THE EXTENDED XLINDLEY DISTRIBUTION: ITS PROPERTIES AND APPLICATION TO LIFETIME DATA

Abstract: This study introduced An Extended Xlindley Distribution (EXD), which was derived from two-parameter Quasi XLindley distribution through the introduction of another parameter β. Some of its mathematical properties were obtained. The Maximum Likelihood Estimation method was used in estimating the parameters of the proposed distribution. The goodness of fit was tested with retrieved two different datasets, compared with other existing model and shows more efficient than the considered models.

Keywords: Quasi XLindley Distribution, Weighted Distribution, Maximum Likelihood Estimation, Moments, Goodness of fit.

Introduction: The simulation of actual events and natural phenomena using probability distributions is one of the fundamental techniques in the study of statistics and probability. Notwithstanding the fact that current probability distributions are insufficient to accurately represent the data generated by natural events, researchers have focused on developing probability distributions because of these characteristics. This affects the generalization and extension of probability distributions in multiple ways.

A new type of probability distribution arose due to the widespread availability of additional components. Merovci et al. (2014). In the statistical literature, many lifespan distributions have been established to provide data modelling in these applied disciplines more flexibility (Beghriche et al., 2023). Lindley Distribution was introduced by Lindley (1958) to analyse failure time data, especially in applications modelling stress - strength reliability.

The probability Distribution Function of Lindley distribution of random variable X, with scale parameter θ is given by:

$$f(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)\mathrm{e}^{-\theta x}; \qquad x>0,\ \theta>0 \tag{1}$$

When hazard rate is unimodal or bathtub shaped then Lindley Distribution is better than Exponential Distribution (Bakouch et al., 2012)..

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Shanker (2015) have comparative study on modeling of lifetime data using one parameter Lindley (1958) distribution and exponential distribution and concluded that there are many lifetime data where exponential distribution gives better fit than Lindley distribution. The findings led to introduce and propose of Two - Parameter XLindley Distribution by Ibrahim, Shah, and Haq (2023) named Quasi-XLindley (QXL) distribution. The PDF of the continuous random variable X of follows QXL is given by:

$$f(x) = \frac{\theta}{1+\alpha}\left(\alpha + \frac{\theta(1+x)}{1+\theta}\right)\mathrm{e}^{-\theta x}; \qquad x > 0; \alpha, \theta > 0 \tag{2}$$

with scale parameter α and θ.

Therefore, the idea of this work is to propose a new distribution called An Extended XLindley Distribution (EXD) by introducing the third (weight) parameter, with the hope that it will attract many applications in different disciplines such as reliability, survival analysis, biology and others. On applying the weighted version, the third parameter indexed, to this distribution, it is expected to be more flexible to describe different lifetime data than its sub-models.

In this paper, a three - parameter weighted quasi XLindley distribution which includes Lindley (1958) distribution, Exponential Lindley distribution and quasi XLindley distribution as particular cases, has been proposed and discussed. Its moments about origin and central moments, coefficient of variation, skewness, kurtosis and index of dispersion have been derived. The hazard rate function and Survival function of the distribution have been derived and their shapes have been discussed for varying values of the parameters. The estimation of its parameters has been discussed using maximum likelihood method. Finally, the goodness of fit and the applications of the distribution have been explained through two failure time data and the fit has been compared with one parameter Lindley distribution and the Quasi XLindley distribution.

Methods

Weighted Quasi XLindley Distribution

This section contained the application of the outlined methods and principles harnessed for the conduct of this research which is a Three Parameter Weighted Quasi Xlindley distribution with scale parameter α, θ and β

The method of moment, the coefficient of variation, Coefficient of Skewness, Coefficient of Kurtosis, Index of Dispersion, the survival function, the hazard function, and maximum likelihood estimation were considered by the study. This study employed the incorporation of parameter into the existing distributions. The pdf of Quasi XLindley distribution for a random variable X with scale parameter α and θ is defined as

$$f(x) = \frac{\theta}{1+\alpha}\left(\alpha + \frac{\theta(1+x)}{1+\theta})\mathrm{e}^{-\theta x}\right); \quad \alpha, \theta > 0; x > 0 \tag{3}$$

Thus, the probability density function of Weighted Quasi Xlindley distribution (WQXL) is obtained given as:

$$f_w(x) = \frac{w(x)f(x)}{E[w(x)]}; \quad x > 0 \tag{4}$$

where

$E[w(x)] = \int_0^\infty w(x)f(x)dx$ is the normalizing factor obtained to make the total probability equal to unity

and

$w(x) = x^\beta$ (as suggested by Rather and Ozel (2020).

It is the weighted version of X called the size-biased version of X and the distribution is called the size-biased distribution (Patil, Rao, Zelen, & Patil, 1987) with the pdf given in equation 4

$$\therefore E(x^\beta) = \int_0^\infty x^\beta f(x)dx \tag{5}$$

substituting equation (3) in (5), we have

$$= \int_0^\infty x^\beta \frac{\theta}{1+\alpha}\left(\alpha + \frac{\theta(1+x)}{1+\theta})\mathrm{e}^{-\theta x}\right) dx$$

Using Mathematica software to solve the integration, we have

$$f_w(x) = \frac{\theta^{(1+\beta)}}{1+\alpha+\beta}\frac{x^\beta(\alpha+\theta+x\theta+\alpha\theta)}{\Gamma(1+\beta)}e^{-x\theta} \tag{6}$$

α and θ are the existing scale while β is the new introduced parameter,

Proposition 1: Suppose a random variable X follows a Weighted Quasi XLindley Distribution, that is, X $\mid \alpha, \theta, \beta \sim WQXL(\alpha, \theta, \beta)$ then the pdf of random variable X is given as:

$$f(x;\alpha,\theta,\beta) = \frac{\theta^{(1+\beta)}}{1+\alpha+\beta}\frac{x^\beta(\alpha+\theta+x\theta+\alpha\theta)}{\Gamma(1+\beta)}e^{-x\theta}; x > 0, \alpha, \theta, \beta > 0 \tag{7}$$

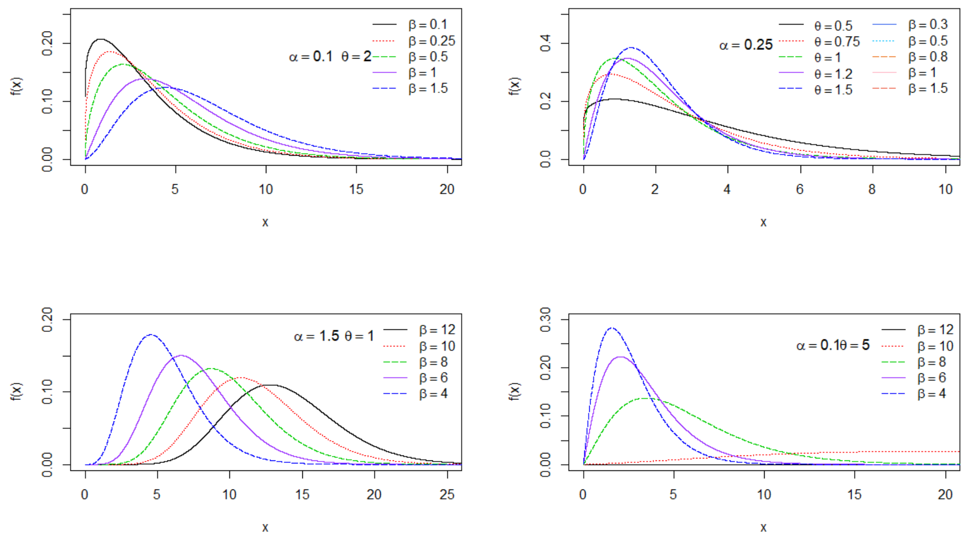

Figure 1: The probability density function of WQXL distribution for various choices of parameters

The graphs presented in Figure 3.1 below shows the pattern of the pdf of the proposed WQXL distribution at various values of α, θ and β . The pdf shows different shapes which makes it flexible to capture different shapes as may be exhibited by different data fields.

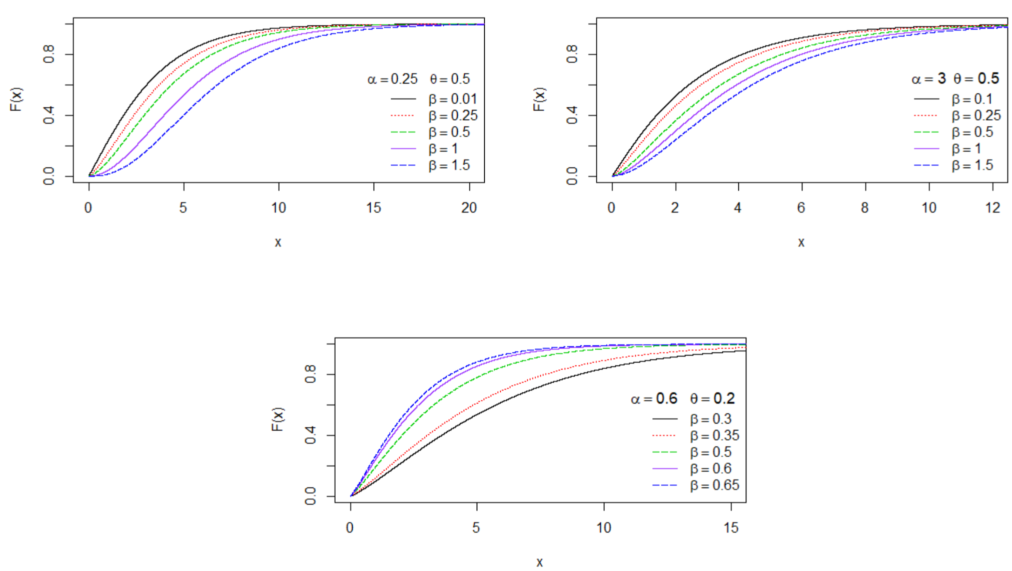

Figure 2: The cumulative distribution function of WQXL distribution for various choices of parameters

Moment

The moments of distribution are used to describe the characteristics or shape of the distribution. The rth moment of a Continuous random variable X:

$$E(X^r) = \mu'_r = \int_0^\infty x^r f(x)dx \tag{8}$$

Where 'r' is a positive integer.

Hence, the r^{th} moment of a random variable X with the WQXL distribution is obtained by putting the (7) into (9) to give the following:

$$E(X^r) = \mu'_r = \frac{\theta^{-r}(1 + r + \alpha + \beta + \theta + \alpha\theta)\Gamma(1 + r + \beta)}{1 + \alpha + \beta + \theta + \alpha\theta)\Gamma(1 + \beta)} \tag{9}$$

The first four moments of WQXL distribution are:

$$\mu'_1 = \frac{(1 + \beta)(2 + \alpha + \beta + \theta + \alpha\theta)}{\theta(1 + \alpha + \beta + \theta + \alpha\theta)}$$

$$\mu'_2 = \frac{(1 + \beta)(2 + \beta)(3 + \alpha + \beta + \theta + \alpha\theta)}{\theta^2(1 + \alpha + \beta + \theta + \alpha\theta)}$$

$$\mu'_3 = \frac{(4 + \alpha + \beta + \theta + \alpha\theta)\Gamma(4 + \beta)}{\theta^3(1 + \alpha + \beta + \theta + \alpha\theta)\Gamma(1 + \beta)}$$

$$\mu'_4 = \frac{(5 + \alpha + \beta + \theta + \alpha\theta)\Gamma(5 + \beta)}{\theta^4(1 + \alpha + \beta + \theta + \alpha\theta)\Gamma(1 + \beta)}$$

Given the random variable X having the WQXL with parameters α, θ and β the variance, coefficient of variation, Skewness, Kurtosis and Index of Variation is given by

$$\sigma^2 = \frac{(1 + \beta)(2 + \beta^2 + 4\theta + \theta^2 + \alpha^2(1 + \theta)^2 + 2\alpha(1 + \theta)(2 + \beta + \theta) + \beta(3 + 2\theta)}{\theta^2(1 + \alpha + \beta + \theta + \alpha\theta)^2}$$

$$CV = \frac{\sqrt{2 + \beta^2 + 4\theta + \theta^2 + \alpha^2(1 + \theta)^2 + 2\alpha(1 + \theta)(2 + \beta + \theta) + \beta(3 + 2\theta)}}{\theta(1 + \alpha + \beta + \theta + \alpha\theta)(2 + \alpha + \beta + \theta + \alpha\theta)}$$

$$S_k = \frac{(4 + \alpha + \beta + \theta + \alpha\theta)\Gamma(4 + \beta)}{\theta^3(1 + \alpha + \beta + \theta + \alpha\theta)\left[\frac{(1+\beta)(2+\beta^2+4\theta+\theta^2+\alpha^2(1+\theta)^2+2\alpha(1+\theta)(2+\beta+\theta)+\beta)(3+2\theta)}{\theta^2(1+\alpha+\beta+\theta+\alpha\theta)^2}\right]^{\frac{3}{2}}\Gamma(1 + \beta)}$$

$$K_s = \frac{(1 + \alpha + \beta + \theta + \alpha\theta)^3(5 + \alpha + \beta + theta + \alpha\theta)\Gamma(5 + \beta)}{(1 + \beta)^2(2 + \beta^2 + 4\theta + \theta^2 + \alpha^2(1 + \theta)^2 + 2\alpha(1 + theta))(2 + \beta + \theta) + \beta(3 + 2\theta))^2\Gamma(1 + \beta)}$$

$$ID_s = \frac{(2 + \beta^2 + 4\theta + \theta^2 + \alpha^2(1 + \theta)(2 + \beta + \theta) + \beta(3 + 2\theta))}{\theta(1 + \alpha + \beta + \theta + \alpha\theta)(2 + \alpha + \beta + \theta + \alpha\theta)}$$

Table 1: Statistical Metrics for $\alpha = 0.5$ and 1.0

Parameters			Statistic				
α	θ	β	μ	σ^2	Sk	Ks	ID
0.5	0.5	0.5	3.818182	15.421488	3.24635	7.66329	4.03896
		1.0	3.846154	29.514793	2.30271	3.99484	7.67385
		1.2	3.855072	35.744793	2.26331	3.41528	9.27215
		2.0	3.882353	63.986159	3.02181	2.32132	16.48128
	1.0	0.5	1.714286	4.025510	2.71879	6.07461	2.34821
		1.0	1.750000	7.437500	2.07066	3.47094	4.25000
		1.2	1.761905	8.964263	2.05592	3.01722	5.08782
		2.0	1.800000	15.960000	2.73753	2.11996	8.86667
	1.2	0.5	1.381579	2.819858	2.59035	5.70205	2.04104
		1.0	1.414729	5.169085	2.00632	3.32899	3.65376
		1.2	1.425926	6.223525	1.99674	2.90575	4.36455
		2.0	1.462264	11.069331	2.65266	2.05891	7.56999
	2.0	0.5	0.750000	1.031250	2.27313	4.80610	1.37500
		1.0	0.772727	1.857438	1.83150	2.95025	2.40374
		1.2	0.780702	2.231207	1.83224	2.60055	2.85795
		2.0	0.807692	3.963018	2.40832	1.88031	4.90659
1.0	0.5	0.5	3.428571	16.102041	2.71879	6.07461	4.69643
		1.0	3.500000	29.750000	2.07066	3.47094	8.50000
		1.2	3.523810	35.857052	2.05592	3.01722	10.17565
		2.0	3.600000	63.840000	2.73753	2.11996	17.73333
	1.0	0.5	1.555556	3.830247	2.65323	6.22178	2.46230
		1.0	1.600000	7.040000	2.05576	3.63184	4.40000
		1.2	1.615385	8.492071	2.03608	3.16369	5.25700
		2.0	1.666667	15.222222	2.59782	2.21947	9.13333
	1.2	0.5	1.258503	2.614724	2.64037	6.28454	2.07764
		1.0	1.296296	4.801097	2.05373	3.68742	3.70370
		1.2	1.309524	5.793084	2.03215	3.21372	4.42381
		2.0	1.354167	10.405816	2.55881	2.25334	7.68429
	2.0	0.5	0.692308	0.890902	2.61876	6.51118	1.28686
		1.0	0.714286	1.632653	2.05439	3.86880	2.28571
		1.2	0.722222	1.971728	2.02638	3.37610	2.73008
		2.0	0.750000	3.562500	2.45387	2.36380	4.75000

Table 2: Statistical Metrics for $\alpha = 1.2$ and 2.0

Parameters			Statistic				
α	θ	β	μ	σ^2	Sk	Ks	ID
1.2	0.5	0.5	3.315789	16.242382	2.59035	5.70205	4.89850
		1.0	3.395349	29.773932	2.00632	3.32899	8.76904
		1.2	3.422222	35.847506	1.99674	2.90575	10.47492
		2.0	3.509434	63.759345	2.65266	2.05891	18.16799
	1.0	0.5	1.510204	3.765202	2.64037	6.28454	2.49317
		1.0	1.555556	6.913580	2.05373	3.68742	4.44444
		1.2	1.571429	8.342041	2.03215	3.21372	5.30857
		2.0	1.625000	14.984375	2.55881	2.25334	9.22115
	1.2	0.5	1.223471	2.550794	2.66223	6.48996	2.08488
		1.0	1.261416	4.687370	2.07163	3.81140	3.71596
		1.2	1.274834	5.659565	2.04609	3.32024	4.43945
		2.0	1.320663	10.195421	2.53510	2.32114	7.71993
	2.0	0.5	0.676056	0.850166	2.74406	7.17596	1.25754
		1.0	0.697368	1.566309	2.13451	4.22168	2.24603
		1.2	0.705128	1.894589	2.09651	3.67332	2.68687
		2.0	0.732558	3.440103	2.47913	2.54849	4.69601
2.0	0.5	0.5	3.000000	16.500000	2.27313	4.80610	5.50000
		1.0	3.090909	29.719008	1.83150	2.95025	9.61497
		1.2	3.122807	35.699304	1.83224	2.60055	11.43180
		2.0	3.230769	63.408284	2.40832	1.88031	19.62637
	1.0	0.5	1.384615	3.563609	2.61876	6.51118	2.57372
		1.0	1.428571	6.530612	2.05439	3.86880	4.57143
		1.2	1.444444	7.886914	2.02638	3.37610	5.46017
		2.0	1.500000	14.250000	2.45387	2.36380	9.50000
	1.2	0.5	1.126761	2.361573	2.74406	7.17596	2.09590
		1.0	1.162281	4.350858	2.13451	4.22168	3.74338
		1.2	1.175214	5.262747	2.09651	3.67332	4.47812
		2.0	1.220930	9.555841	2.47913	2.54849	7.82669
	2.0	0.5	0.631579	0.735976	3.18247	9.68443	1.16530
		1.0	0.650000	1.377500	2.41228	5.53358	2.11923
		1.2	0.656863	1.673629	2.34056	4.77612	2.54791
		2.0	0.681818	3.080579	2.58928	3.23308	4.51818

Table 1 & 2 shows the nature of the Mean (μ_3), variance (σ^2), coefficient of skewness (S_k), coefficient of kurtosis (Ks) and Index of Dispersion (ID) of the WXLD for varying values of the parameters. The coefficient of skewness (S_k) indicates a positively skewed distribution that is moderately to highly asymmetric and coefficient of kurtosis (K_s) indicates a range of tail behaviors from slightly platykurtic (1.88031) to highly leptokurtic (9.68443).

Maximum Likelihood Estimation (MLEs)

The Maximum Likelihood Estimate (MLE) is a widely used method for estimating the parameters of an assumed probability distribution. This is because of MLE estimators have desirable properties such as consistency, asymptotic efficiency, and invariance. To obtain the maximum likelihood estimators of the parameters of the WQXLD, let x_1,x_2,...,$x_n$be a random sample of size n from the WQXLD with the log-likelihood function is defined as

$$L(\alpha,\theta,\beta;x_i) = \prod_{i=1}^{n} f(x_i;\alpha,\theta,\beta,) \tag{10}$$

This implies that

$$L(\alpha,\theta,\beta;x_i) = \left(\frac{\theta^{(1+\beta)}}{1+\alpha+\beta}\right)\prod_{i=1}^{n}\frac{x_i^{\beta}(\alpha+\theta+x\theta+\alpha\theta)}{\Gamma(1+\beta)}e^{-n\theta\bar{x}} \tag{11}$$

$$lnL(\alpha,\theta,\beta;x_i) = ln\left(\frac{\theta^{(1+\beta)}}{1+\alpha+\beta}\right)\sum_{i=1}^{n} ln\left(\frac{x_i^{\beta}(\alpha+\theta+x\theta+\alpha\theta)}{\Gamma(1+\beta)}e^{-n\theta\bar{x}}\right) \tag{12}$$

By simplifying, we have

$$\ell(\alpha,\theta,\beta;x_i) = (1+\beta)ln(\theta) - ln(1+\alpha+\beta) + \beta\sum_{i=1}^{n} ln(x_i) + ln(\alpha+\theta+x\theta+\alpha\theta) - ln(\Gamma(1+\beta)) - n\theta$$

let

$$g(\alpha,\theta,\beta,x_i) = ln(x_i) + ln(\alpha+\theta+x\theta+\alpha\theta) - ln(\Gamma(1+\beta)) - n\theta$$

the likelihood function becomes

$$\ell(\alpha,\theta,\beta,x_i) = (1+\beta)ln(\theta) - ln(1+\alpha+\beta) + \beta\sum_{i=1}^{n} ln(x_i) + g(\alpha,\theta,\beta,x_i)$$

To find the MLE $\hat{\alpha}$, $\hat{\theta}$, $\hat{\beta}$ take the partial derivatives of the log-likelihood function with respect to α, θ and β, and set them to zero are then the solution to non-linear.
To differentiate with respect to θ, we ignore the terms without θ, we have

$$\ell(\alpha,\theta,\beta,x_i) = (1+\beta)ln(\theta) + g(\alpha,\theta,\beta,x_i)$$

$$\frac{\partial\ell}{\partial\theta} = \sum_{i=1}^{n}\left(\frac{1+\beta}{\theta} + \frac{1+x_i+\alpha}{\alpha+\theta+x_i\theta+\alpha\theta} - n\right)$$

To differentiate with respect to α, we ignore the terms without α, we have

$$\ell(\alpha,\theta,\beta,x_i) = g(\alpha,\theta,\beta,x_i) - ln(1+\alpha+\beta)$$
$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{1+\alpha+\beta} + \sum_{i=1}^{n} \frac{1}{\alpha+\theta+x_i\theta+\alpha\theta}$$

To differentiate with respect to β, we ignore the terms without β, we have:

$$\ell(\alpha,\theta,\beta,x_i) = (1+\beta)ln(\theta) - ln(1+\alpha+\beta) + \beta\sum_{i=1}^{n} ln(x_i) + g(\alpha,\theta,\beta,x_i)$$
$$\frac{\partial \ell}{\partial \beta} = n\ln(\theta) - \psi(1+\beta) + \sum_{i=1}^{n} \ln(x_i)$$

These three natural log likelihood do not seem to be solved directly. However, the Fisher's scoring method was used to solve these equations and an Hessian Matrix of the log-likelihood function lnL, which consists of second-order partial derivatives of lnL with respect to the parameters α,θ, β. was obtained

$$I(\alpha,\theta,\beta) = \begin{bmatrix} \frac{\partial^2 lnL}{\partial\alpha^2} & \frac{\partial^2 lnL}{\partial\alpha\partial\theta} & \frac{\partial^2 lnL}{\partial\alpha\partial\beta} \\ \frac{\partial^2 lnL}{\partial\theta\partial\alpha} & \frac{\partial^2 lnL}{\partial\theta^2} & \frac{\partial^2 lnL}{\partial\theta\partial\beta} \\ \frac{\partial^2 lnL}{\partial\beta\partial\alpha} & \frac{\partial^2 lnL}{\partial\beta\partial\theta} & \frac{\partial^2 lnL}{\partial\beta^2} \end{bmatrix}$$

where α_0, θ_0 and β_0 are the initial values of α, θ and β respectively. These equations were solved iteratively till sufficiently close values of $\hat{\alpha}$, $\hat{\theta}$ and $\hat{\beta}$ are obtained.

Reliability Analysis

Survival Function

The survival function is a function that gives the probability that a patient, device, or other object of interest will survive past certain time (Jiang & Guterman, 2024). The survival function is the complementary cumulative distribution of the lifetime. Let the lifetime X be a continuous random variable with cumulative density function F(x) and probability density function f(x) on the interval [0, ∞].its survival function is

$$S(X) = 1 - F(x)$$

Therefore, the survival function for WQXL is given as:

$$S(x,\alpha,\theta,\beta) = 1 - F(x,\alpha,\theta,\beta)$$

$$S(x,\alpha,\theta,\beta) = 1 + \frac{x^\beta\theta^\beta(x\theta)^{-\beta}\left(-(1+\alpha+\beta+\theta+\alpha\theta)\Gamma(1+\beta) + (\alpha+\theta+\alpha\theta)\Gamma(1+\beta,x\theta) + \Gamma(2+\beta,x\theta)\right)}{(1+\alpha+\beta+\theta+\alpha\theta)\Gamma(1+\beta)}$$

Figure 3.3 presented the shapes of the Survival Function of the proposed WQXL distribution at selected values of α, θ and β. The plots also show a decreasing rate of survival at time x.

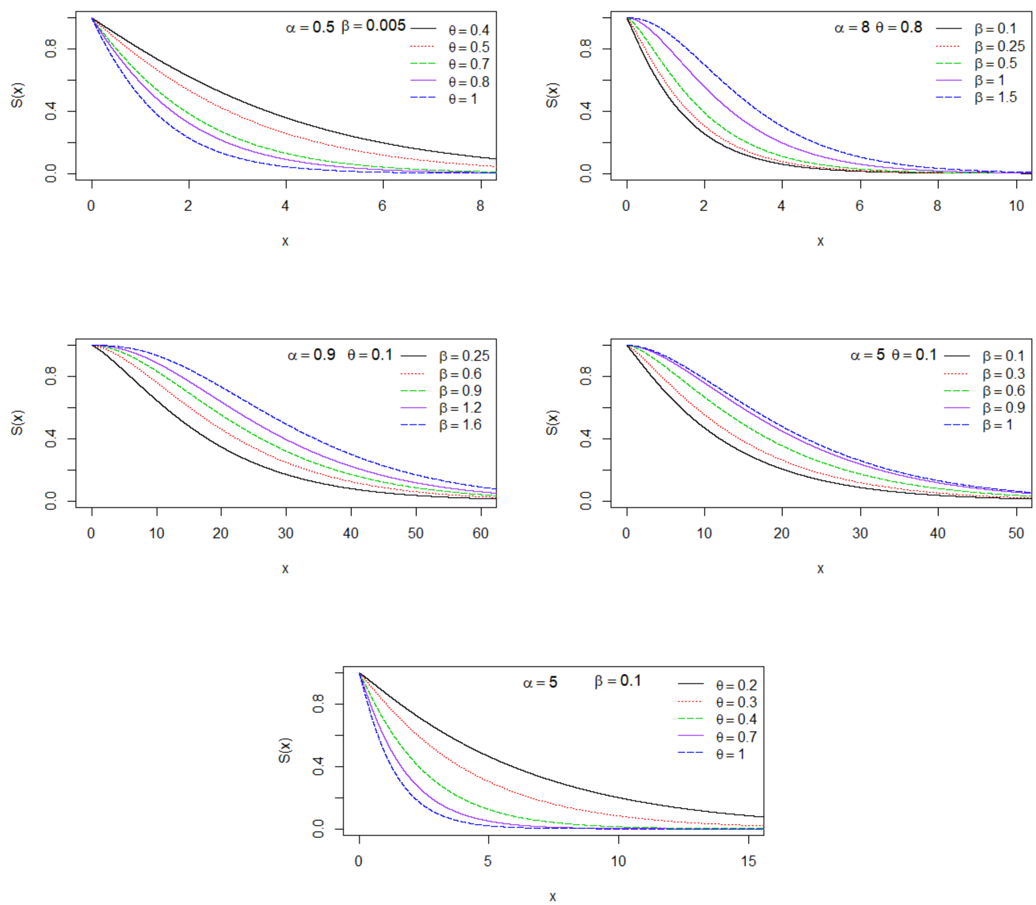

Figure 3: The Survival Function of WQXL distribution for various choices of parameter

Hazard Function

The hazard function h(x) of an event is the probability of the failure of the event at time x. It is the probability that an event will fail at a given time x. The hazard rate function h(x) is given as:

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{f(x)}{S(x)} \tag{13}$$

the Hazard rate function of WQXL distribution is given as

$$h(x, \alpha, \theta, \beta) = \frac{e^{-x\theta}\theta\left(\alpha + \frac{(1+x)\theta}{1+\theta}\right)}{(1+\alpha)\left(1 + \frac{x^{\beta}\theta^{\beta}(x\theta)^{-\beta}(-(1+\alpha+\beta+\theta+\alpha\theta)\Gamma(1+\beta)+(\alpha+\theta+\alpha\theta)\Gamma(1+\beta,x\theta)+\Gamma(2+\beta,x\theta))}{(1+\alpha+\beta+\theta+\alpha\theta)\Gamma(1+\beta)}\right)} \tag{14}$$

Figure 4 presented the shapes of the Hazard Function of the proposed WQXL distribution at selected values of α, θ and β. The plots show an increasing pattern which suggests that items are more likely to fail at the beginning after which they maintain a constant failure rate. This means that the failure may likely happen during the useful life of the item and failure occurs at random.

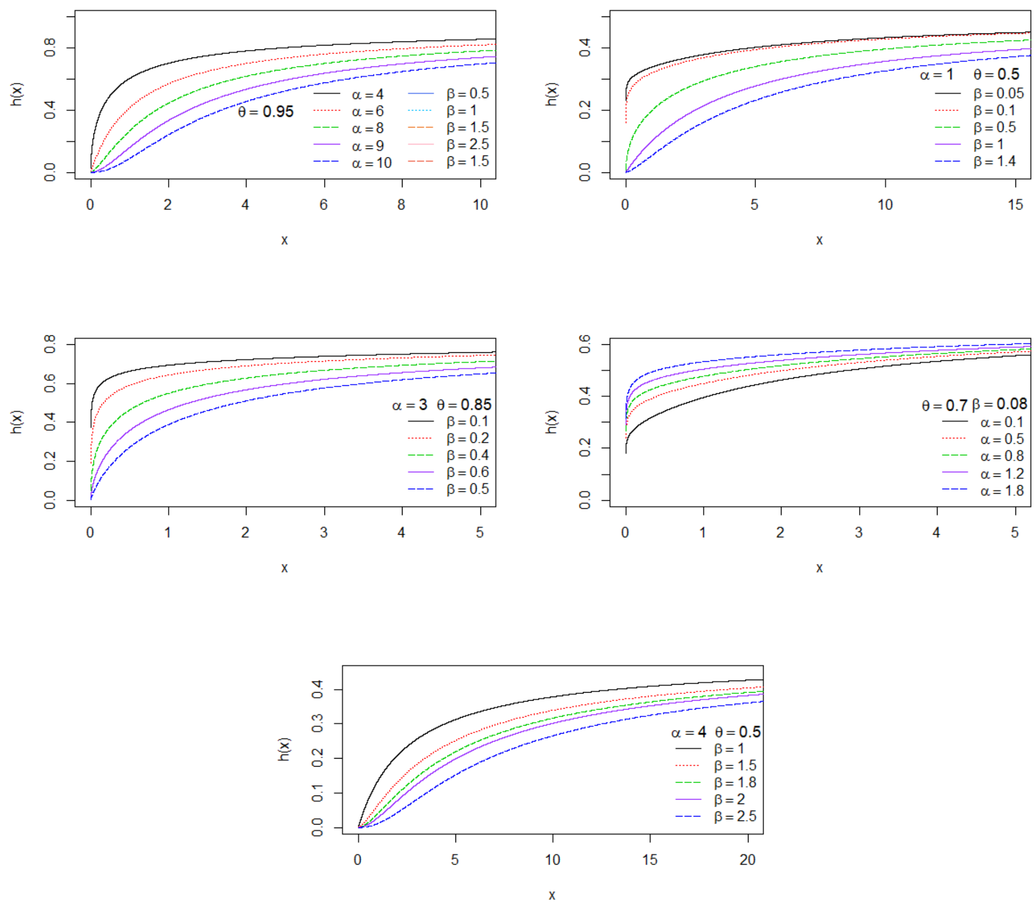

Figure 4: The Hazard Rate Function of WQXL distribution for various choices of parameter

Models Diagnostic Method

The model selection criteria considered in this thesis are the AIC (Akaike Information Criterion) by Akaike (1974), AICC (Corrected Akaike Information Criterion) by Kletting and Glatting (2009), HQIC (Hannan-Quinn Information Criterion) by Maïnassara and Kokonendji (2016) and BIC (Bayesian Information Criterion) by Weakliem (1999). Where the AIC, AICC, HQIC and BIC are obtained as follows:

$$AIC = 2k - 2ln(L) \tag{15}$$

$$AICC = AIC + \frac{k(k-1)}{n-k-1} \tag{16}$$

$$HQIC = -2\ln(L) + 6\ln(\ln(n)) \tag{17}$$

$$BIC = kln(n) - 2ln(L) \tag{18}$$

Where k is the number of parameters in the distribution and n is the number of observations.

Applications **Dataset 1**

The dataset represents the remission times (in months) of a random sample of 128 bladder cancer patients reported by by Lee and Wang (2003)

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52,4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80,25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54,3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32,10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90,2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41,7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12,1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62,7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46,4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02,2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07,21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Dataset 2

The following data set represents the tree circumferences in Marshall, Minnesota and reported by Shakil et al. (2010)

1.8, 1.8, 1.9, 2.4, 3.1, 3.4, 3.7, 3.7, 3.8, 3.9, 4.0, 4.1, 4.9, 5.1, 5.1, 5.2, 5.3, 5.5, 8.3, 13.7

Table 3: Parameter estimates and goodness of fit test statistics for data set1

Model	MLE	SE	-2logL	AIC	AICC	HQIC	BIC
WQXLD	$\hat{\alpha} = 8.0088129$ $\hat{\theta} = 0.1751757$ $\hat{\beta} = 0.3675707$	$\hat{\alpha} = 14.08806507$ $\hat{\theta} = 0.02289153$ $\hat{\beta} = 0.18971355$	**791.4412**	**797.4412**	**797.6348**	**800.9176**	**801.1453**
QXLD	$\hat{\alpha} = 9.3656364$ $\hat{\theta} = 0.1159141$	$\hat{\alpha} = 17.61272918$ $\hat{\theta} = 0.01835956$	828.8488	832.8488	832.9448	835.1664	838.5529
ATPLD	$\hat{\alpha} = 9.4726004$ $\hat{\theta} = 0.1067767$ $\hat{\beta} = 0.00000152$	$\hat{\alpha} = 1.238039$ $\hat{\theta} = 2.958895$ $\hat{\beta} = 2.642266$	828.6838	834.6838	834.8774	838.1602	838.3879
MQLD	$\hat{\alpha} = 0.1734$ $\hat{\theta} = 4.2845703$ $\hat{\beta} = 7.2913652$	$\hat{\alpha} = 2.18535413$ $\hat{\theta} = 0.1152096$ $\hat{\beta} = 4.157229884$	826.1754	832.1754	832.3689	835.6518	835.8794
WQLD	$\hat{\alpha} = 9.3661663$ $\hat{\theta} = 0.1352041$ $\hat{\beta} = 1.1588063$	$\hat{\alpha} = 10.57737185$ $\hat{\theta} = 0.02030864$ $\hat{\beta} = 0.13066425$	827.1847	831.1847	831.2807	833.5023	836.8888

Table 4: Parameter estimates and goodness of fit test statistics for data set2

Model	MLE	SE	-2logL	AIC	AICC	HQIC	BIC
WQXLD	$\hat{\alpha} = 1570.209396$ $\hat{\theta} = 2.248653$ $\hat{\beta} = 8.504710$	$\hat{\alpha} = 0.0003355662$ $\hat{\theta} = 5.105475$ $\hat{\beta} = 2.153904$	**48.51669**	**54.51669**	**55.92846**	**55.19676**	**54.60574**
QXLD	$\hat{\alpha} = 0.0010$ $\hat{\theta} = 0.158$	$\hat{\alpha} = 1.334$ $\hat{\theta} = 0.3791782$	94.15925	98.07925	98.77513	98.462	100.07
ATPLD	$\hat{\alpha} = 0.0100$ $\hat{\theta} = 0.9257282$ $\hat{\beta} = 4.1981806$	$\hat{\alpha} = 1.1102399$ $\hat{\theta} = 0.2994372$ $\hat{\beta} = 1.2783275$	85.97328	91.1676	92.6676	91.75073	94.15479
MQLD	$\hat{\alpha} = 0.001101142$ $\hat{\theta} = 0.925568985$ $\hat{\beta} = 4.197450679$	$\hat{\alpha} = 3.3458615$ $\hat{\theta} = 0.2993867$ $\hat{\beta} = 1.2780963$	85.16757	91.16757	92.66757	91.7507	91.15903
WQLD	$\hat{\alpha} = 0.1$ $\hat{\theta} = 0.926923$ $\hat{\beta} = 3.206593$	$\hat{\alpha} = 8.5449816$ $\hat{\theta} = 0.05449816$ $\hat{\beta} = 1.5449816$	91.17018	91.17018	92.67018	91.75332	94.15738

Table 3 which represents the goodness of fit tested with the retrieved data of 128 patients with Bladder cancer to recovery, it was observed that the WQXLD yielded the smallest values in terms of AIC, AICC, HQIC, BIC and log likelihood values compared with all the competing models. This suggests that the WQXLD model provides more accurate estimates as well as better fits among all the considered distributions which have been fitted to the patients with Bladder cancer data set.

Table 4 which represents the goodness of fit test with the retrieved data of 20 Tree Circumferences in Marshall, Minnesota, it was observed that the WQXLD yielded the smallest values in terms of AIC, AICC, HQIC, BIC and log likelihood values compared with all the competing models. This suggests that the WQXLD model provides more accurate estimates as well as better fits among all the considered distributions which have been fitted to the tree Circumferences data set.

Conclusion

This paper aims to derive a new distribution known as Weighted Quasi XLindley Distribution (*WQXLD*) by introducing another parameter β. The parameter is the weighted version of X called the size-biased version of X. The mathematical properties such as the method of moment, the coefficient of variation, the survival function, the hazard function were discussed and Maximum likelihood estimate was used to estimate the parameters. Finally, two datasets were used to check real life applicability of the WQXLD. The negative log-likelihood, criterion values such as *AIC*, *AICC*, *HQIC* and *BIC* were used for comparison with *QXL*, *ATPLD*, *WQLD*, and *MQLD*. The results from table 3 and 4 revealed that *WQXLD* produced low values in terms of *-LL*, *AIC*, *AICC*, *HQIC* and *BIC* which shows that comparatively *WQXLD* is more efficient than the existing models considered. Hence, the *WQXLD* can be considered an important life time distribution for modelling data.

Future Research

Having consider WQXLD as an important lifetime distribution for modeling data, it is therefore suggested that future research should include more goodness of fit test and simulation of various sample size.

Conflict of Interest Authors declare that there is no conflict of interest regarding this article.

References

Aderoju, S. (2021). Samade probability distribution: its properties and application to real lifetime data. *Asian Journal of Probability and Statistics, 14* (1), 1–11.

Ahmad, Z., Hamedani, G. G., & Butt, N. S. (2019). Recent developments in distribution theory: a brief survey and some new generalized classes of distributions. *Pakistan Journal of Statistics and Operation Research*, 87–110.

Akaike, H. (1974). A new look at the statistical model identification. *IEEE transactions on automatic control, 19* (6), 716–723.

Alizadeh, M., Ranjbar, V., Eftekharian, A., & Kharazmi, O. (2021). An extended lindley distribution with application to lifetime data with bayesian estimation. *Statistics, Optimization & Information Computing, 9* (4), 849–870.

Bakouch, H. S., Ristic, M. M., Asgharzadeh, A., Esmaily, L., & Al-Zahrani, B. M. (2012). An exponentiated exponential binomial distribution with application. *Statistics & Probability Letters, 82* (6), 1067–1081.

Beghriche, A., Tashkandy, Y. A., Bakr, M. E., Halim, Z., Gemeay, A. M., Hossain, M. M., et al. (2023). The inverse xlindley distribution: Properties and application. *IEEE Access*.

Bhati, D., & Ravi, S. (2018). On generalized log-moyal distribution: a new heavy tailed size distribution. *Insurance: Mathematics and Economics, 79*, 247–259.

Chouia, S., & Zeghdoudi, H. (2021). The xlindley distribution: Properties and application. *Journal of Statistical Theory and Applications, 20*(2), 318–327.

Frühwirth-Schnatter, S. (2006). *Finite mixture and markov switching models.* Springer. Gómez-Déniz, E., & Calderín-Ojeda, E. (2011). The discrete lindley distribution: properties and applications.*Journal of statistical computation and simulation, 81* (11), 1405–1416.

Gupta, R. C., & Keating, J. P. (1986). Relations for reliability measures under length biased sampling. *Scandinavian Journal of Statistics*, 49–56.

Gupta, R. C., & Kirmani, S. (1990). The role of weighted distributions in stochastic modeling. *Communications in Statistics-Theory and methods*, 19 (9), 3147–3162.

Ibrahim, M., Shah, M. K. A., & Haq, M. Ahsan-ul. (2023). New two-parameter xlindley distribution with statistical properties, simulation and applications on lifetime data. International *Journal of Modelling and Simulation*, 1–14.

Jiang, F., & Guterman, E. (2024). Survival analysis. *Statistical Methods in Epilepsy*, 124–142.

Kletting, P., & Glatting, G. (2009). Model selection for time-activity curves: the corrected akaike information criterion and the f-test. *Zeitschrift für medizinische Physik, 19* (3), 200–206.

Lee, E. T. (2003). Statistical methods for survival data analysis. *John Wiley & Sons.*

Lindley, D. V. (1958). Fiducial distributions and bayes' theorem. *Journal of the Royal Statistical Society. Series B (Methodological)*, 102–107.

Maïnassara, Y. B., & Kokonendji, C. C. (2016). Modified schwarz and hannan–quinn information criteria for weak varma models. *Statistical Inference for Stochastic Processes, 19*, 199–217.

Meriem, B., Gemeay, A. M., Almetwally, E. M., Halim, Z., Alshawarbeh, E., Abdulrahman, A. T., et al. (2022). [retracted] the power xlindley distribution: Statistical inference, fuzzy reliability, and covid-19 application. *Journal of Function Spaces, 2022* (1), 9094078.

Merovci, F., Sharma, V. K., et al. (2014). The beta-lindley distribution: properties and applications. *Journal of applied mathematics*, 2014 .

Murthy, D. P., Xie, M., & Jiang, R. (2004). *Weibull models.* John Wiley & Sons.

Oluyede, B. O. (1999). On inequalities and selection of experiments for length biased distributions. *Probability in the Engineering and Informational Sciences, 13* (2), 169–185.

Patil, G. P., Rao, C. R., Zelen, M., & Patil, G. P. (1987). *Weighted distributions.* Citeseer.

R Core Team. (2023). R: A language and environment for statistical computing. R Foundation for Statistical Computing. https://www.R-project.org/

Rather, A., & Ozel, G. (2020). The weighted power lindley distribution with applications on the life time data. *Pakistan Journal of Statistics and operation research, 16* (2).

Saghir, A., Hamedani, G. G., Tazeem, S., & Khadim, A. (2017). Weighted distributions: A brief review, perspective and characterizations. *International journal of Statistics and Probability*

Shakil, M., Kibria, B. G., & Singh, J. N. (2010). A new family of distributions based on the generalized pearson differential equation with some applications. *Austrian journal of statistics, 39*(3), 259–278.

Shanker, R. (2015). Shanker distribution and its applications. *International journal of statistics and Applications, 5* (6), 338–348.

Shanker, R., & Mishra, A. (2013). A two-parameter lindley distribution. *Statistics in Transition new series, 14* (1), 45–56.

Shanker, R., Shukla, K. K., & Leonida, T. A. (2019). Weighted quasi lindley distribution with properties and applications. *International Journal of Statistics and Applications, 9* (1), 8–20.

Shanker, R., Shukla, K. K., Shanker, R., & Leonida, T. A. (2017). A three-parameter lindley distribution. *American Journal of Mathematics and Statistics, 7* (1), 15–26.

Sharma, V. K., Singh, S. K., Singh, U., & Agiwal, V. (2015). The inverse lindley distribution: a stress-strength reliability model with application to head and neck cancer data. *Journal of Industrial and Production Engineering, 32* (3), 162–173.

Tahir, M. H., & Cordeiro, G. M. (2016). Compounding of distributions: a survey and new generalized classes. *Journal of Statistical Distributions and Applications, 3* , 1–35.

Tharshan, R., & Wijekoon, P. (2021). A modification of the quasi lindley distribution. Open *Journal of Statistics*, 11 (3), 369–392.

Weakliem, D. L. (1999). A critique of the bayesian information criterion for model selection. *Sociological Methods & Research, 27* (3), 359–397.

Zichuan, M., Hussain, S., Iftikhar, A., Ilyas, M., Ahmad, Z., Khan, D. M., et al. (2020). A new extended-x family of distributions: Properties and applications. *Computational and mathematical methods in medicine, 2020* (1), 4650520.