A Note on k-Circulant Matrices with the Generalized Padovan Numbers

Abstract. In this paper, we obtain explicit forms of the sum of entries, the maximum column sum matrix norm, the maximum row sum matrix norm, Euclidean norm, eigenvalues and determinant of k-circulant matrix with the generalized Padovan numbers. We also study the spectral norm of this k-circulant matrix.

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1. Introduction

In this section, we recall definitions and some properties of the generalized Padovan sequence. A generalized Padovan sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = V_{n-2} + V_{n-3} \tag{1.1}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} + V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

Binet formula of generalized padovan numbers can be given as

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1 = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0,\ b_2 = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0,\ b_3 = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0. \tag{1.2}$$

Here, α, β and γ are the roots of the cubic equation $x^3 - x - 1 = 0$. Moreover

$$\begin{aligned}
\alpha &= \left(\frac{1}{2}+\sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2}-\sqrt{\frac{23}{108}}\right)^{1/3} = 1.324\,717\,957\,24 \\
\beta &= \omega\left(\frac{1}{2}+\sqrt{\frac{23}{108}}\right)^{1/3} + \omega^2\left(\frac{1}{2}-\sqrt{\frac{23}{108}}\right)^{1/3} \\
\gamma &= \omega^2\left(\frac{1}{2}+\sqrt{\frac{23}{108}}\right)^{1/3} + \omega\left(\frac{1}{2}-\sqrt{\frac{23}{108}}\right)^{1/3}
\end{aligned}$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3)$$

Note that

$$\begin{aligned}
\alpha+\beta+\gamma &= 0, \\
\alpha\beta+\alpha\gamma+\beta\gamma &= -1, \\
\alpha\beta\gamma &= 1.
\end{aligned}$$

Now we define four special cases of the sequence $\{V_n\}$. Padovan (Cordonnier) sequence$\{P_n\}_{n\geq 0}$, Perrin (Padovan-Lucas) sequence $\{E_n\}_{n\geq 0}$, Padovan-Perrin sequence $\{S_n\}_{n\geq 0}$ and modified Padovan sequence $\{A_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$\begin{aligned}
P_{n+3} &= P_{n+1} + P_n, \quad && P_0 = 1, P_1 = 1, P_2 = 1, \\
E_{n+3} &= E_{n+1} + E_n, \quad && E_0 = 3, E_1 = 0, E_2 = 2, \\
S_{n+3} &= S_{n+1} + S_n, \quad && S_0 = 0, S_1 = 0, S_2 = 1, \\
A_{n+3} &= A_{n+1} + A_n, \quad && A_0 = 3, A_1 = 1, A_2 = 3.
\end{aligned}$$

Note that the case $V_n = R_n$, $R_0 = 1, R_1 = 0, R_2 = 1$ (or $V_n = R_n$, $R_0 = 0, R_1 = 1, R_2 = 0$) is called the sequence of the Van der Laan numbers, in the literature.

The sequences $\{P_n\}_{n\geq 0}$, $\{E_n\}_{n\geq 0}$, $\{S_n\}_{n\geq 0}$ and $\{A_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}
P_{-n} &= -P_{-(n-1)} + P_{-(n-3)} \\
E_{-n} &= -E_{-(n-1)} + E_{-(n-3)} \\
S_{-n} &= -S_{-(n-1)} + S_{-(n-3)} \\
A_{-n} &= -A_{-(n-1)} + A_{-(n-3)}
\end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Note that P_n and S_n are two variants of the same sequence in [16]. In fact, the following are basically all variants of the same sequence in [16] which P_n and S_n belong: A000931, A078027, A096231, A124745, A133034, A134816, A164001, A182097, A228361 and probably A020720 (however, each one has its own special features and deserves its own entry). E_n is the sequence A001608 in [16] and A_n is the sequence A276276 in [16].

For more details for the generalized Padovan numbers, see Soykan [25].

The following Theorem presents sum formula of generalized Padovan numbers.

THEOREM 1.1. *Let x be a nonzero real or complex number. For $n \geq 0$, we have the following formula: If $x^3 + x^2 - 1 \neq 0$, then*

$$\sum_{k=0}^{n} x^k V_k = \frac{\Theta_1(x)}{\Theta(x)}$$

where

$$\Theta_1(x) = x^{n+3}V_{n+3} + x^{n+2}V_{n+2} - (x^2-1)x^{n+1}V_{n+1} - x^2V_2 - xV_1 + (x^2-1)V_0,$$
$$\Theta(x) = x^3 + x^2 - 1.$$

Proof. Take $r = 0,\ s = 1, t = 1$ in [17, Theorem 2.1. (a)]. □

The following Theorem presents sum formulas of generalized Padovan numbers.

THEOREM 1.2. *For $n \geq 0$, we have the following formulas:*

- **(a):** $\sum_{i=0}^{n} V_i = V_{n+3} + V_{n+2} - V_2 - V_1.$
- **(b):** $\sum_{i=0}^{n} iV_i = (n-2)V_{n+3} + (n-3)V_{n+2} - 2V_{n+1} + 3V_2 + 4V_1 + 2V_0.$
- **(c):** $\sum_{i=0}^{n} V_i^2 = -V_{n+3}^2 - V_{n+2}^2 - 2V_{n+1}^2 + 2V_{n+2}V_{n+3} + 2V_{n+1}V_{n+3} + V_2^2 + V_1^2 + 2V_0^2 - 2V_0V_2 - 2V_1V_2.$
- **(d):** $\sum_{i=0}^{n} iV_i^2 = -(n+6)V_{n+3}^2 - (n+5)V_{n+2}^2 - 2(n+4)V_{n+1}^2 + 2(n+4)V_{n+3}V_{n+2} + 2(n+5)V_{n+3}V_{n+1} - 2V_{n+1}V_{n+2} + 5V_2^2 + 4V_1^2 + 6V_0^2 - 6V_1V_2 - 8V_0V_2 + 2V_0V_1.$

Proof.

- **(a):** Take $x = 1, r = 0,\ s = 1, t = 1$ in [17, Theorem 2.1. (a)] or take $r = 0,\ s = 1, t = 1$ in [20, Theorem 2.1. (a)].
- **(b):** Take $x = 1, r = 0,\ s = 1, t = 1$ in [22, Theorem 2.1. (a)] or take $r = 0,\ s = 1, t = 1$ in [24, Theorem 2.1. (a)].
- **(c):** Take $x = 1, r = 0,\ s = 1, t = 1$ in [19, Theorem 3.1 (a)]. See also [18, Theorem 2.1].
- **(d):** Take $x = 1, r = 0,\ s = 1, t = 1$ in [21, Theorem 2.1. (a)] or take $r = 0,\ s = 1, t = 1$ in [23, Theorem 2.1. (a)]. □

Note that, using the recurrence relation $V_{n+3} = V_{n+1} + V_n$, we can write the above theorem as follows.

THEOREM 1.3. *For $n \geq 0$, we have the following formulas:*

- **(a):** $\sum_{i=0}^{n} V_i = V_{n+2} + V_{n+1} + V_n - V_2 - V_1 = \frac{\Theta_1}{\Theta}.$

(b): $\sum_{i=0}^{n} iV_i = (n-3)V_{n+2} + (n-4)V_{n+1} + (n-2)V_n + 3V_2 + 4V_1 + 2V_0 = \dfrac{\Psi_1}{\Psi}$.

(c): $\sum_{i=0}^{n} V_i^2 = -V_{n+2}^2 - V_{n+1}^2 - V_n^2 + 2V_{n+1}V_{n+2} + 2V_nV_{n+2} + V_2^2 + V_1^2 + 2V_0^2 - 2V_0V_2 - 2V_1V_2 = \dfrac{\Delta_1}{\Delta}$.

(d): $\sum_{i=0}^{n} iV_i^2 = -(n+5)V_{n+2}^2 - (n+4)V_{n+1}^2 - (n+6)V_n^2 + 2(n+3)V_{n+2}V_{n+1} + 2(n+4)V_{n+2}V_n - 2V_nV_{n+1} + 5V_2^2 + 4V_1^2 + 6V_0^2 - 6V_1V_2 - 8V_0V_2 + 2V_0V_1 = \dfrac{\Omega_1}{\Omega}$.

From the last Theorem, we have the following corollary which gives sum formulas of Padovan numbers (take $V_n = P_n$ with $P_0 = 1, P_1 = 1, P_2 = 1$).

Corollary 1.4. *For $n \geq 0$, Padovan numbers have the following properties:*

(a): $\sum_{i=0}^{n} P_i = P_{n+2} + P_{n+1} + P_n - 2$.

(b): $\sum_{i=0}^{n} iP_i = (n-3)P_{n+2} + (n-4)P_{n+1} + (n-2)P_n + 9$.

(c): $\sum_{i=0}^{n} P_i^2 = -P_{n+2}^2 - P_{n+1}^2 - P_n^2 + 2P_{n+1}P_{n+2} + 2P_nP_{n+2}$.

(d): $\sum_{i=0}^{n} iP_i^2 = -(n+5)P_{n+2}^2 - (n+4)P_{n+1}^2 - (n+6)P_n^2 + 2(n+3)P_{n+2}P_{n+1} + 2(n+4)P_{n+2}P_n - 2P_nP_{n+1} + 3$.

Taking $V_n = E_n$ with $E_0 = 3, E_1 = 0, E_2 = 2$ in the last Theorem, we have the following corollary which presents sum formulas of Perrin numbers.

Corollary 1.5. *For $n \geq 0$, Perrin numbers have the following properties:*

(a): $\sum_{i=0}^{n} E_i = E_{n+2} + E_{n+1} + E_n - 2$.

(b): $\sum_{i=0}^{n} iE_i = (n-3)E_{n+2} + (n-4)E_{n+1} + (n-2)E_n + 12$.

(c): $\sum_{i=0}^{n} E_i^2 = -E_{n+2}^2 - E_{n+1}^2 - E_n^2 + 2E_{n+1}E_{n+2} + 2E_nE_{n+2} + 10$.

(d): $\sum_{i=0}^{n} iE_i^2 = -(n+5)E_{n+2}^2 - (n+4)E_{n+1}^2 - (n+6)E_n^2 + 2(n+3)E_{n+2}E_{n+1} + 2(n+4)E_{n+2}E_n - 2E_nE_{n+1} + 26$.

From the last Theorem, we have the following corollary which gives sum formulas of Padovan-Perrin numbers (take $V_n = S_n$ with $S_0 = 0, S_1 = 0, S_2 = 1$).

Corollary 1.6. *For $n \geq 0$, Padovan-Perrin numbers have the following properties:*

(a): $\sum_{i=0}^{n} S_i = S_{n+2} + S_{n+1} + S_n - 1$.

(b): $\sum_{i=0}^{n} iS_i = (n-3)S_{n+2} + (n-4)S_{n+1} + (n-2)S_n + 3$.

(c): $\sum_{i=0}^{n} S_i^2 = -S_{n+2}^2 - S_{n+1}^2 - S_n^2 + 2S_{n+1}S_{n+2} + 2S_nS_{n+2} + 1$.

(d): $\sum_{i=0}^{n} iS_i^2 = -(n+5)S_{n+2}^2 - (n+4)S_{n+1}^2 - (n+6)S_n^2 + 2(n+3)S_{n+2}S_{n+1} + 2(n+4)S_{n+2}S_n - 2S_nS_{n+1} + 5$.

Taking $V_n = A_n$ with $A_0 = 3, A_1 = 1, A_2 = 3$ in the last Theorem, we have the following corollary which presents sum formulas of modified Padovan numbers.

Corollary 1.7. *For $n \geq 0$, modified Padovan numbers have the following properties:*

(a): $\sum_{i=0}^{n} A_i = A_{n+2} + A_{n+1} + A_n - 4.$

(b): $\sum_{i=0}^{n} iA_i = (n-3)A_{n+2} + (n-4)A_{n+1} + (n-2)A_n + 19.$

(c): $\sum_{i=0}^{n} A_i^2 = -A_{n+2}^2 - A_{n+1}^2 - A_n^2 + 2A_{n+1}A_{n+2} + 2A_nA_{n+2} + 4.$

(d): $\sum_{i=0}^{n} iA_i^2 = -(n+5)A_{n+2}^2 - (n+4)A_{n+1}^2 - (n+6)A_n^2 + 2(n+3)A_{n+2}A_{n+1} + 2(n+4)A_{n+2}A_n - 2A_nA_{n+1} + 19.$

2. Main Results

Next, we recall some information on k-circulant matrix and Frobenius norm, spectral norm, maximum column length norm and maximum row length norm. Let $n \geq 2$ be an integer and k be any real or complex number. An $n \times n$ matrix $C_k = (c_{ij}) \in M_{n\times n}(\mathbb{C})$ is called a k-circulant matrix if it is of the form

$$C_k = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ kc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ kc_{n-2} & kc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ kc_2 & kc_3 & kc_4 & \cdots & c_0 & c_1 \\ kc_1 & kc_2 & kc_3 & \cdots & kc_{n-1} & c_0 \end{pmatrix}_{n\times n}.$$

k-circulant matrix C_k is denoted by $C_k = \mathrm{Circ}_k(c_0, c_1, ..., c_{n-1})$.

If $k = 1$ then 1-circulant matrix is called as circulant matrix and denoted by $C = \mathrm{Circ}(c_0, c_1, ..., c_{n-1})$. Circulant matrix was first proposed by Davis in [4].

The Frobenius (or Euclidean) norm and spectral norm of a $m \times n$ matrix $A = (a_{ij})_{m\times n} \in M_{m\times n}(\mathbb{C})$ are defined respectively as follows:

$$\|A\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2} \quad \text{and} \quad \|A\|_2 = \left(\max_{1\leq i\leq n} |\lambda_i(A^*A)|\right)^{1/2}$$

where $\lambda_i(A^*A)$'s are the eigenvalues of the matrix A^*A and A^* is the conjugate of transpose of the matrix A. The following inequality holds for any matrix $A = (a_{ij})_{m\times n} \in M_{n\times n}(\mathbb{C})$ (see [30, Theorem 1 and Table 1]):

$$\frac{1}{\sqrt{n}} \|A\|_F \leq \|A\|_2 \leq \|A\|_F . \tag{2.1}$$

It follows that

$$\|A\|_2 \leq \|A\|_F \leq \sqrt{n}\, \|A\|_2 .$$

In literature there are other types of norms of matrices. The maximum column sum matrix norm of $n \times n$ matrix $A = (a_{ij})$ is $\|A\|_1 = \max\limits_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|$ and the maximum row sum matrix norm is $\|A\|_\infty = \max\limits_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$. The maximum column length norm $c_1(A)$ and the maximum row length norm $r_1(A)$ of

$m \times n$ matrix $A = (a_{ij})$ are defined as follows:

$$c_1(A) = \max_{1 \le j \le n} \left(\sum_{i=1}^{n} |a_{ij}|^2 \right)^{1/2} \quad \text{and} \quad r_1(A) = \max_{1 \le i \le n} \left(\sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

There is a relation between $\|.\|_2$, $c_1(.)$ and $r_1(.)$ norms:

Lemma 2.1. *[8]For any matrices $A = (a_{ij})_{m\times n} \in M_{m\times n}(\mathbb{C})$ and $B = (b_{ij})_{m\times n} \in M_{m\times n}(\mathbb{C})$, we have*

$$\|A \circ B\|_2 \le r_1(A)c_1(B)$$

and

$$\|A \circ B\|_2 \le \|A\|_2 \|B\|_2$$

and

$$\|A \otimes B\|_2 = \|A\|_2 \|B\|_2$$

where $A \circ B$ is the Hadamard product which is defined by

$$A \circ B = (a_{ij}b_{ij}),$$

$A \otimes B$ is the Kronecker product which is defined by

$$A \otimes B = (a_{ij}B).$$

For more details on norm of matrices, see for example [7]. In the following Table 1, we present a few special study on the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (k-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences which require sum formulas of second powers of numbers in m-step Fibonacci sequences $(m = 2, 3, 4)$.

Table 1. Papers on the norms.

Name of sequence	Papers
second order↓	second order↓
Fibonacci, Lucas	[5,6,11]
Pell, Pell-Lucas	[1,26]
Jacobsthal, Jacobsthal-Lucas	[12,27,28,29]
third order↓	third order↓
Tribonacci, Tribonacci-Lucas	[13,14]
Padovan, Perrin	[3,10,15]
fourth order↓	fourth order↓
Tetranacci, Tetranacci-Lucas	[9]

We need the following two lemmas for our calculations.

Lemma 2.2. *[2, Lemma 4]Let $C_k = \mathrm{Circ}_k(c_0, c_1, ..., c_{n-1})$ be a $n \times n$ k-circulant matrix. Then we have*

$$\lambda_j(C_k) = \sum_{p=0}^{n-1} k^{\frac{p}{n}} \omega^{-jp} c_p = \sum_{p=0}^{n-1} \left(k^{\frac{1}{n}} \omega^{-j}\right)^p c_p$$

where $\omega = \exp(2\pi i/n) = e^{\frac{2\pi i}{n}}$, $j = 0, 1, 2, ..., n-1$. Moreover, in this case

$$c_p = \frac{1}{n} \sum_{j=0}^{n-1} \left(k^{\frac{1}{n}} \omega^{-j}\right)^{-p} \lambda_j(C_k), \qquad p = 0, 1, 2, ..., n-1.$$

Lemma 2.3. *[7]Let A be a $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$. Then, A is a normal matrix if and only if the eigenvalues of AA^* are $|\lambda_1|^2, |\lambda_2|^2, |\lambda_3|^2, ..., |\lambda_n|^2$ where A^* is the conjugate of transpose of the matrix A.*

Next, we define k-circulant matrix with generalized Padovan numbers entries. Throughout this paper, the k-circulant matrix, whose entries are the generalized Padovan numbers, will be denoted by $C_n(V)_k = \mathrm{Circ}_k(V_0, V_1, ..., V_{n-1})$:

Definition 2.4. *A $n \times n$ k-circulant matrix with generalized Padovan numbers entries is defined by*

$$C_n(V)_k = \mathrm{Circ}_k(V_0, V_1, ..., V_{n-1}) = \begin{pmatrix} V_0 & V_1 & V_2 & \cdots & V_{n-2} & V_{n-1} \\ kV_{n-1} & V_0 & V_1 & \cdots & V_{n-3} & V_{n-2} \\ kV_{n-2} & kV_{n-1} & V_0 & \cdots & V_{n-4} & V_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ kV_1 & kV_2 & kV_3 & \cdots & kV_{n-1} & V_0 \end{pmatrix}_{n \times n}. \tag{2.2}$$

We call this matrix as generalized Padovan k-circulant matrix. We consider four special cases of generalized Padovan k-circulant matrix, namely Padovan k-circulant matrix: $C_n(P)_k = \mathrm{Circ}_k(P_0, P_1, ..., P_{n-1})$, Perrin k-circulant matrix: $C_n(E)_k = \mathrm{Circ}_k(E_0, E_1, ..., E_{n-1})$, Padovan-Perrin k-circulant matrix: $C_n(S)_k = \mathrm{Circ}_k(S_0, S_1, ..., S_{n-1})$ and modified Padovan k-circulant matrix: $C_n(A)_k = \mathrm{Circ}_k(A_0, A_1, ..., A_{n-1})$.

We denote the sum of entries of $C_n(V)_k$ as $S(C_n(V)_k)$.

Lemma 2.5. *The sum of entries of $C_n(V)_k$ is*

$$S(C_n(V)_k) = (kn-3k+3)V_{n+2} + (kn-4k+4)V_{n+1} + 2(1-k)V_n + (3k-n-3)V_2 + (4k-n-4)V_1 + 2(k-1)V_0$$

Proof. From the definition of $C_n(V)_k$, using Theorem 1.3, we obtain

$$\begin{aligned} S(C_n(V)_k) &= nV_0 + ((n-1)+k)V_1 + ((n-2)+2k)V_2 + ... + (1+(n-1)k)V_{n-1} \\ &= \sum_{i=0}^{n-1}(n-i)V_i + k\sum_{i=1}^{n-1} iV_i \\ &= n\sum_{i=0}^{n-1} V_i + (k-1)\sum_{i=1}^{n-1} iV_i \\ &= (kn-3k+3)V_{n+2} + (kn-4k+4)V_{n+1} + 2(1-k)V_n \\ &\quad +(3k-n-3)V_2 + (4k-n-4)V_1 + 2(k-1)V_0. \quad \square \end{aligned}$$

Taking $V_n = P_n$ with $P_0 = 1, P_1 = 1, P_2 = 1$, $V_n = E_n$ with $E_0 = 3, E_1 = 0, E_2 = 2$, $V_n = S_n$ with $S_0 = 0, S_1 = 0, S_2 = 1$ and $V_n = A_n$ with $A_0 = 3, A_1 = 1, A_2 = 3$, respectively in the last Lemma, we obtain the following corollary.

Corollary 2.6. *We have the following results:*

(a): *The sum of entries of $C_n(P)_k$ is*

$$S(C_n(P)_k) = (kn - 3k + 3)P_{n+2} + (kn - 4k + 4)P_{n+1} + 2(1 - k)P_n + (9k - 2n - 9).$$

(b): *The sum of entries of $C_n(E)_k$ is*

$$S(C_n(E)_k) = (kn - 3k + 3)E_{n+2} + (kn - 4k + 4)E_{n+1} + 2(1 - k)E_n + (12k - 2n - 12).$$

(c): *The sum of entries of $C_n(S)_k$ is*

$$S(C_n(S)_k) = (kn - 3k + 3)S_{n+2} + (kn - 4k + 4)S_{n+1} + 2(1 - k)S_n + (3k - n - 3).$$

(d): *The sum of entries of $C_n(A)_k$ is*

$$S(C_n(A)_k) = (kn - 3k + 3)A_{n+2} + (kn - 4k + 4)A_{n+1} + 2(1 - k)A_n + (19k - 4n - 19).$$

Next, we present the maximum column sum matrix norm $\|C_n(V)_k\|_1$ and the maximum row sum matrix norm $\|C_n(V)_k\|_\infty$ of the matrix $C_n(V)_k = (c_{ij})$ under certain condition on the generalized Padovan sequence V_n and k.

Theorem 2.7. *Suppose that $V_p \geq 0$ for all the nonnegative integers p. Then we have the following formulas: If $k \geq 1$ then*

$$\|C_n(V)_k\|_1 = \|C_n(V)_k\|_\infty = kV_{n+2} + kV_{n+1} - kV_2 - kV_1 + (1 - k)V_0$$

and if $k < 1$ then

$$\|C_n(V)_k\|_1 = \|C_n(V)_k\|_\infty = V_{n+2} + V_{n+1} - V_2 - V_1$$

Proof. Suppose that $k \geq 1$. Then from the definition of the matrix $C_n(V)_k = (c_{ij})$, using Theorem 1.3, we can write

$$\begin{aligned}
\|C_n(V)_k\|_1 &= \max_{1\leq j\leq n} \sum_{i=1}^{n} |c_{ij}| = \max_{1\leq j\leq n} \{|c_{1j}| + |c_{2j}| + |c_{3j}| + ... + |c_{nj}|\} \\
&= |c_{11}| + |c_{21}| + |c_{31}| + ... + |c_{n1}| \\
&= V_0 + kV_{n-1} + kV_{n-2} + ... + kV_3 + kV_2 + kV_1 \\
&= (V_0 - kV_0 - kV_n) + k\sum_{i=0}^{n} V_i \\
&= kV_{n+2} + kV_{n+1} - kV_2 - kV_1 + (1 - k)V_0.
\end{aligned}$$

Similarly, we have

$$\|C_n(V)_k\|_\infty = kV_{n+2} + kV_{n+1} - kV_2 - kV_1 + (1 - k)V_0.$$

Suppose now that $k < 1$. Then from the definition of the matrix $C_n(V)_k = (c_{ij})$, using Theorem 1.3, we can write

$$\begin{aligned}\|C_n(V)_k\|_1 &= \max_{1\leq j\leq n}\sum_{i=1}^{n}|c_{ij}| = \max_{1\leq j\leq n}\{|c_{1j}|+|c_{2j}|+|c_{3j}|+...+|c_{nj}|\} \\ &= |c_{1n}|+|c_{2n}|+|c_{3n}|+...+|c_{nn}| \\ &= V_{n-1}+V_{n-2}+...+V_3+V_2+V_1+V_0 \\ &= -V_n+\sum_{i=0}^{n}V_i \\ &= V_{n+2}+V_{n+1}-V_2-V_1.\end{aligned}$$

Similarly, we have

$$\|C_n(V)_k\|_\infty = V_{n+2}+V_{n+1}-V_2-V_1. \ \Box$$

Taking $V_n = P_n$ with $P_0 = 1, P_1 = 1, P_2 = 1$, $V_n = E_n$ with $E_0 = 3, E_1 = 0, E_2 = 2$, $V_n = S_n$ with $S_0 = 0, S_1 = 0, S_2 = 1$ and $V_n = A_n$ with $A_0 = 3, A_1 = 1, A_2 = 3$, respectively in the last theorem, we obtain the following corollary.

Corollary 2.8. *We have the following results:*

(a): *If $k \geq 1$ then*

$$\|C_n(P)_k\|_1 = \|C_n(P)_k\|_\infty = kP_{n+2}+kP_{n+1}+(1-3k),$$

and if $k < 1$ then

$$\|C_n(P)_k\|_1 = \|C_n(P)_k\|_\infty = P_{n+2}+P_{n+1}-2.$$

(b): *If $k \geq 1$ then*

$$\|C_n(E)_k\|_1 = \|C_n(E)_k\|_\infty = kE_{n+2}+kE_{n+1}+(3-5k),$$

and if $k < 1$ then

$$\|C_n(E)_k\|_1 = \|C_n(E)_k\|_\infty = E_{n+2}+E_{n+1}-2.$$

(c): *If $k \geq 1$ then*

$$\|C_n(S)_k\|_1 = \|C_n(S)_k\|_\infty = kS_{n+2}+kS_{n+1}-k,$$

and if $k < 1$ then

$$\|C_n(S)_k\|_1 = \|C_n(S)_k\|_\infty = S_{n+2}+S_{n+1}-1.$$

(d): *If $k \geq 1$ then*

$$\|C_n(A)_k\|_1 = \|C_n(A)_k\|_\infty = kA_{n+2}+kA_{n+1}+(3-7k),$$

and if $k < 1$ then

$$\|C_n(A)_k\|_1 = \|C_n(A)_k\|_\infty = A_{n+2} + A_{n+1} - 4.$$

Now, we determine the Euclidean (Frobenius) norm of k-circulant matrix $C_n(V)_k$.

THEOREM 2.9. *The Euclidean (Frobenius) norm of k-circulant matrix $C_n(V)_k$ is:*

$$\|C_n(V)_k\|_F = \sqrt{n\left(\varphi_1(V)\right) + \varphi_2(V)}$$

where

$\varphi_1(V) = -V_{n+2}^2 - V_{n+1}^2 - V_n^2 - V_n^2 + 2V_{n+1}V_{n+2} + 2V_nV_{n+2} + V_2^2 + V_1^2 + 2V_0^2 - 2V_0V_2 - 2V_1V_2,$

$\varphi_2(V) = (|k|^2-1)(-(n+5)V_{n+2}^2-(n+4)V_{n+1}^2-2(n+3)V_n^2+2(n+3)V_{n+2}V_{n+1}+2(n+4)V_{n+2}V_n-2V_nV_{n+1}+5V_2^2 + 4V_1^2 + 6V_0^2 - 6V_1V_2 - 8V_0V_2 + 2V_0V_1).$

Proof. From the definition of the Euclidean norm of a matrix, using Theorem 1.3, we obtain

$$\begin{aligned}
\left(\|C_n(V)_k\|_F\right)^2 &= \sum_{i=1,j=1}^{n} |c_{ij}|^2 \\
&= \sum_{i=0}^{n-1}(n-i)V_i^2 + |k|^2 \sum_{i=1}^{n-1} iV_i^2 \\
&= n\sum_{i=0}^{n-1} V_i^2 + (|k|^2 - 1)\sum_{i=1}^{n-1} iV_i^2 \\
&= n\left(\varphi_1(V)\right) + \varphi_2(V)
\end{aligned}$$

where $\varphi_1(V)$ and $\varphi_2(V)$ are as in the statement of the theorem. Now, it follows that

$$\|C_n(V)_k\|_F = \sqrt{n\left(\varphi_1(V)\right) + \varphi_2(V)}. \quad \square$$

Note that

$$\varphi_1(V) = \sum_{i=0}^{n-1} V_i^2$$

and

$$\varphi_2(V) = (|k|^2 - 1)\sum_{i=1}^{n-1} iV_i^2.$$

Taking $V_n = P_n$ with $P_0 = 1, P_1 = 1, P_2 = 1$, $V_n = E_n$ with $E_0 = 3, E_1 = 0, E_2 = 2$, $V_n = S_n$ with $S_0 = 0, S_1 = 0, S_2 = 1$ and $V_n = A_n$ with $A_0 = 3, A_1 = 1, A_2 = 3$, respectively in the last Theorem, we obtain the following corollary.

COROLLARY 2.10. *We have the following results:*

(a): *The Euclidean (Frobenius) norm of k-circulant matrix $C_n(P)_k$ is:*

$$\|C_n(P)_k\|_F = \sqrt{n\left(\varphi_1(P)\right) + \varphi_2(P)}$$

where

$\varphi_1(P) = -P_{n+2}^2 - P_{n+1}^2 - P_n^2 - P_n^2 + 2P_{n+1}P_{n+2} + 2P_nP_{n+2}$,

$\varphi_2(P) = (|k|^2 - 1)(-(n+5)P_{n+2}^2 - (n+4)P_{n+1}^2 - 2(n+3)P_n^2 + 2(n+3)P_{n+2}P_{n+1} + 2(n+4)P_{n+2}P_n - 2P_nP_{n+1} + 3)$.

(b): *The Euclidean (Frobenius) norm of k-circulant matrix* $C_n(E)_k$ *is:*

$$\|C_n(E)_k\|_F = \sqrt{n\left(\varphi_1(E)\right) + \varphi_2(E)}$$

where

$\varphi_1(E) = -E_{n+2}^2 - E_{n+1}^2 - E_n^2 - E_n^2 + 2E_{n+1}E_{n+2} + 2E_nE_{n+2} + 10$,

$\varphi_2(E) = (|k|^2 - 1)(-(n+5)E_{n+2}^2 - (n+4)E_{n+1}^2 - 2(n+3)E_n^2 + 2(n+3)E_{n+2}E_{n+1} + 2(n+4)E_{n+2}E_n - 2E_nE_{n+1} + 26)$.

(c): *The Euclidean (Frobenius) norm of k-circulant matrix* $C_n(S)_k$ *is:*

$$\|C_n(S)_k\|_F = \sqrt{n\left(\varphi_1(S)\right) + \varphi_2(S)}$$

where

$\varphi_1(S) = -S_{n+2}^2 - S_{n+1}^2 - S_n^2 - S_n^2 + 2S_{n+1}S_{n+2} + 2S_nS_{n+2} + 1$,

$\varphi_2(S) = (|k|^2 - 1)(-(n+5)S_{n+2}^2 - (n+4)S_{n+1}^2 - 2(n+3)S_n^2 + 2(n+3)S_{n+2}S_{n+1} + 2(n+4)S_{n+2}S_n - 2S_nS_{n+1} + 5)$.

(d): *The Euclidean (Frobenius) norm of k-circulant matrix* $C_n(A)_k$ *is:*

$$\|C_n(A)_k\|_F = \sqrt{n\left(\varphi_1(A)\right) + \varphi_2(A)}$$

where

$\varphi_1(A) = -A_{n+2}^2 - A_{n+1}^2 - A_n^2 - A_n^2 + 2A_{n+1}A_{n+2} + 2A_nA_{n+2} + 4$,

$\varphi_2(A) = (|k|^2 - 1)(-(n+5)A_{n+2}^2 - (n+4)A_{n+1}^2 - 2(n+3)A_n^2 + 2(n+3)A_{n+2}A_{n+1} + 2(n+4)A_{n+2}A_n - 2A_nA_{n+1} + 19)$.

The following theorem gives us the eigenvalues of the matrix in (2.2).

THEOREM 2.11. *The eigenvalues of* $C_n(V)_k$ *are*

$$\lambda_j(C_n(V)) = \frac{\Phi_j(V)}{(k^{\frac{1}{n}}\omega^{-j})^3 + (k^{\frac{1}{n}}\omega^{-j})^2 - 1},$$

where

$\Phi_j(V) = kV_n - V_0 - k^{\frac{1}{n}}(-kV_{n+1} + V_1)\omega^{-j} + k^{\frac{2}{n}}(kV_{n+2} - kV_n - V_2 + V_0)\omega^{-2j}$

and

$$\begin{aligned} \omega &= \exp(2\pi i/n) = e^{\frac{2\pi i}{n}}, \\ j &= 0, 1, 2, 3, ..., n-1. \end{aligned}$$

Proof. By using Lemma 2.2, we obtain

$$\begin{aligned}\lambda_j(C_n(V)_k) &= \sum_{p=0}^{n-1} k^{\frac{p}{n}}\omega^{-jp}V_p \\ &= -k\omega^{-jn}V_n + \sum_{p=0}^{n} k^{\frac{p}{n}}\omega^{-jp}V_p \\ &= -k\omega^{-jn}V_n + \sum_{p=0}^{n}(k^{\frac{1}{n}}\omega^{-j})^p V_p.\end{aligned}$$

Now using Theorem 1.1 (by putting $x = k^{\frac{1}{n}}\omega^{-j}$) and recurrence relation $V_{n+3} = V_{n+1} + V_n$, we obtain required result. □

Taking $V_n = P_n$ with $P_0 = 1, P_1 = 1, P_2 = 1$, $V_n = E_n$ with $E_0 = 3, E_1 = 0, E_2 = 2$, $V_n = S_n$ with $S_0 = 0, S_1 = 0, S_2 = 1$ and $V_n = A_n$ with $A_0 = 3, A_1 = 1, A_2 = 3$, respectively in the last Theorem, we obtain the following corollary.

Corollary 2.12. *We have the following results:*

(a): *The eigenvalues of* $C_n(P)_k$ *are*

$$\lambda_j(C_n(P)) = \frac{\Phi_j(P)}{(k^{\frac{1}{n}}\omega^{-j})^3 + (k^{\frac{1}{n}}\omega^{-j})^2 - 1},$$

(b): *the eigenvalues of* $C_n(E)_k$ *are*

$$\lambda_j(C_n(E)) = \frac{\Phi_j(E)}{(k^{\frac{1}{n}}\omega^{-j})^3 + (k^{\frac{1}{n}}\omega^{-j})^2 - 1},$$

(c): *The eigenvalues of* $C_n(S)_k$ *are*

$$\lambda_j(C_n(S)) = \frac{\Phi_j(S)}{(k^{\frac{1}{n}}\omega^{-j})^3 + (k^{\frac{1}{n}}\omega^{-j})^2 - 1},$$

(d): *The eigenvalues of* $C_n(A)_k$ *are*

$$\lambda_j(C_n(A)) = \frac{\Phi_j(A)}{(k^{\frac{1}{n}}\omega^{-j})^3 + (k^{\frac{1}{n}}\omega^{-j})^2 - 1},$$

where

$\Phi_j(P) = kP_n - 1 - k^{\frac{1}{n}}(-kP_{n+1} + 1)\omega^{-j} + k^{\frac{2}{n}}(kP_{n+2} - kP_n)\omega^{-2j}$,
$\Phi_j(E) = kE_n - 3 - k^{\frac{1}{n}}(-kE_{n+1})\omega^{-j} + k^{\frac{2}{n}}(kE_{n+2} - kE_n + 1)\omega^{-2j}$,
$\Phi_j(S) = kS_n - k^{\frac{1}{n}}(-kS_{n+1})\omega^{-j} + k^{\frac{2}{n}}(kS_{n+2} - kS_n - 1)\omega^{-2j}$,
$\Phi_j(A) = kA_n - 3 - k^{\frac{1}{n}}(-kA_{n+1} + 1)\omega^{-j} + k^{\frac{2}{n}}(kA_{n+2} - kA_n)\omega^{-2j}$,
$\omega = \exp(2\pi i/n) = e^{\frac{2\pi i}{n}}$, $j = 0, 1, 2, 3, ..., n-1$.

The following theorem presents the upper and lower bounds of the spectral norm of $C_n(V)_k$.

Theorem 2.13. *Let* $C_n(V)_k = \mathrm{Circ}_k(V_0, V_1, ..., V_{n-1})$ *be a k-circulant matrix. Then if* $|k| \geq 1$ *then*

$$\sqrt{\varphi_1(V)} \leq \|C_n(V)_k\|_2 \leq \sqrt{V_0^2 + |k|^2\left(-V_0^2 + \varphi_1(V)\right)}\sqrt{1 - V_0^2 + \varphi_1(V)},$$

and if $|k| < 1$ *then*

$$|k| \sqrt{\varphi_1(V)} \le \|C_n(V)_k\|_2 \le \sqrt{n\,(\varphi_1(V))}$$

where $\varphi_1(V)$ *is as in Theorem 2.9.*

Proof. Note that we can write $\varphi_1(V)$ as in the following forms.

$$\begin{aligned}
\varphi_1(V) &= \sum_{i=0}^{n-1} V_i^2 \\
&= -V_{n+2}^2 - V_{n+1}^2 - V_n^2 - V_n^2 + 2V_{n+1}V_{n+2} + 2V_nV_{n+2} + V_2^2 + V_1^2 + 2V_0^2 - 2V_0V_2 - 2V_1V_2, \\
\varphi_1(V) &= V_0^2 + \sum_{i=1}^{n-1} V_i^2 \Rightarrow -V_0^2 + \varphi_1(V) = \sum_{i=1}^{n-1} V_i^2.
\end{aligned}$$

From Theorem 2.9, we know that the Euclidean (Frobenius) norm of k-circulant matrix $C_n(V)_k$ is

$$\begin{aligned}
(\|C_n(V)_k\|_F)^2 &= \sum_{i=0}^{n-1}(n-i)V_i^2 + |k|^2 \sum_{i=1}^{n-1} iV_i^2 \\
&= n\sum_{i=0}^{n-1} V_i^2 + (|k|^2 - 1)\sum_{i=1}^{n-1} iV_i^2.
\end{aligned}$$

If $|k| \ge 1$, then we get, using Theorem 1.3,

$$(\|C_n(V)_k\|_F)^2 \ge \sum_{i=0}^{n-1}(n-i)V_i^2 + \sum_{i=1}^{n-1} iV_i^2 = n\sum_{i=0}^{n-1} V_i^2 = n\,(\varphi_1(V))$$

i.e.

$$\|C_n(V)_k\|_F \ge \sqrt{n\,(\varphi_1(V))}.$$

It follows that

$$\frac{\|C_n(V)_k\|_F}{\sqrt{n}} \ge \sqrt{\varphi_1(V)}.$$

Then by (2.1), we obtain

$$\|C_n(V)_k\|_2 \ge \sqrt{\varphi_1(V)}.$$

Similarly, If $|k| < 1$, then we obtain

$$\begin{aligned}
\|C_n(V)_k\|_F^2 &= \sum_{i=0}^{n-1}(n-i)V_i^2 + |k|^2 \sum_{i=1}^{n-1} iV_i^2 \\
&\ge \sum_{i=0}^{n-1}(n-i)\,|k|^2\,V_i^2 + |k|^2 \sum_{i=1}^{n-1} iV_i^2 = n\,|k|^2 \sum_{i=0}^{n-1} V_i^2 \\
&= n\,|k|^2\,(\varphi_1(V)).
\end{aligned}$$

i.e.

$$\|C_n(V)_k\|_F \ge \sqrt{n\,|k|^2\,(\varphi_1(V))}.$$

It follows that

$$\frac{\|C_n(V)_k\|_F}{\sqrt{n}} \geq |k| \sqrt{\varphi_1(V)}.$$

Then by considering (2.1), we get

$$\|C_n(V)_k\|_2 \geq |k| \sqrt{(\varphi_1(V))}.$$

Now, for $|k| \geq 1$, we give the upper bound for the spectral norm of the matrix $C_n(V)_k$ as follows. Let the matrices B and C be as

$$B = \begin{pmatrix} V_0 & 1 & 1 & \cdots & 1 & 1 \\ kV_{n-1} & V_0 & 1 & \cdots & 1 & 1 \\ kV_{n-2} & kV_{n-1} & V_0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ kV_1 & kV_2 & kV_3 & \cdots & kV_{n-1} & V_0 \end{pmatrix}_{n\times n}$$

and

$$C = \begin{pmatrix} 1 & V_1 & V_2 & \cdots & V_{n-2} & V_{n-1} \\ 1 & 1 & V_1 & \cdots & V_{n-3} & V_{n-2} \\ 1 & 1 & 1 & \cdots & V_{n-4} & V_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}_{n\times n}$$

such that $C_n(V)_k = B \circ C$. Then we obtain

$$r_1(B) = \max_{1\leq i\leq n} \left(\sum_{j=1}^{n} |b_{ij}|^2 \right)^{1/2} = \sqrt{V_0^2 + |k|^2 \sum_{j=1}^{n-1} V_j^2} = \sqrt{V_0^2 + |k|^2 \left(-V_0^2 + \varphi_1(V)\right)},$$

$$c_1(C) = \max_{1\leq j\leq n} \left(\sum_{i=1}^{n} |c_{ij}|^2 \right)^{1/2} = \sqrt{1 + \sum_{i=1}^{n-1} V_i^2} = \sqrt{1 - V_0^2 + \varphi_1(V)}.$$

By Lemma 2.1, we have

$$\|C_n(V)_k\|_2 \leq r_1(B)c_1(C) = \sqrt{V_0^2 + |k|^2 \left(-V_0^2 + \varphi_1(V)\right)}\sqrt{1 - V_0^2 + \varphi_1(V)}.$$

For $|k| < 1$, we give the upper bound for the spectral norm of the matrix $C_n(V)_k$ as follows. We define the matrices D and E as

$$D = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ k & 1 & 1 & \cdots & 1 & 1 \\ k & k & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ k & k & k & \cdots & k & 1 \end{pmatrix}_{n\times n}$$

and

$$E = \begin{pmatrix} V_0 & V_1 & V_2 & \cdots & V_{n-2} & V_{n-1} \\ V_{n-1} & V_0 & V_1 & \cdots & V_{n-3} & V_{n-2} \\ V_{n-2} & V_{n-1} & V_0 & \cdots & V_{n-4} & V_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ V_1 & V_2 & V_3 & \cdots & V_{n-1} & V_0 \end{pmatrix}_{n\times n}$$

such that $C_n(V)_k = D \circ E$. Then we obtain

$$r_1(D) = \max_{1\le i\le n}\left(\sum_{j=1}^{n} |d_{ij}|^2\right)^{1/2} = \sqrt{n},$$

and

$$c_1(E) = \max_{1\le j\le n}\left(\sum_{i=1}^{n} |e_{ij}|^2\right)^{1/2} = \sqrt{\sum_{i=0}^{n-1} V_i^2} = \sqrt{\varphi_1(V)}.$$

By Lemma 2.1, we have

$$\|C_n(V)_k\|_2 \le r_1(D)c_1(E) = \sqrt{n\,(\varphi_1(V))}.$$

This completes the proof. □

We consider four special cases of the above theorem.

Firstly, the following corollary gives the upper and lower bounds of the spectral norm of $C_n(P)_k$.

COROLLARY 2.14. *Let $C_n(P)_k = \text{Circ}_k(P_0, P_1, ..., P_{n-1})$ be Padovan k-circulant matrix. Then if $|k| \ge 1$ then*

$$\sqrt{\varphi_1(P)} \le \|C_n(P)_k\|_2 \le \sqrt{P_0^2 + |k|^2\,(-P_0^2 + \varphi_1(P))}\sqrt{1 - P_0^2 + \varphi_1(P)},$$

and if $|k| < 1$ then

$$|k|\,\sqrt{\varphi_1(P)} \le \|C_n(P)_k\|_2 \le \sqrt{n\,(\varphi_1(P))}$$

where $\varphi_1(P)$ is as in Corollary 2.10.

Proof. Take $V_n = P_n$, $P_0 = 1, P_1 = 1, P_2 = 1$ in Theorem 2.13. □

Secondly, the following corollary gives the upper and lower bounds of the spectral norm of $C_n(E)_k$.

COROLLARY 2.15. *Let $C_n(E)_k = \text{Circ}_k(E_0, E_1, ..., E_{n-1})$ be Perrin k-circulant matrix. Then if $|k| \ge 1$ then*

$$\sqrt{\varphi_1(E)} \le \|C_n(E)_k\|_2 \le \sqrt{E_0^2 + |k|^2\,(-E_0^2 + \varphi_1(E))}\sqrt{1 - E_0^2 + \varphi_1(E)},$$

and if $|k| < 1$ then

$$|k|\,\sqrt{\varphi_1(E)} \le \|C_n(E)_k\|_2 \le \sqrt{n\,(\varphi_1(E))}$$

where $\varphi_1(E)$ is as in Corollary 2.10.

Proof. Take $V_n = E_n$, $E_0 = 3, E_1 = 0, E_2 = 2$ in Theorem 2.13. □

Thirdly, the following corollary gives the upper and lower bounds of the spectral norm of $C_n(S)_k$.

COROLLARY 2.16. *Let* $C_n(S)_k = \mathrm{Circ}_k(S_0, S_1, ..., S_{n-1})$ *be Padovan-Perrin k-circulant matrix. Then if* $|k| \geq 1$ *then*

$$\sqrt{\varphi_1(S)} \leq \|C_n(S)_k\|_2 \leq \sqrt{S_0^2 + |k|^2(-S_0^2 + \varphi_1(S))}\sqrt{1 - S_0^2 + \varphi_1(S)},$$

and if $|k| < 1$ *then*

$$|k|\sqrt{\varphi_1(S)} \leq \|C_n(S)_k\|_2 \leq \sqrt{n(\varphi_1(S))}$$

where $\varphi_1(S)$ *is as in Corollary 2.10.*

Proof. Take $V_n = S_n$, $S_0 = 0, S_1 = 0, S_2 = 1$ in Theorem 2.13. □

Fourthly, the following corollary gives the upper and lower bounds of the spectral norm of $C_n(A)_k$.

COROLLARY 2.17. *Let* $C_n(A)_k = \mathrm{Circ}_k(A_0, A_1, ..., A_{n-1})$ *be modified Padovan k-circulant matrix. Then if* $|k| \geq 1$ *then*

$$\sqrt{\varphi_1(A)} \leq \|C_n(A)_k\|_2 \leq \sqrt{A_0^2 + |k|^2(-A_0^2 + \varphi_1(A))}\sqrt{1 - A_0^2 + \varphi_1(A)},$$

and if $|k| < 1$ *then*

$$|k|\sqrt{\varphi_1(A)} \leq \|C_n(A)_k\|_2 \leq \sqrt{n(\varphi_1(A))}$$

where $\varphi_1(A)$ *is as in Corollary 2.10.*

Proof. Take $V_n = A_n$ with $A_0 = 3, A_1 = 1, A_2 = 3$ in Theorem 2.13. □

Next, we present the determinant of $C_n(V)_k$.

THEOREM 2.18. *The determinant of* $C_n(V)_k$ *is given by*

$$\det(C_n(V)_k) = \frac{\Lambda_1^n\left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n + \left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(kE_n + (k - E_{-n})k^2 - 1)}$$

where

$$\begin{aligned}
\Lambda_1 &= kV_n - V_0,\\
\Lambda_2 &= k^{\frac{1}{n}}(-kV_{n+1} + V_1),\\
\Lambda_3 &= k^{\frac{2}{n}}(kV_{n+2} - kV_n - V_2 + V_0).
\end{aligned}$$

Proof. By considering identities

$$\begin{aligned}
\prod_{k=0}^{n-1}(x - y\omega^{-k}) &= x^n - y^n\\
\prod_{j=0}^{n-1}(x - y\omega^{-j} + z\omega^{-2j}) &= x^n\left(1 - \left(\frac{y - \sqrt{y^2 - 4xz}}{2x}\right)^n - \left(\frac{y + \sqrt{y^2 - 4xz}}{2x}\right)^n + \left(\frac{z}{x}\right)^n\right)
\end{aligned}$$

and

$$(k^{\frac{1}{n}}\omega^{-j})^3 + (k^{\frac{1}{n}}\omega^{-j})^2 - 1 = (\alpha k^{\frac{1}{n}}\omega^{-j} - 1)(\beta k^{\frac{1}{n}}\omega^{-j} - 1)(\gamma k^{\frac{1}{n}}\omega^{-j} - 1),$$

we see that

$$\prod_{j=0}^{n-1}\left((k^{\frac{1}{n}}\omega^{-j})^3+(k^{\frac{1}{n}}\omega^{-j})^2-1\right)=(-1)^{n+1}(kE_n+(k-E_{-n})k^2-1)$$

and

$$\prod_{j=0}^{n-1}\Phi_j(V)=\Lambda_1^n\left(1-\left(\frac{\Lambda_2-\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n-\left(\frac{\Lambda_2+\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n+\left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right),$$

where

$$\begin{aligned}\omega &= \exp(2\pi i/n),\\ \Phi_j(V) &= kV_n-V_0-k^{\frac{1}{n}}(-kV_{n+1}+V_1)\omega^{-j}+k^{\frac{2}{n}}(kV_{n+2}-kV_n-V_2+V_0)\omega^{-2j}\end{aligned}$$

and

$$\begin{aligned}\Lambda_1 &= kV_n-V_0,\\ \Lambda_2 &= k^{\frac{1}{n}}(-kV_{n+1}+V_1),\\ \Lambda_3 &= k^{\frac{2}{n}}(kV_{n+2}-kV_n-V_2+V_0).\end{aligned}$$

From Theorem 2.11, we have

$$\begin{aligned}\det(C_n(V)_k) &= \prod_{j=0}^{n-1}\lambda_j(C_n(V)_k)\\ &= \prod_{j=0}^{n-1}\frac{\Phi_j(V)}{(k^{\frac{1}{n}}\omega^{-j})^3+(k^{\frac{1}{n}}\omega^{-j})^2-1}\\ &= \frac{\prod_{j=0}^{n-1}\Phi_j(V)}{\prod_{j=0}^{n-1}\left((k^{\frac{1}{n}}\omega^{-j})^3+(k^{\frac{1}{n}}\omega^{-j})^2-1\right)}\\ &= \frac{\Lambda_1^n\left(1-\left(\frac{\Lambda_2-\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n-\left(\frac{\Lambda_2+\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n+\left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(kE_n+(k-E_{-n})k^2-1)}\end{aligned}$$

which completes the proof. □

We consider four special cases of the above theorem.

Firstly, the following corollary gives the determinant of $C_n(P)_k$.

Corollary 2.19. *The determinant of $C_n(P)_k$ is given by*

$$\det(C_n(P)_k)=\frac{\Lambda_1^n\left(1-\left(\frac{\Lambda_2-\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n-\left(\frac{\Lambda_2+\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n+\left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(kE_n+(k-E_{-n})k^2-1)}$$

where

$$\begin{aligned}\Lambda_1 &= kP_n - 1,\\ \Lambda_2 &= k^{\frac{1}{n}}(-kP_{n+1} + 1),\\ \Lambda_3 &= k^{\frac{2}{n}}(kP_{n+2} - kP_n).\end{aligned}$$

Proof. Take $V_n = P_n$, $P_0 = 1, P_1 = 1, P_2 = 1$ in Theorem 2.18. □

Secondly, the following corollary gives the determinant of $C_n(E)_k$.

COROLLARY 2.20. *The determinant of* $C_n(E)_k$ *is given by*

$$\det(C_n(E)_k) = \frac{\Lambda_1^n\left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n + \left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(kE_n + (k - E_{-n})k^2 - 1)}$$

where

$$\begin{aligned}\Lambda_1 &= kE_n - E_0,\\ \Lambda_2 &= k^{\frac{1}{n}}(-kE_{n+1} + E_1),\\ \Lambda_3 &= k^{\frac{2}{n}}(kE_{n+2} - kE_n - E_2 + E_0).\end{aligned}$$

$$\begin{aligned}\Lambda_1 &= kE_n - 3,\\ \Lambda_2 &= k^{\frac{1}{n}}(-kE_{n+1}),\\ \Lambda_3 &= k^{\frac{2}{n}}(kE_{n+2} - kE_n + 1).\end{aligned}$$

Proof. Take $V_n = E_n$, $E_0 = 3, E_1 = 0, E_2 = 2$ in Theorem 2.18. □

Thirdly, the following corollary gives the determinant of $C_n(S)_k$.

COROLLARY 2.21. *The determinant of* $C_n(S)_k$ *is given by*

$$\det(C_n(S)_k) = \frac{\Lambda_1^n\left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n + \left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(kE_n + (k - E_{-n})k^2 - 1)}$$

where

$$\begin{aligned}\Lambda_1 &= kS_n - S_0,\\ \Lambda_2 &= k^{\frac{1}{n}}(-kS_{n+1} + S_1),\\ \Lambda_3 &= k^{\frac{2}{n}}(kS_{n+2} - kS_n - S_2 + S_0).\end{aligned}$$

$$\begin{aligned}\Lambda_1 &= kS_n,\\ \Lambda_2 &= k^{\frac{1}{n}}(-kS_{n+1}),\\ \Lambda_3 &= k^{\frac{2}{n}}(kS_{n+2} - kS_n - 1).\end{aligned}$$

Proof. Take $V_n = S_n$ with $S_0 = 0, S_1 = 0, S_2 = 1$ in Theorem 2.18. □

Fourthly, the following corollary gives the determinant of $C_n(A)_k$.

COROLLARY 2.22. *The determinant of $C_n(A)_k$ is given by*

$$\det(C_n(A)_k) = \frac{\Lambda_1^n \left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n + \left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(kE_n + (k - E_{-n})k^2 - 1)}$$

where

$$\begin{aligned} \Lambda_1 &= kA_n - 3, \\ \Lambda_2 &= k^{\frac{1}{n}}(-kA_{n+1} + 1), \\ \Lambda_3 &= k^{\frac{2}{n}}(kA_{n+2} - kA_n). \end{aligned}$$

Proof. Take $V_n = A_n$ with $A_0 = 3, A_1 = 1, A_2 = 3$ in Theorem 2.18. □

References

[1] Alptekin, E. G.Pell, Pell-Lucas ve Modified Pell sayıları ile tanımlı circulant ve semicirculant matrisler (Doctoral dissertation, Selçuk Üniversitesi Fen Bilimleri Enstitüsü), 2005.

[2] Cline, R.E., Plemmons, R. J., Worm, G., Generalized Inverses of Certain Toeplitz Matrices, Linear Algebra and its Applications 8, 25–33, 1974.

[3] Coskun, A., Taskara, N., On the Some Properties of Circulant Matrices with Third Order Linear Recurrent Sequences, Mathematical Sciences and Applications E-Notes, 6(1), 12-18, 2018.

[4] Davis, P. J., Circulant Matrices, JohnWiley&Sons, New York, 1979.

[5] Deveci, O., Karaduman, E., Campbell, C.M., The Fibonacci-Circulant Sequences and Their Applications, Iranian Journal of Science and Technology Transaction A-Science, 41(A4), 1033-1038, 2007.

[6] Deveci, O., On The Fibonacci-Circulant p-Sequences, Utilitas Mathematica, 108, 107-124, 2018.

[7] Horn, R.A., Johnson, C.R., Matrix Analysis, Cambridge University Press, Cambridge, 2nd edition, 2012.

[8] Horn, R.A., Johnson, C.R., Topics in Matrix Analysis, Cambridge University Press, Cambridge, 1991.

[9] Özkoç, A. Ardıyok, E., Circulant and Negacyclic Matrices Via Tetranacci Numbers. Honam Mathematical J. 38(4), 725-738, 2016.

[10] Pacheenburawana, A., Sintunavarat, W., On the Spectral Norms of r-Circulant Matrices with the Padovan and Perrin Sequences, Journal of Mathematical Analysis, 9(3), 110-122, 2018

[11] Polatlı, E., On Geometric Circulant Matrices Whose Entries are Bi-Periodic Fibonacci and Bi-Periodic Lucas Numbers, Universal Journal of Mathematics and Applications, 3(3), 102-108, 2020. https://doi.org/10.32323/ujma.669276

[12] Radicic, B., On k-Circulant Matrices Involving the Jacobsthal Numbers, Revista De La Union Matematica Argentina, 60(2), 431–442, 2019.

[13] Raza, Z., Riaz, M., Ali, M.A., Some Inequalities on the Norms of Special Matrices with Generalized Tribonacci and Generalized Pell-Padovan Sequences, arXiv, 2015, http://arxiv.org/abs/1407.1369v2

[14] Raza, Z., Ali, M.A., On the Norms of Circulant, r-Circulant, Semi-Circulant and Hankel Matrices with Tribonacci Sequence, arxiv, http://arxiv.org/abs/1407.1369v1, 2014.

[15] Sintunavarat, W., The Upper Bound Estimation for the Spectral Norm of r-Circulant and Symmetric r-Circulant Matrices with the Padovan Sequence, J. Nonlinear Sci. Appl. 9, 92-101, 2016.

[16] Sloane, N.J.A., The on-line encyclopedia of integer sequences, http://oeis.org/

[17] Soykan Y., Generalized Tribonacci Numbers: Summing Formulas, Int. J. Adv. Appl. Math. and Mech. 7(3), 57-76, 2020.

[18] Soykan, Y., A Closed Formula for the Sums of Squares of Generalized Tribonacci numbers, Journal of Progressive Research in Mathematics, 16(2), 2932-2941, 2020.

[19] Soykan, Y., On the Sums of Squares of Generalized Tribonacci Numbers: Closed Formulas of $\sum_{k=0}^{n} x^k V_k^2$, Archives of Current Research International, 20(4), 22-47, 2020. DOI: 10.9734/ACRI/2020/v20i430187

[20] Soykan, Y., Summing Formulas For Generalized Tribonacci Numbers, Universal Journal of Mathematics and Applications, 3(1), 1-11, 2020. ISSN 2619-9653, DOI: https://doi.org/10.32323/ujma.637876

[21] Soykan, Y., A Study On the Sums of Squares of Generalized Tribonacci Numbers: Closed Form Formulas of $\sum_{k=0}^{n} kx^k V_k^2$, Journal of Scientific Perspectives, 5(1), 1-23, 2021. DOI: https://doi.org/10.26900/jsp.5.1.02

[22] Soykan, Y., A Study on Sum Formulas for Generalized Tribonacci Numbers: Closed Forms of the Sum Formulas $\sum_{k=0}^{n} kx^k W_k$ and $\sum_{k=1}^{n} kx^k W_{-k}$, Journal of Progressive Research in Mathematics, 17(1), 1-40, 2020.

[23] Soykan, Y., Formulae For The Sums of Squares of Generalized Tribonacci Numbers: Closed Form Formulas of $\sum_{k=0}^{n} kV_k^2$, IOSR Journal of Mathematics, 16(4), 1-18, 2020. DOI: 10.9790/5728-1604010118

[24] Soykan, Y., On Sum Formulas for Generalized Tribonacci Sequence, Journal of Scientific Research & Reports, 26(7), 27-52, 2020. ISSN: 2320-0227, DOI: 10.9734/JSRR/2020/v26i730283

[25] Soykan Y., On Generalized Padovan Numbers, Preprints 2021, 2021100101 (doi: 10.20944/preprints202110.0101.v1).

[26] Turkmen, R., Gökbaş, H., On the Spectral Norm of r-Circulant Matrices with the Pell and Pell-Lucas Numbers, J. Inequal. Appl. 2016:65, 2016.

[27] Uygun Ş., Some Bounds for the Norms of Circulant Matrices with the k-Jacobsthal and k-Jacobsthal Lucas Numbers, Journal of Mathematics Research, 8(6), 133-138, 2016.

[28] Uygun Ş., Yaşamalı, S., On the Bounds for the Norms of Circulant Matrices with the Jacobsthal and Jacobsthal–Lucas Numbers, Notes on Number Theory and Discrete Mathematics, 23(1), 91-98, 2017.

[29] Uygun Ş., Yaşamalı, S., On the Bounds for the Norms of r-Circulant Matrices with the Jacobsthal and Jacobsthal–Lucas Numbers, International Journal of Pure and Applied Mathematics, 112(1), 93-102, 2017.

[30] G. Zielke, G., Some Remarks on Matrix Norms, Condition Numbers, and Error Estimates for Linear Equations, Linear Algebra Appl. 110, 29-41, 1988.