mathcal{S}_n^2 + 2\mathcal{S}_{n+1}\mathcal{S}_{n+2} + 2\mathcal{S}_n\mathcal{S}_{n+2} + 1,$

$\varphi_2(\mathcal{S}) = (|k|^2 - 1)(-(n+5)\mathcal{S}_{n+2}^2 - (n+4)\mathcal{S}_{n+1}^2 - 2(n+3)\mathcal{S}_n^2 + 2(n+3)\mathcal{S}_{n+2}\mathcal{S}_{n+1} + 2(n+4)\mathcal{S}_{n+2}\mathcal{S}_n - 2\mathcal{S}_n\mathcal{S}_{n+1} + 5).$

(d): *The Frobenius (Euclidean) norm of k-circulant matrix* $\mathcal{C}_n(\mathcal{A})_k$ *expressed as:*

$$\|\mathcal{C}_n(\mathcal{A})_k\|_F = \sqrt{n\left(\varphi_1(\mathcal{A})\right) + \varphi_2(\mathcal{A})}$$

where

$\varphi_1(\mathcal{A}) = -\mathcal{A}_{n+2}^2 - \mathcal{A}_{n+1}^2 - \mathcal{A}_n^2 - \mathcal{A}_n^2 + 2\mathcal{A}_{n+1}\mathcal{A}_{n+2} + 2\mathcal{A}_n\mathcal{A}_{n+2} + 4,$

$\varphi_2(\mathcal{A}) = (|k|^2 - 1)(-(n+5)\mathcal{A}_{n+2}^2 - (n+4)\mathcal{A}_{n+1}^2 - 2(n+3)\mathcal{A}_n^2 + 2(n+3)\mathcal{A}_{n+2}\mathcal{A}_{n+1} + 2(n+4)\mathcal{A}_{n+2}\mathcal{A}_n - 2\mathcal{A}_n\mathcal{A}_{n+1} + 19).$

The eigenvalues of the matrix in equation (2.2) are given by the following theorem.

Theorem 2.11. *The eigenvalues of* $\mathcal{C}_n(\mathcal{V})_k$ *are given by the following expression:*

$$\lambda_j(\mathcal{C}_n(\mathcal{V})) = \frac{\Phi_j(\mathcal{V})}{(k^{\frac{1}{n}}\omega^{-j})^3 + (k^{\frac{1}{n}}\omega^{-j})^2 - 1},$$

where

$\Phi_j(\mathcal{V}) = k\mathcal{V}_n - \mathcal{V}_0 - k^{\frac{1}{n}}(-k\mathcal{V}_{n+1} + \mathcal{V}_1)\omega^{-j} + k^{\frac{2}{n}}(k\mathcal{V}_{n+2} - k\mathcal{V}_n - \mathcal{V}_2 + \mathcal{V}_0)\omega^{-2j}$

and

$$\omega = \exp(2\pi i/n) = e^{\frac{2\pi i}{n}},\ j = 0, 1, 2, 3, ..., n-1.$$

Proof. Utilizing Lemma 2.2, we derive the following expression for the eigenvalues:

$$\begin{aligned}
\lambda_j(\mathcal{C}_n(\mathcal{V})_k) &= \sum_{p=0}^{n-1} k^{\frac{p}{n}}\omega^{-jp}\mathcal{V}_p \\
&= -k\omega^{-jn}\mathcal{V}_n + \sum_{p=0}^{n-1} k^{\frac{p}{n}}\omega^{-jp}\mathcal{V}_p \\
&= -k\omega^{-jn}\mathcal{V}_n + \sum_{\mathcal{P}=0}^{n} (k^{\frac{1}{n}}\omega^{-j})^p\mathcal{V}_p.
\end{aligned}$$

Now, applying Theorem 1.1 (by setting $x = k^{\frac{1}{n}}\omega^{-j}$) and utilizing the recurrence relation $\mathcal{V}_{n+3} = \mathcal{V}_{n+1} + \mathcal{V}_n$, we derive the required result. □

By setting $\mathcal{V}_n = \mathcal{P}_n$, where $\mathcal{P}_0 = 1, \mathcal{P}_1 = 1, \mathcal{P}_2 = 1$, $\mathcal{V}_n = \mathcal{E}_n$, where $\mathcal{E}_0 = 3, \mathcal{E}_1 = 0, \mathcal{E}_2 = 2$, $\mathcal{V}_n = \mathcal{S}_n$, where $\mathcal{S}_0 = 0, \mathcal{S}_1 = 0, \mathcal{S}_2 = 1$ and $\mathcal{V}_n = \mathcal{A}_n$, where $\mathcal{A}_0 = 3, \mathcal{A}_1 = 1, \mathcal{A}_2 = 3$, respectively into the previous theorem, we derive the following corollary.

Corollary 2.12. *The results obtained from our analysis are as follows:*

(a): *The eigenvalues of* $\mathcal{C}_n(\mathcal{P})_k$ *are given by:*

$$\lambda_j(\mathcal{C}_n(\mathcal{P})) = \frac{\Phi_j(\mathcal{P})}{(k^{\frac{1}{n}}\omega^{-j})^3 + (k^{\frac{1}{n}}\omega^{-j})^2 - 1},$$

(b): *The eigenvalues of* $\mathcal{C}_n(\mathcal{E})_k$ *are given by:*

$$\lambda_j(\mathcal{C}_n(\mathcal{E})) = \frac{\Phi_j(\mathcal{E})}{(k^{\frac{1}{n}}\omega^{-j})^3 + (k^{\frac{1}{n}}\omega^{-j})^2 - 1},$$

(c): *The eigenvalues of* $\mathcal{C}_n(\mathcal{S})_k$ *are given by:*

$$\lambda_j(\mathcal{C}_n(\mathcal{S})) = \frac{\Phi_j(\mathcal{S})}{(k^{\frac{1}{n}}\omega^{-j})^3 + (k^{\frac{1}{n}}\omega^{-j})^2 - 1},$$

(d): *The eigenvalues of* $\mathcal{C}_n(\mathcal{A})_k$ *are given by:*

$$\lambda_j(\mathcal{C}_n(\mathcal{A})) = \frac{\Phi_j(\mathcal{A})}{(k^{\frac{1}{n}}\omega^{-j})^3 + (k^{\frac{1}{n}}\omega^{-j})^2 - 1},$$

where

$$\Phi_j(\mathcal{P}) = k\mathcal{P}_n - 1 - k^{\frac{1}{n}}(-k\mathcal{P}_{n+1} + 1)\omega^{-j} + k^{\frac{2}{n}}(k\mathcal{P}_{n+2} - k\mathcal{P}_n)\omega^{-2j},$$

$$\Phi_j(\mathcal{E}) = k\mathcal{E}_n - 3 - k^{\frac{1}{n}}(-k\mathcal{E}_{n+1})\omega^{-j} + k^{\frac{2}{n}}(k\mathcal{E}_{n+2} - k\mathcal{E}_n + 1)\omega^{-2j},$$

$$\Phi_j(\mathcal{S}) = k\mathcal{S}_n - k^{\frac{1}{n}}(-k\mathcal{S}_{n+1})\omega^{-j} + k^{\frac{2}{n}}(k\mathcal{S}_{n+2} - k\mathcal{S}_n - 1)\omega^{-2j},$$

$$\Phi_j(\mathcal{A}) = k\mathcal{A}_n - 3 - k^{\frac{1}{n}}(-k\mathcal{A}_{n+1} + 1)\omega^{-j} + k^{\frac{2}{n}}(k\mathcal{A}_{n+2} - k\mathcal{A}_n)\omega^{-2j},$$

$$\omega = \exp(2\pi i/n) = e^{\frac{2\pi i}{n}},\ j = 0, 1, 2, 3, ..., n-1.$$

The upper and lower bounds for the spectral norm of $\mathcal{C}_n(\mathcal{V})_k$ are given by the following theorem.

THEOREM 2.13. *Consider* $\mathcal{C}_n(\mathcal{V})_k$ *as a k-circulant matrix defined by* $\mathrm{Circ}_k(\mathcal{V}_0, \mathcal{V}_1, ..., \mathcal{V}_{n-1})$. *Then, for* $|k| \geq 1$,

$$\sqrt{\varphi_1(\mathcal{V})} \leq \|d_n(\mathcal{V})_k\|_2 \leq \sqrt{\mathcal{V}_0^2 + |k|^2\left(-\mathcal{V}_0^2 + \varphi_1(\mathcal{V})\right)}\sqrt{1 - \mathcal{V}_0^2 + \varphi_1(\mathcal{V})},$$

and for $|k| < 1$,

$$|k|\sqrt{\varphi_1(\mathcal{V})} \leq \|d_n(\mathcal{V})_k\|_2 \leq \sqrt{n\left(\varphi_1(\mathcal{V})\right)}$$

where $\varphi_1(\mathcal{V})$ *is defined as in Theorem 2.9.*

Proof. It can be noted that $\varphi_1(\mathcal{V})$ can be represented in the forms below.

$$\begin{aligned}
\varphi_1(\mathcal{V}) &= \sum_{i=0}^{n-1}\mathcal{V}_i^2 \\
&= -\mathcal{V}_{n+2}^2 - \mathcal{V}_{n+1}^2 - \mathcal{V}_n^2 - \mathcal{V}_n^2 + 2\mathcal{V}_{n+1}\mathcal{V}_{n+2} + 2\mathcal{V}_n\mathcal{V}_{n+2} + \mathcal{V}_2^2 + \mathcal{V}_1^2 + 2\mathcal{V}_0^2 - 2\mathcal{V}_0\mathcal{V}_2 - 2\mathcal{V}_1\mathcal{V}_2, \\
\varphi_1(\mathcal{V}) &= \mathcal{V}_0^2 + \sum_{i=1}^{n-1}\mathcal{V}_i^2 \Rightarrow -\mathcal{V}_0^2 + \varphi_1(\mathcal{V}) = \sum_{i=1}^{n-1}\mathcal{V}_i^2.
\end{aligned}$$

According to Theorem 2.9, the Frobenius (Euclidean) norm of the k-circulant matrix $\mathcal{C}_n(\mathcal{V})_k$ is given by

$$\begin{aligned}\left(\left\|\mathcal{C}_n(\mathcal{V})_k\right\|_F\right)^2 &= \sum_{i=0}^{n-1}(n-i)\mathcal{V}_i^2+|k|^2\sum_{i=1}^{n-1}i\mathcal{V}_i^2\\ &= n\sum_{i=0}^{n-1}\mathcal{V}_i^2+(|k|^2-1)\sum_{i=1}^{n-1}i\mathcal{V}_i^2.\end{aligned}$$

If $|k|\geq 1$, then we obtain, using Theorem 1.3,

$$\left(\left\|\mathcal{C}_n(\mathcal{V})_k\right\|_F\right)^2 \geq \sum_{i=0}^{n-1}(n-i)\mathcal{V}_i^2+\sum_{i=1}^{n-1}i\mathcal{V}_i^2=n\sum_{i=0}^{n-1}\mathcal{V}_i^2=n\left(\varphi_1(\mathcal{V})\right)$$

i.e.

$$\left\|\mathcal{C}_n(\mathcal{V})_k\right\|_F \geq \sqrt{n\left(\varphi_1(\mathcal{V})\right)}.$$

It follows that

$$\frac{\left\|\mathcal{C}_n(\mathcal{V})_k\right\|_F}{\sqrt{n}} \geq \sqrt{\varphi_1(\mathcal{V})}.$$

Therefore, by applying equation (2.1), it can be concluded that

$$\left\|\mathcal{C}_n(\mathcal{V})_k\right\|_2 \geq \sqrt{\varphi_1(\mathcal{V})}.$$

Moreover, if $|k|<1$, we derive

$$\begin{aligned}\left\|\mathcal{C}_n(\mathcal{V})_k\right\|_F^2 &= \sum_{i=0}^{n-1}(n-i)\mathcal{V}_i^2+|k|^2\sum_{i=1}^{n-1}i\mathcal{V}_i^2\\ &\geq \sum_{i=0}^{n-1}(n-i)\left|k\right|^2\mathcal{V}_i^2+|k|^2\sum_{i=1}^{n-1}i\mathcal{V}_i^2=n\left|k\right|^2\sum_{i=0}^{n-1}\mathcal{V}_i^2\\ &= n\left|k\right|^2\left(\varphi_1(\mathcal{V})\right).\end{aligned}$$

i.e.

$$\left\|\mathcal{C}_n(\mathcal{V})_k\right\|_F \geq \sqrt{n\left|k\right|^2\left(\varphi_1(\mathcal{V})\right)}.$$

It can be concluded that

$$\frac{\left\|\mathcal{C}_n(\mathcal{V})_k\right\|_F}{\sqrt{n}} \geq |k|\sqrt{\varphi_1(\mathcal{V})}.$$

Then by considering (2.1), we get

$$\left\|\mathcal{C}_n(\mathcal{V})_k\right\|_2 \geq |k|\sqrt{(\varphi_1(\mathcal{V}))}.$$

For $|k| \geq 1$, we now present the upper bound of the spectral norm of the matrix $\mathcal{C}_n(\mathcal{V})_k$. Suppose matrices B and C are defined as

$$B = \begin{pmatrix} \mathcal{V}_0 & 1 & 1 & \cdots & 1 & 1 \\ k\mathcal{V}_{n-1} & \mathcal{V}_0 & 1 & \cdots & 1 & 1 \\ k\mathcal{V}_{n-2} & k\mathcal{V}_{n-1} & \mathcal{V}_0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ k\mathcal{V}_1 & k\mathcal{V}_2 & k\mathcal{V}_3 & \cdots & k\mathcal{V}_{n-1} & \mathcal{V}_0 \end{pmatrix}_{n\times n}$$

and

$$C = \begin{pmatrix} 1 & \mathcal{V}_1 & \mathcal{V}_2 & \cdots & \mathcal{V}_{n-2} & \mathcal{V}_{n-1} \\ 1 & 1 & \mathcal{V}_1 & \cdots & \mathcal{V}_{n-3} & \mathcal{V}_{n-2} \\ 1 & 1 & 1 & \cdots & \mathcal{V}_{n-4} & \mathcal{V}_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}_{n\times n}$$

respectively, such that $\mathcal{C}_n(\mathcal{V})_k = B \circ C$. Hence, we derive

$$r_1(B) = \max_{1\leq i\leq n} \left(\sum_{j=1}^{n} |b_{ij}|^2\right)^{1/2} = \sqrt{\mathcal{V}_0^2 + |k|^2 \sum_{j=1}^{n-1} \mathcal{V}_j^2} = \sqrt{\mathcal{V}_0^2 + |k|^2\left(-\mathcal{V}_0^2 + \varphi_1(\mathcal{V})\right)},$$

$$c_1(C) = \max_{1\leq j\leq n} \left(\sum_{i=1}^{n} |c_{ij}|^2\right)^{1/2} = \sqrt{1 + \sum_{i=1}^{n-1} \mathcal{V}_i^2} = \sqrt{1 - \mathcal{V}_0^2 + \varphi_1(\mathcal{V})}.$$

By Lemma 2.1, we have

$$\|\mathcal{C}_n(\mathcal{V})_k\|_2 \leq r_1(B)c_1(C) = \sqrt{\mathcal{V}_0^2 + |k|^2\left(-\mathcal{V}_0^2 + \varphi_1(\mathcal{V})\right)}\sqrt{1 - \mathcal{V}_0^2 + \varphi_1(\mathcal{V})}.$$

For $|k| < 1$, We now provide the upper bound for the spectral norm of the matrix $\mathcal{C}_n(\mathcal{V})_k$. Let matrices D and E be defined as

$$D = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ k & 1 & 1 & \cdots & 1 & 1 \\ k & k & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ k & k & k & \cdots & k & 1 \end{pmatrix}_{n\times n}$$

and

$$E = \begin{pmatrix} \mathcal{V}_0 & \mathcal{V}_1 & \mathcal{V}_2 & \cdots & \mathcal{V}_{n-2} & \mathcal{V}_{n-1} \\ \mathcal{V}_{n-1} & \mathcal{V}_0 & \mathcal{V}_1 & \cdots & \mathcal{V}_{n-3} & \mathcal{V}_{n-2} \\ \mathcal{V}_{n-2} & \mathcal{V}_{n-1} & \mathcal{V}_0 & \cdots & \mathcal{V}_{n-4} & \mathcal{V}_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathcal{V}_1 & \mathcal{V}_2 & \mathcal{V}_3 & \cdots & \mathcal{V}_{n-1} & \mathcal{V}_0 \end{pmatrix}_{n\times n}$$

respectively, such that $\mathcal{C}_n(\mathcal{V})_k = D \circ E$. From this, we derive

$$r_1(D) = \max_{1\leq i\leq n}\left(\sum_{j=1}^{n}|d_{ij}|^2\right)^{1/2} = \sqrt{n},$$

and

$$c_1(E) = \max_{1\leq j\leq n}\left(\sum_{i=1}^{n}|\mathcal{E}_{ij}|^2\right)^{1/2} = \sqrt{\sum_{i=0}^{n-1}\mathcal{V}_i^2} = \sqrt{\varphi_1(\mathcal{V})}.$$

By Lemma 2.1, we have

$$\|\mathcal{C}_n(\mathcal{V})_k\|_2 \leq r_1(D)c_1(E) = \sqrt{n\left(\varphi_1(\mathcal{V})\right)}.$$

Thus, the proof is finalized. □

We examine four specific cases of the previously theorem. To start, the following corollary establishes the upper and lower bounds for the spectral norm of $d_n(\mathcal{P})_k$.

Corollary 2.14. *Let* $\mathcal{C}_n(\mathcal{P})_k = \mathrm{Circ}_k(\mathcal{P}_0, \mathcal{P}_1, ..., \mathcal{P}_{n-1})$ *be Padovan k-circulant matrix. Then if* $|k| \geq 1$ *then*

$$\sqrt{\varphi_1(\mathcal{P})} \leq \|\mathcal{C}_n(\mathcal{P})_k\|_2 \leq \sqrt{\mathcal{P}_0^2 + |k|^2\left(-\mathcal{P}_0^2 + \varphi_1(\mathcal{P})\right)}\sqrt{1 - \mathcal{P}_0^2 + \varphi_1(\mathcal{P})},$$

and if $|k| < 1$ *then*

$$|k|\sqrt{\varphi_1(\mathcal{P})} \leq \|\mathcal{C}_n(\mathcal{P})_k\|_2 \leq \sqrt{n\left(\varphi_1(\mathcal{P})\right)}$$

where $\varphi_1(\mathcal{P})$ *is as in Corollary 2.10.*

Proof. Consider $\mathcal{V}_n = \mathcal{P}_n$, where $\mathcal{P}_0 = 1, \mathcal{P}_1 = 1$, and $\mathcal{P}_2 = 1$ in Theorem 2.13. □

Secondly, we present a corollary that establishes the upper and lower bounds of the spectral norm for the matrix $\mathcal{C}_n(\mathcal{E})_k$.

Corollary 2.15. *Consider* $\mathcal{C}_n(\mathcal{E})_k = \mathrm{Circ}_k(\mathcal{E}_0, \mathcal{E}_1, ..., \mathcal{E}_{n-1})$*, which denotes a Padovan k-circulant matrix. Then, for* $|k| \geq 1$*,*

$$\sqrt{\varphi_1(\mathcal{E})} \leq \|\mathcal{C}_n(\mathcal{E})_k\|_2 \leq \sqrt{\mathcal{E}_0^2 + |k|^2\left(-\mathcal{E}_0^2 + \varphi_1(\mathcal{E})\right)}\sqrt{1 - \mathcal{E}_0^2 + \varphi_1(\mathcal{E})},$$

and for $|k| < 1$*,*

$$|k|\sqrt{\varphi_1(\mathcal{E})} \leq \|\mathcal{C}_n(\mathcal{E})_k\|_2 \leq \sqrt{n\left(\varphi_1(\mathcal{E})\right)}$$

where $\varphi_1(\mathcal{E})$ *is as in Corollary 2.10.*

Proof. Take $\mathcal{V}_n = \mathcal{E}_n$, $\mathcal{E}_0 = 3, \mathcal{E}_1 = 0, \mathcal{E}_2 = 2$ in Theorem 2.13. □

Thirdly, the subsequent corollary provides the upper and lower bounds for the spectral norm of the matrix $\mathcal{C}_n(\mathcal{S})_k$.

COROLLARY 2.16. *Let* $\mathcal{C}_n(\mathcal{S})_k = \mathrm{Circ}_k(\mathcal{S}_0, \mathcal{S}_1, ..., \mathcal{S}_{n-1})$ *be Padovan-Perrin k-circulant matrix. Then, for* $|k| \geq 1$,

$$\sqrt{\varphi_1(\mathcal{S})} \leq \|\mathcal{C}_n(\mathcal{S})_k\|_2 \leq \sqrt{\mathcal{S}_0^2 + |k|^2\left(-\mathcal{S}_0^2 + \varphi_1(\mathcal{S})\right)}\sqrt{1 - \mathcal{S}_0^2 + \varphi_1(\mathcal{S})},$$

and for $|k| < 1$,

$$|k|\sqrt{\varphi_1(\mathcal{S})} \leq \|\mathcal{C}_n(\mathcal{S})_k\|_2 \leq \sqrt{n\left(\varphi_1(\mathcal{S})\right)}$$

where $\varphi_1(\mathcal{S})$ *is as in Corollary 2.10.*

Proof. Consider $\mathcal{V}_n = \mathcal{S}_n$, $\mathcal{S}_0 = 0, \mathcal{S}_1 = 0, \mathcal{S}_2 = 1$ in Theorem 2.13. □

Fourthly, the subsequent corollary provides the upper and lower bounds for the spectral norm of the matrix $\mathcal{C}_n(\mathcal{A})_k$.

COROLLARY 2.17. *Let* $\mathcal{C}_n(\mathcal{A})_k = \mathrm{Circ}_k(\mathcal{A}_0, \mathcal{A}_1, ..., \mathcal{A}_{n-1})$ *be modified Padovan k-circulant matrix. Then if* $|k| \geq 1$ *then*

$$\sqrt{\varphi_1(\mathcal{A})} \leq \|\mathcal{C}_n(\mathcal{A})_k\|_2 \leq \sqrt{\mathcal{A}_0^2 + |k|^2\left(-\mathcal{A}_0^2 + \varphi_1(\mathcal{A})\right)}\sqrt{1 - \mathcal{A}_0^2 + \varphi_1(\mathcal{A})},$$

and if $|k| < 1$ *then*

$$|k|\sqrt{\varphi_1(\mathcal{A})} \leq \|\mathcal{C}_n(\mathcal{A})_k\|_2 \leq \sqrt{n\left(\varphi_1(\mathcal{A})\right)}$$

where $\varphi_1(\mathcal{A})$ *is as in Corollary 2.10.*

Proof. Consider $\mathcal{V}_n = \mathcal{A}_n$ with $\mathcal{A}_0 = 3, \mathcal{A}_1 = 1, \mathcal{A}_2 = 3$ in Theorem 2.13. □

We now compute the determinant of $\mathcal{C}_n(\mathcal{V})_k$, as stated in the following theorem.

THEOREM 2.18. *The determinant of* $\mathcal{C}_n(\mathcal{V})_k$ *is expressed as*

$$\det(\mathcal{C}_n(\mathcal{V})_k) = \frac{\Lambda_1^n\left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n + \left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(k\mathcal{E}_n + (k - \mathcal{E}_{-n})k^2 - 1)}$$

where

$$\begin{aligned}
\Lambda_1 &= k\mathcal{V}_n - \mathcal{V}_0, \\
\Lambda_2 &= k^{\frac{1}{n}}(-k\mathcal{V}_{n+1} + \mathcal{V}_1), \\
\Lambda_3 &= k^{\frac{2}{n}}(k\mathcal{V}_{n+2} - k\mathcal{V}_n - \mathcal{V}_2 + \mathcal{V}_0).
\end{aligned}$$

Proof. Upon examining identities

$$\begin{aligned}
\prod_{k=0}^{n-1}(x - y\omega^{-k}) &= x^n - y^n \\
\prod_{j=0}^{n-1}(x - y\omega^{-j} + z\omega^{-2j}) &= x^n\left(1 - \left(\frac{y - \sqrt{y^2 - 4xz}}{2x}\right)^n - \left(\frac{y + \sqrt{y^2 - 4xz}}{2x}\right)^n + \left(\frac{z}{x}\right)^n\right)
\end{aligned}$$

and

$$(k^{\frac{1}{n}}\omega^{-j})^3 + (k^{\frac{1}{n}}\omega^{-j})^2 - 1 = (\alpha_1 k^{\frac{1}{n}}\omega^{-j} - 1)(\alpha_2 k^{\frac{1}{n}}\omega^{-j} - 1)(\alpha_3 k^{\frac{1}{n}}\omega^{-j} - 1),$$

we have that

$$\prod_{j=0}^{n-1}\left((k^{\frac{1}{n}}\omega^{-j})^3+(k^{\frac{1}{n}}\omega^{-j})^2-1\right)=(-1)^{n+1}(k\mathcal{E}_n+(k-\mathcal{E}_{-n})k^2-1)$$

and

$$\prod_{j=0}^{n-1}\Phi_j(\mathcal{V})=\Lambda_1^n\left(1-\left(\frac{\Lambda_2-\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n-\left(\frac{\Lambda_2+\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n+\left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right),$$

where

$$\begin{aligned}\omega &= \exp(2\pi i/n),\\ \Phi_j(\mathcal{V}) &= k\mathcal{V}_n-\mathcal{V}_0-k^{\frac{1}{n}}(-k\mathcal{V}_{n+1}+\mathcal{V}_1)\omega^{-j}+k^{\frac{2}{n}}(k\mathcal{V}_{n+2}-k\mathcal{V}_n-\mathcal{V}_2+\mathcal{V}_0)\omega^{-2j}\end{aligned}$$

and

$$\begin{aligned}\Lambda_1 &= k\mathcal{V}_n-\mathcal{V}_0,\\ \Lambda_2 &= k^{\frac{1}{n}}(-k\mathcal{V}_{n+1}+\mathcal{V}_1),\\ \Lambda_3 &= k^{\frac{2}{n}}(k\mathcal{V}_{n+2}-k\mathcal{V}_n-\mathcal{V}_2+\mathcal{V}_0).\end{aligned}$$

Using Theorem 2.11, we get

$$\begin{aligned}\det(\mathcal{C}_n(\mathcal{V})_k) &= \prod_{j=0}^{n-1}\lambda_j(\mathcal{C}_n(\mathcal{V})_k)\\ &= \prod_{j=0}^{n-1}\frac{\Phi_j(\mathcal{V})}{(k^{\frac{1}{n}}\omega^{-j})^3+(k^{\frac{1}{n}}\omega^{-j})^2-1}\\ &= \frac{\prod_{j=0}^{n-1}\Phi_j(\mathcal{V})}{\prod_{j=0}^{n-1}\left((k^{\frac{1}{n}}\omega^{-j})^3+(k^{\frac{1}{n}}\omega^{-j})^2-1\right)}\\ &= \frac{\Lambda_1^n\left(1-\left(\frac{\Lambda_2-\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n-\left(\frac{\Lambda_2+\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n+\left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(k\mathcal{E}_n+(k-\mathcal{E}_{-n})k^2-1)}\end{aligned}$$

thereby concluding the proof. □

We now examine four particular instances of the preceding theorem.

To begin, the corollary below provides the determinant of $\mathcal{C}_n(\mathcal{P})_k$.

Corollary 2.19. *The determinant of* $\mathcal{C}_n(\mathcal{P})_k$ *can be expressed as*

$$\det(\mathcal{C}_n(\mathcal{P})_k)=\frac{\Lambda_1^n\left(1-\left(\frac{\Lambda_2-\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n-\left(\frac{\Lambda_2+\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n+\left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(k\mathcal{E}_n+(k-\mathcal{E}_{-n})k^2-1)}$$

where

$$\begin{aligned}\Lambda_1 &= k\mathcal{P}_n - 1,\\ \Lambda_2 &= k^{\frac{1}{n}}(-k\mathcal{P}_{n+1} + 1),\\ \Lambda_3 &= k^{\frac{2}{n}}(k\mathcal{P}_{n+2} - k\mathcal{P}_n).\end{aligned}$$

Proof. Consider $\mathcal{V}_n = \mathcal{P}_n$, $\mathcal{P}_0 = 1, \mathcal{P}_1 = 1, \mathcal{P}_2 = 1$ in Theorem 2.18. □

Secondly, the next corollary presents the determinant of $\mathcal{C}_n(\mathcal{E})_k$.

Corollary 2.20. *The determinant of $\mathcal{C}_n(\mathcal{E})_k$ can be expressed as*

$$\det(\mathcal{C}_n(\mathcal{E})_k) = \frac{\Lambda_1^n\left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n + \left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(k\mathcal{E}_n + (k - \mathcal{E}_{-n})k^2 - 1)}$$

where

$$\begin{aligned}\Lambda_1 &= k\mathcal{E}_n - \mathcal{E}_0,\\ \Lambda_2 &= k^{\frac{1}{n}}(-k\mathcal{E}_{n+1} + \mathcal{E}_1),\\ \Lambda_3 &= k^{\frac{2}{n}}(k\mathcal{E}_{n+2} - k\mathcal{E}_n - \mathcal{E}_2 + \mathcal{E}_0).\end{aligned}$$

$$\begin{aligned}\Lambda_1 &= k\mathcal{E}_n - 3,\\ \Lambda_2 &= k^{\frac{1}{n}}(-k\mathcal{E}_{n+1}),\\ \Lambda_3 &= k^{\frac{2}{n}}(k\mathcal{E}_{n+2} - k\mathcal{E}_n + 1).\end{aligned}$$

Proof. Consider $\mathcal{V}_n = \mathcal{E}_n$, $\mathcal{E}_0 = 3, \mathcal{E}_1 = 0, \mathcal{E}_2 = 2$ in Theorem 2.18. □

Thirdly, the next corollary presents the determinant of $\mathcal{C}_n(\mathcal{S})_k$.

Corollary 2.21. *The determinant of $\mathcal{C}_n(\mathcal{S})_k$ can be expressed as*

$$\det(\mathcal{C}_n(\mathcal{S})_k) = \frac{\Lambda_1^n\left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n + \left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(k\mathcal{E}_n + (k - \mathcal{E}_{-n})k^2 - 1)}$$

where

$$\begin{aligned}\Lambda_1 &= k\mathcal{S}_n - \mathcal{S}_0,\\ \Lambda_2 &= k^{\frac{1}{n}}(-k\mathcal{S}_{n+1} + \mathcal{S}_1),\\ \Lambda_3 &= k^{\frac{2}{n}}(k\mathcal{S}_{n+2} - k\mathcal{S}_n - \mathcal{S}_2 + \mathcal{S}_0).\end{aligned}$$

$$\begin{aligned}\Lambda_1 &= k\mathcal{S}_n,\\ \Lambda_2 &= k^{\frac{1}{n}}(-k\mathcal{S}_{n+1}),\\ \Lambda_3 &= k^{\frac{2}{n}}(k\mathcal{S}_{n+2} - k\mathcal{S}_n - 1).\end{aligned}$$

Proof. Consider $\mathcal{V}_n = \mathcal{S}_n$ with $\mathcal{S}_0 = 0, \mathcal{S}_1 = 0, \mathcal{S}_2 = 1$ in Theorem 2.18. $\square$

Fourthly, the next corollary presents the determinant of $\mathcal{C}_n(\mathcal{A})_k$.

Corollary 2.22. *The determinant of* $\mathcal{C}_n(\mathcal{A})_k$ *can be expressed as*

$$\det(\mathcal{C}_n(\mathcal{A})_k) = \frac{\Lambda_1^n \left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n + \left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(k\mathcal{E}_n + (k - \mathcal{E}_{-n})k^2 - 1)}$$

where

$$\begin{aligned} \Lambda_1 &= k\mathcal{A}_n - 3, \\ \Lambda_2 &= k^{\frac{1}{n}}(-k\mathcal{A}_{n+1} + 1), \\ \Lambda_3 &= k^{\frac{2}{n}}(k\mathcal{A}_{n+2} - k\mathcal{A}_n). \end{aligned}$$

Proof. Consider $\mathcal{V}_n = \mathcal{A}_n$ with $\mathcal{A}_0 = 3, \mathcal{A}_1 = 1, \mathcal{A}_2 = 3$ in Theorem 2.18. $\square$

3. Conclusion

In this paper, we present several such explicit formulas for the norms, eigenvalues and determinants of the so-called k-circulant matrices on the basis of generalized Padovan numbers. It is impossible not to appreciate the significance in both theory and practice of mathematics concerning such technologically oriented matrix structures. In particular, one of the attractive aspects of these matrices is the significance for which they are meant in practical applications; If a k-circulant matrix is employed in cryptography, then it is capable of transmitting secure information and creating data encryption designs. Meanwhile for numerical analysis, methods In this sense, it means complimentary approaches may be suggestive in a strategy that seeks to enhance the effectiveness of computer processes. Therefore, the objective of this paper is to encourage the development of additional more scientific applications by considering the interrelations of these new matrices with other sequences of numbers. It will be expected that this examinations will enable new constructive mathematical models to be built and development of existing ones which will facilitate innovative methodologies across several disciplines. Furthermore, these findings are likely to enhance further advancement into the characteristics of such new types of matrices and enable the expansion of the boundaries of modern science in this area and in the adjacent areas as well.

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