AN EXTENDED INVERSE LINDLEY DISTRIBUTION; MODEL, PROPERTIES AND APPLICATIONS

ABSTRACT

This research work proposed a new extension of the inverse Lindley distribution called exponential-inverse Lindley distribution. The study derived some Statistical and mathematical properties of the distribution such as its ordinary moments, moment generating and characteristics function. It also considered the survival and hazard functions and the distribution of ordered statistics. Some plots of the distribution revealed that it is a flexible and skewed distribution. The implications of the plots for the survival function indicate that the exponential-inverse Lindley distribution could be used to model time or age-dependent events, where survival rate decreases with time. The research also conducted a simulation study to check the consistency of the model parameters using maximum likelihood estimation. From the results of the simulation study, it was revealed that the average estimates tend to be closer to the true parameters when sample size increases and the biases and mean square errors all decrease as sample size increases which is in agreement with the theory of estimation

KEYWORDS: Inverse Lindley distribution, Exponential-inverse Lindley Distribution, Statistical Properties, Parameters Estimation, Method of Likelihood Estimation

1. INTRODUCTION

Lindley (1958) defined and investigated a probability distribution in context of fiducial statistic as a counter example of Bayesian theory which was later called "Lindley distribution". Details of the fundamental properties of the Lindley distribution with its applications are available in Ieren *et al*. (2019).

The inverse Lindley distribution is another form of the Lindley distribution and research has shown that authors mainly focused on the Lindley distribution and little research has been done on the inverse Lindley distribution. Sharma *et al*. (2015) discussed the properties of inverse Lindley distribution with application to stress strength reliability analysis. Sharma *et al*. (2016) introduced a two parameter extension of inverse Lindley distribution (generalized inverse Lindley distribution). Also, Alkarni (2015) proposed a three parameter inverse Lindley distribution (extended inverse Lindley distribution) with application to maximum flood level data.

Several families of distributions have been proposed in literature such as quadratic rank transmutation map by Shaw and Buckley (2007), Exponentiated T-X by Alzaghal *et al*. (2013), Weibull-X by Alzaatreh *et al*. (2013), Weibull-G by Bourguignon *et al*. (2014), a Lomax-G family by Cordeiro *et al.* (2014), a new Weibull-G family by Tahir *et al.* (2016), a Lindley-G family by Cakmakyapan and Ozel (2016), a Gompertz-G family by Alizadeh *et al.* (2017) and Odd Lindley-G family by Gomes-Silva *et al*. (2017) and odd Lomax-G family by Cordeiro *et al.* (2019). These families have been used to study compound distribution such as exponential-Lindley distribution by Ieren and Balogun (2021), transmuted odd Lindley-Rayleigh distribution by Umar et al. (2021), transmuted Kumaraswamy distribution by Khan *et al*. (2016), Power Lindley distribution by Ghitany *et al.* (2013), bivariate generalized Rayleigh distribution by Abdel-Hady (2013), Lomax-Frechet distribution by Gupta *et al*. (2015), Weibull-Frechet by Afify *et al*. (2016) and transmuted odd generalized exponential-exponential distribution Abdullahi *et al*. (2018)

The cumulative distribution function (c.d.f) and probability density function (pdf) of the Inverse Lindley distribution (ILD) are defined as:

$$G(x)=\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}} \tag{1}$$

and

$$g(x)=\frac{\theta^2}{\theta+1}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}} \tag{2}$$

respectively, for $x>0$ and $\theta>0$ where θ is the scale parameter of ILD.

2. The Exponential-Inverse Lindley Distribution (EILD).

According to Bouguignon *et al.*, (2014), the cumulative distribution function (cdf) and the probability density function (pdf) (for $x>0$) of the exponential-G family of distributions with an additional shape parameter ($\alpha>0$) are defined by:

$$F(x)=1-\exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]\right\} \tag{3}$$

and

$$f(x)=\frac{\alpha g(x)}{\left[1-G(x)\right]^2}\exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]\right\} \tag{4}$$

respectively, where $g(x)$ and $G(x)$ represent the pdf and the cdf of the continuous distribution to be modified respectively.

Using equation (1) and (2) in (3) and (4) and simplifying, the cdf and pdf of the EILD are obtained as follows:

$$F(x)=1-\exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]\right\}$$

$$F(x)=1-\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\} \quad (5)$$

And

$$f(x)=\frac{\alpha g(x)}{[1-G(x)]^2}\exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]\right\}$$

$$f(x)=\frac{\alpha\frac{\theta^2}{\theta+1}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}}{\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^2}\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}$$

$$f(x)=\frac{\alpha\theta^2\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}}{(\theta+1)\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^2}\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\} \quad (6)$$

respectively.

For $x>0$, $\alpha,\theta>0$ where $\alpha>0$ is a shape parameter and $\theta>0$ is a scale parameter. Hence equation (5) and (6) are the cdf and pdf of the EILD.

For some chosen values of the parameters α, β and θ, some possible shapes for the *pdf* and the *cdf* of the EILD as shown in figure 1 and 2 below:

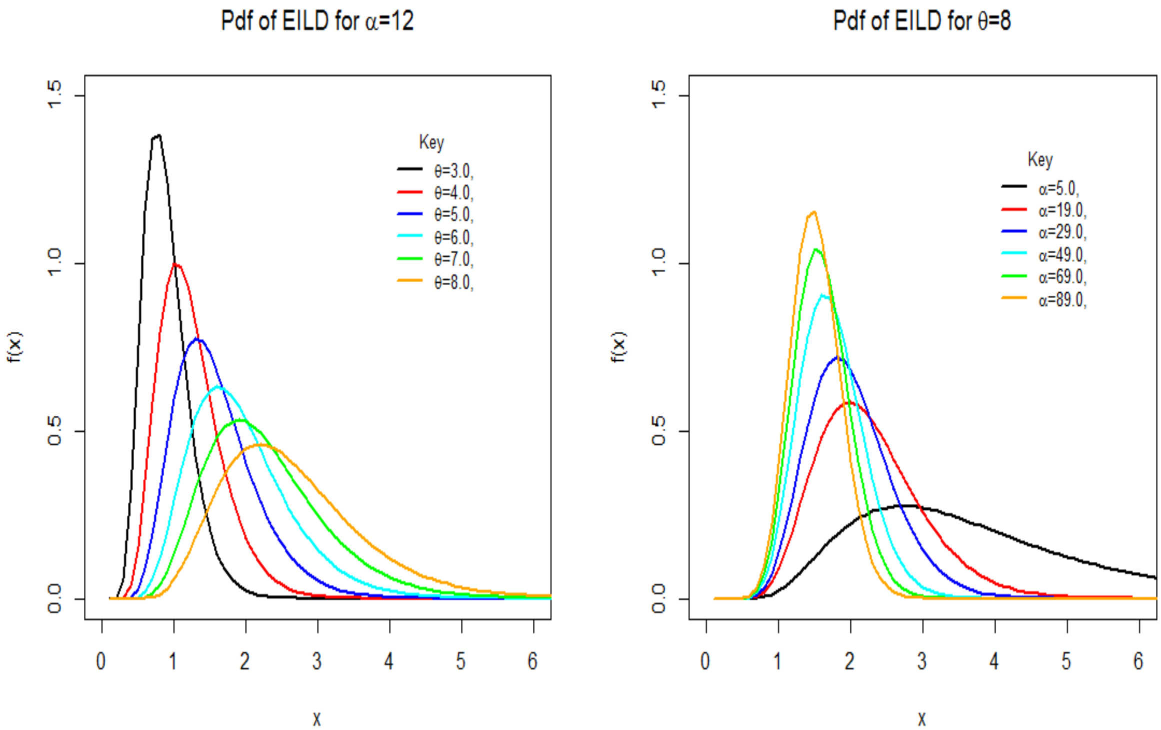


Figure 1: PDF of the EILD for different parameter values.

Figure **1** indicates that the EILD is a skewed or flexible distribution depending on the parameter values. This means that the distribution can be appropriate for datasets with different shapes.

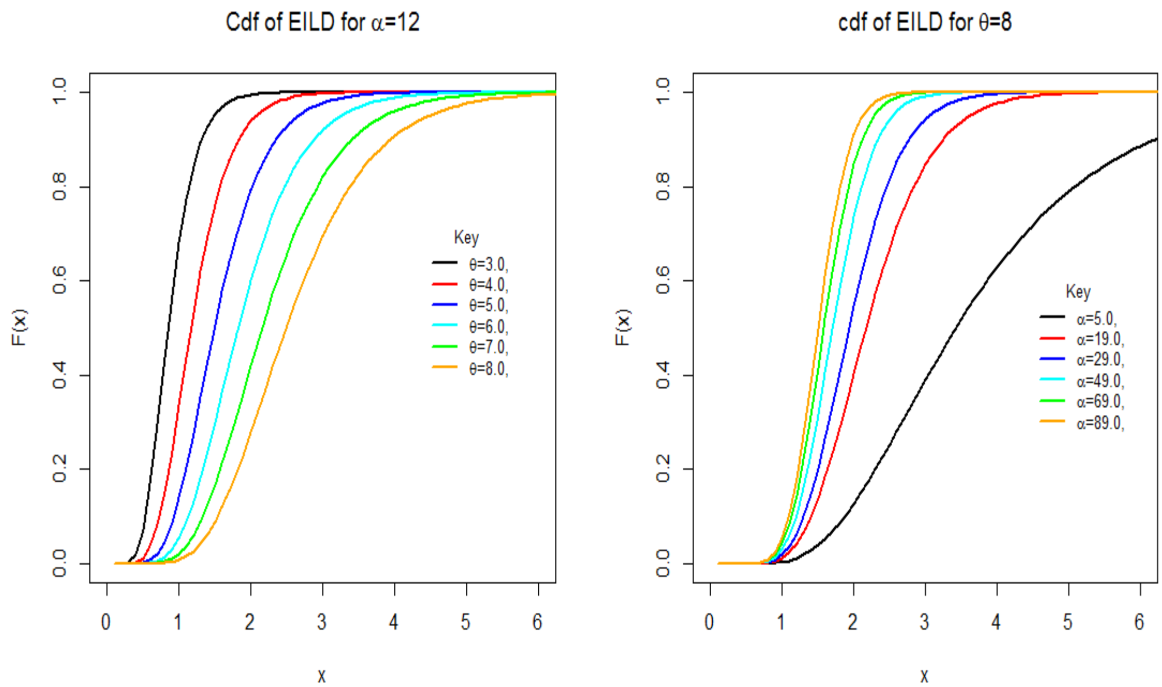


Figure 2: CDF of the EILD for different parameter values.

From the above *cdf* plot, the *cdf* increases when *X* increases, and approaches 1 when *X* becomes large, as expected.

3. Some Properties of the Proposed Distribution

3.1 Moments

Let X denote a continuous random variable, the n^{th} moment of X is given by;

$$\mu_n' = E\left[X^n\right] = \int_0^\infty x^n f(x)\,dx \tag{7}$$

Using the pdf of the EILD as given in equation (6).

$$f(x) = \frac{\alpha\theta^2\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}}{(\theta+1)\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^2}\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\} \tag{8}$$

Before substituting (8) in (7), we perform the expansion, simplification and linear representation of the pdf as follows:

First, by using power series expansion on the last term in the pdf above, we have:

$$\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}=\sum_{k=0}^{\infty}\frac{(-1)^k\alpha^k}{k!}\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]^k \tag{9}$$

By using the result in equation (9) and simplifying, equation (8) becomes

$$f(x)=\sum_{k=0}^{\infty}\frac{(-1)^k\alpha^{k+1}\theta^2}{(\theta+1)k!}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}\left[\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^k\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^{-(k+2)} \tag{10}$$

Using binomial expansion on the last term in equation (10) gives

$$\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^{-(k+2)}=\sum_{l=0}^{\infty}(-1)^l\binom{-k-2}{l}\left[\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^l \tag{11}$$

By substituting the result of equation (11) in (10), we obtain

$$f(x)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{-k-2}{l}\frac{(-1)^{k+l}\alpha^{k+1}\theta^2}{(\theta+1)k!}\left(\frac{1+x}{x^3}\right)e^{-(k+l+1)\frac{\theta}{x}}\left[\left(1+\frac{\theta}{(\theta+1)x}\right)\right]^{k+l} \tag{12}$$

Again using binomial expansion on the last term in (12) gives:

$$\left(1+\frac{\theta}{(\theta+1)x}\right)^{k+l}=\sum_{m=0}^{\infty}\binom{k+l}{m}\left(\frac{\theta}{\theta+1}\right)^m x^{-m} \tag{13}$$

Making use of the result (13) in equation (12) and simplifying, we obtain:

$$f(x)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{k+l}{m}\left(\frac{\theta}{\theta+1}\right)^m\binom{-k-2}{l}\frac{(-1)^{k+l}\alpha^{k+1}\theta^2}{(\theta+1)k!}(1+x)x^{-m-3}e^{-(k+l+1)\frac{\theta}{x}} \tag{14}$$

Now, let $W_{klm}=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{k+l}{m}\left(\frac{\theta}{\theta+1}\right)^m\binom{-k-2}{l}\frac{(-1)^{k+l}\alpha^{k+1}\theta^2}{(\theta+1)k!}$ be a constant, which implies that the pdf in equation (14) can also be written in its simple and linear form as:

$$f(x)=W_{klm}(1+x)x^{-m-3}e^{-(k+l+1)\frac{\theta}{x}} \tag{15}$$

Now, using the simplified of the pdf of the EILD in equation (15), the n^{th} ordinary moment of the EILD is derived as follows:

$$\mu_n' = E\left(X^n\right) = \int_0^\infty x^n f(x)dx = W_{klm}\int_0^\infty x^{n-m-3}(1+x)e^{-(k+l+1)\frac{\theta}{x}}dx \quad (16)$$

Making use of integration by substitution method in equation (16), we perform the following operations:

Let $u = \theta(k+l+1)x^{-1} \Rightarrow x = \left(\frac{u}{\theta(k+l+1)}\right)^{-1}$ such that $\frac{du}{dx} = -\theta(k+l+1)x^{-2} \Rightarrow dx = -\frac{du}{\theta(k+l+1)x^{-2}}$.

Substituting for x, u and dx in equation (16) and simplifying; we have:

$$\mu_n' = E\left(X^n\right) = W_{klm}(-1)\left(\frac{1}{\theta(k+l+1)}\right)^{m-n+1}\left[\left(\frac{1}{\theta(k+l+1)}\right)\int_0^\infty u^{m-n+2-1}e^{-u}du + \int_0^\infty u^{m-n+1-1}e^{-u}du\right] \quad (17)$$

Now, recall that $\int_0^\infty t^{k-1}e^{-t}dt = \Gamma(k)$ and that $\int_0^\infty t^k e^{-t}dt = \int_0^\infty t^{k+1-1}e^{-t}dt = \Gamma(k+1)$

Using the statement above, the n^{th} ordinary moment of X for the EILD is obtained as:

$$\mu_n' = E\left(X^n\right) = W_{klm}(-1)\left(\frac{1}{\theta(k+l+1)}\right)^{m-n+1}\left[\left(\frac{1}{\theta(k+l+1)}\right)\Gamma(m-n+2) + \Gamma(m-n+1)\right]$$

$$\mu_n' = E\left(X^n\right) = W_{klm}(-1)\left[\frac{\Gamma(m-n+2)}{(\theta(k+l+1))^{m-n+2}} + \frac{\Gamma(m-n+1)}{(\theta(k+l+1))^{m-n+1}}\right] \quad (18)$$

Again recall that W_{klm} is a constant and making use of its value as defined previously, the expression for the nth ordinary moment of EILD becomes:

$$\mu_n' = E\left(X^n\right) = \sum_{k=0}^\infty\sum_{l=0}^\infty\sum_{m=0}^\infty \binom{k+l}{m}\left(\frac{\theta}{\theta+1}\right)^m \binom{-k-2}{l}\frac{(-1)^{k+l+1}\alpha^{k+1}\theta^2}{(\theta+1)k!}\left[\frac{\Gamma(m-n+2)}{(\theta(k+l+1))^{m-n+2}} + \frac{\Gamma(m-n+1)}{(\theta(k+l+1))^{m-n+1}}\right] \quad (19)$$

3.2 Moment Generating Function

The moment generating function of a random variable X can be obtained as

$$M_x(t) = E\left[e^{tx}\right] = \int_{-\infty}^\infty e^{tx} f(x)dx \quad (20)$$

Using power series expansion in equation (20) and simplifying the integral gives:

$$M_x(t) = \sum_{r=0}^\infty \frac{t^r}{r!}\left[\sum_{k=0}^\infty\sum_{l=0}^\infty\sum_{m=0}^\infty \binom{k+l}{m}\left(\frac{\theta}{\theta+1}\right)^m \binom{-k-2}{l}\frac{(-1)^{k+l+1}\alpha^{k+1}\theta^2}{(\theta+1)k!}\left[\frac{\Gamma(m-r+2)}{(\theta(k+l+1))^{m-r+2}} + \frac{\Gamma(m-r+1)}{(\theta(k+l+1))^{m-r+1}}\right]\right] \quad (21)$$

3.3 Characteristics Function

A representation for the characteristics function is given by

$$\phi_x(t) = E\left(e^{itx}\right) = \int_0^{\infty} e^{itx} f(x)\,dx \qquad (22)$$

Hence, simple algebra and use of power series expansion in (22) above yields:

$$\phi_x(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \binom{k+l}{m} \left(\frac{\theta}{\theta+1}\right)^m \binom{-k-2}{l} \frac{(-1)^{k+l+1} \alpha^{k+1} \theta^2}{(\theta+1)k!} \left[\frac{\Gamma(m-r+2)}{(\theta(k+l+1))^{m-r+2}} + \frac{\Gamma(m-r+1)}{(\theta(k+l+1))^{m-r+1}} \right] \right] \qquad (23)$$

3.4 Quantile Function

Let $Q(u) = F^{-1}(u)$ be the quantile function (qf) of $F(x)$ for $0 < u < 1$.
Taking *F(x)* to be the *cdf* of the Exponential-inverse Lindley distribution (EILD) and inverting it as above will give us the quantile function as follows.
Inverting $F(x) = u$

$$F(x) = 1 - \exp\left\{ -\alpha \left[\frac{\left(1 + \frac{\theta}{(\theta+1)x}\right) e^{-\frac{\theta}{x}}}{1 - \left(1 + \frac{\theta}{(\theta+1)x}\right) e^{-\frac{\theta}{x}}} \right] \right\} = u \qquad (24)$$

Simplifying equation (24) above gives the quantile function as:

$$Q(u) = \left\{ -1 - \frac{1}{\theta} - \frac{1}{\theta} W_{-1}\left((\theta+1) \frac{\ln(1-u)}{[\ln(1-u) - \alpha]} e^{-(\theta+1)} \right) \right\}^{-1} \qquad (25)$$

where u is a uniform variate on the unit interval *(0,1)* and $W_{-1}(.)$ represents the negative branch of the Lambert function.

3.5 Skewness and Kurtosis

The quantile based measures of skewness and kurtosis will be employed due to non-existence of the classical measures in some cases. The Bowley's measure of skewness (Kennedy and Keeping, 1962.) based on quartiles is given by;

$$SK = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \qquad (26)$$

And the Moores' (1998) kurtosis is on octiles and is given by;

$$KT = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) - Q\left(\frac{3}{8}\right) + \left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{1}{4}\right)} \qquad (27)$$

4. Reliability Analysis

Mathematically, the survival function is given by:

$$S(x) = 1-F(x) \tag{28}$$

Now, taking $F(x)$ to be the *cdf* of the proposed EILD and substituting produces;

$$S(x)=\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\} \tag{29}$$

Below is a plot of the survival function at chosen parameter values in figure **3**

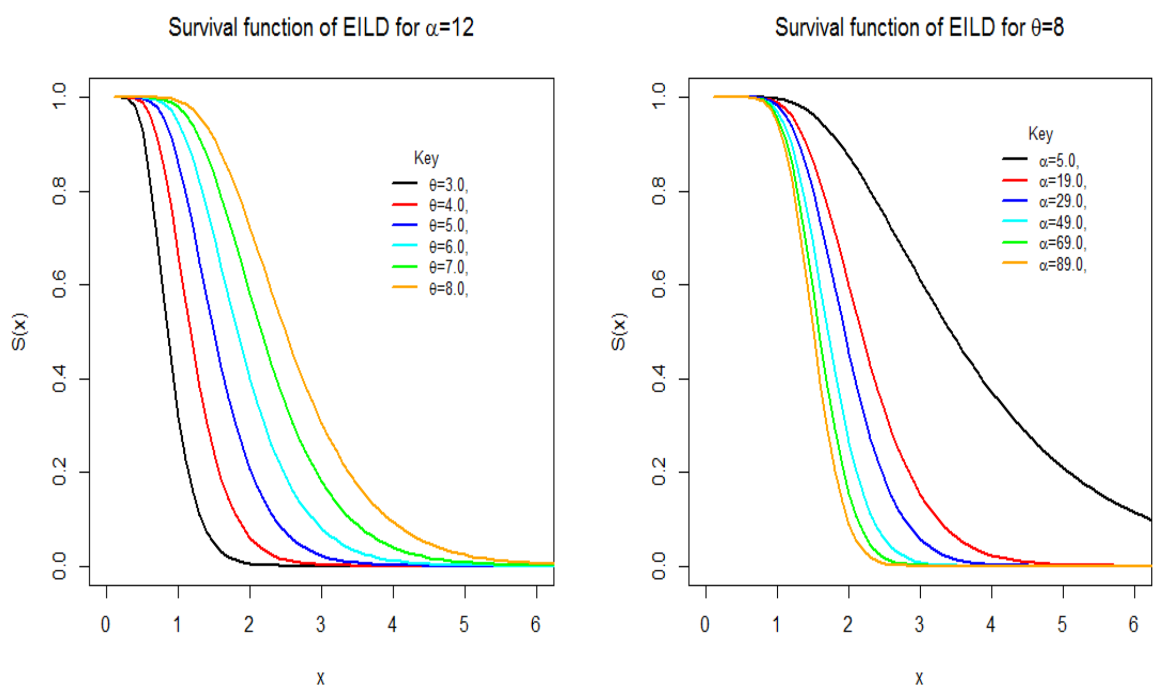


Figure 3: The survival function of the EILD.

The figure above revealed that the probability of survival for any random variable following a EILD decreases with time, that is, as life gets older, probability of life decreases. This implies that the EILD can be used to model random variables whose survival rate decreases as their age grows.

The hazard function is defined as;

$$h(x)=\frac{f(x)}{1-F(x)}=\frac{f(x)}{S(x)} \tag{30}$$

Taking $f(x)$ and $F(x)$ to be the *pdf* and *cdf* of the proposed Exponential-inverse Lindley distribution (EILD) and substituting in equation (30) gives the hazard function of EILD as:

$$h(x)=\frac{\alpha\theta e^{-\frac{\theta}{x}}}{(\theta+1)}\left(\frac{1+x}{x^{3}}\right)\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^{-2} \quad (31)$$

The following is a plot of the hazard function at chosen parameter values in figure **4**

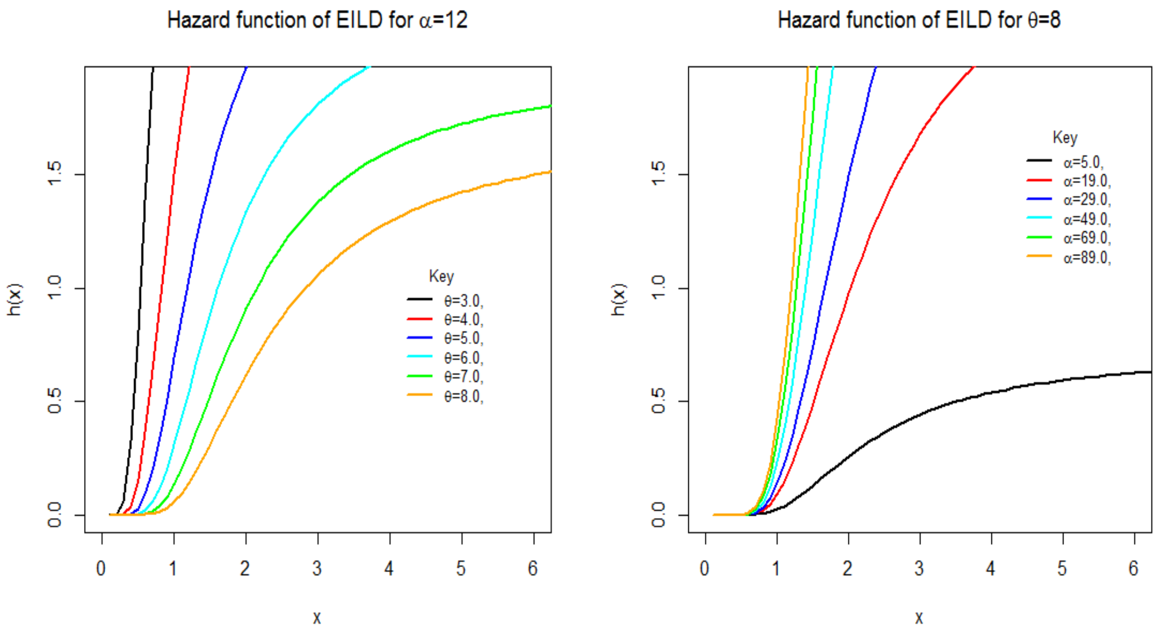


Figure 4: The hazard function of the EILD.

The figure above revealed that the probability of failure for any random variable following an EILD increases with time, that is, as time goes on, probability of death increases. It also decreases slowly for some parameter values. This implies that the EILD can be used to model random variables whose failure rate increases with time.

5 Order Statistics

Suppose $X_1, X_2, ..., X_n$ is a random sample from a distribution with *pdf*, *f(x)*, and let $X_{1:n}, X_{2n}, ..., X_{i:n}$ denote the corresponding order statistic obtained from this sample. The *pdf*, $f_{i:n}(x)$ of the i^{th} order statistic can be defined as;

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}f(x)F(x)^{i-1}\left[1-F(x)\right]^{n-i} \quad (32)$$

Taking *f(x)* and *F(x)* to be the *pdf* and *cdf* of the EILD respectively and using (3) and (4), the *pdf* of the i^{th} order statistics $X_{i:n}$ for the EILD can be expressed from (32) as;

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}\left[\frac{\alpha(\theta+1)^{-1}\theta^2\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}}{\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^2}\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}\right]\left[1-\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}\right]^{i+k-1} \quad (33)$$

Hence, the *pdf* of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the EILD are given by;

$$f_{1:n}(x)=n\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\left[\frac{\alpha(\theta+1)^{-1}\theta^2\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}}{\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^2}\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}\right]\left[1-\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}\right]^{k} \quad (34)$$

and

$$f_{n:n}(x)=n\left[\frac{\alpha(\theta+1)^{-1}\theta^2\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}}{\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^2}\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}\right]\left[1-\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}\right]^{n-1} \quad (35)$$

respectively.

6. Estimation of Parameters of the EILD

Let $X_1, X_2, \ldots\ldots, X_n$ be a sample of size 'n' independently and identically distributed random variables from the EILD with unknown parameters, α *and* θ defined previously. The pdf of the EILD as defined previously gives the likelihood function as:

$$L(X|\alpha,\theta)\propto\left(\frac{\alpha\theta^2}{(\theta+1)}\right)^n\prod_{i=1}^{n}\left(\frac{\left(\frac{1+x_i}{x_i^3}\right)e^{-\frac{\theta}{x_i}}}{\left[1-\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}\right]^2}\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}}{1-\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}}\right]\right\}\right) \quad (36)$$

Let the log-likelihood function, $l=\log L(X/\alpha,\theta)$, therefore

$$l=n\log\alpha+2n\log\theta-n\log(\theta+1)+\sum_{i=1}^{n}\log\left(\frac{1+x_i}{x_i^3}\right)-\theta\sum_{i=1}^{n}x_i^{-1}-2\sum_{i=1}^{n}\log\left[1-\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}\right]-\alpha\sum_{i=1}^{n}\left[\frac{\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}}{1-\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}}\right] \quad (37)$$

Differentiating l partially with respect to α *and* θ respectively gives;

$$\frac{\partial l}{\partial \alpha}=\frac{n}{\alpha}-\alpha\sum_{i=1}^{n}\left[\frac{\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}}{1-\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}}\right] \quad (38)$$

$$\frac{\partial l(\kappa)}{\partial \theta}=\frac{2n}{\theta}-\frac{n}{(\theta+1)}-\sum_{i=1}^{n}x_i^{-1}-\sum_{i=1}^{n}\left\{\frac{\left(\frac{1}{x}+\frac{\theta}{(\theta+1)^2 x_i}\left(\frac{\theta+1-x_i}{x_i}\right)\right)\mathrm{e}^{-\frac{\theta}{x_i}}}{\left[1-\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}\right]}\right\}-\alpha\sum_{i=1}^{n}\left\{\frac{\left(\frac{1}{x}+\frac{\theta}{(\theta+1)^2 x_i}\left(\frac{\theta+1-x_i}{x_i}\right)\right)\mathrm{e}^{-\frac{\theta}{x_i}}}{\left[1-\left(\left(1+\frac{\theta}{(\theta+1)x_i}\right)\mathrm{e}^{-\frac{\theta}{x_i}}\right)\right]^2}\right\} \quad (39)$$

The solution of the non-linear system of equations; $\frac{\partial l}{\partial \alpha}=0 \ and \ \frac{\partial l}{\partial \theta}=0$ will give us the maximum likelihood estimates of parameters α *and* θ. However, the solution cannot be obtained analytically except numerically with the aid of suitable statistical software like R, SAS, Maple, e.t.c when data sets are available.

7. Simulation study for EILD

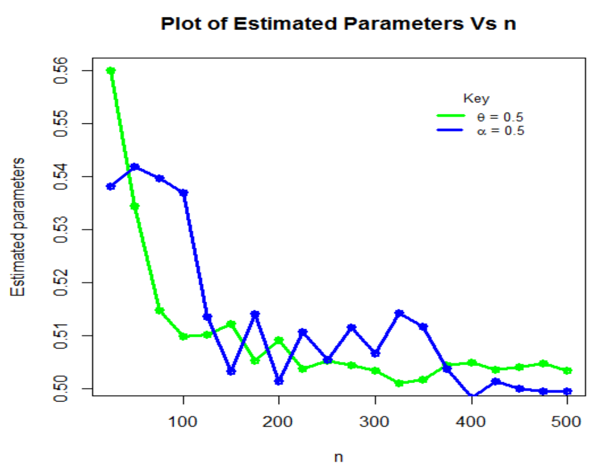
We perform a Monte Carlo (MC) simulation study with the objective to assess the behavior of the MLEs of EILD via the optim() R-function with the argument method = "L-BFGS-B". It is used for maximizing the log-likelihood function of a probabilistic model. We consider 500 MC replicates under different sample sizes *n* = 25, 50. . . 500. These samples are obtained using the inverse CDF (also known as quantile function). The SS is conducted for three different combination of θ and α. These combination values are given by (i) $\theta=0.5 \ and \ \alpha=0.5$, (ii) $\theta=2.5 \ and \ \alpha=0.5$, and (iii) $\theta=0.5 \ and \ \alpha=2.5$.

The judgment about the performances of $\hat{\theta}_{MLE}$ and $\hat{\alpha}_{MLE}$ is made by considering two evaluation criteria. These criteria are the Mean square error (MSE) and Bias.

For every sample size, the average MLEs, mean square errors (MSE), Biases and Absolute biases were computed. The results obtained after performing the MC simulation are provided in Tables 1-3 and displayed graphically in Figures 5–7.

Table 1: Simulation results for the EILD for $\theta = 0.5$ and $\alpha = 0.5$

N	Measures /Criteria	Parameters θ	Parameters α	N	Measures/ Criteria	Parameters θ	Parameters α
n=25	MLEs	0.5600	0.5382	n=200	MLEs	0.5091	0.5014
	Biases	0.0600	0.0382		Biases	0.0091	0.0014
	MSEs	0.0279	0.3045		MSEs	0.0028	0.0222
n=50	MLEs	0.5344	0.5419	n=300	MLEs	0.5034	0.5067
	Biases	0.0344	0.0419		Biases	0.0034	0.0067
	MSEs	0.0148	0.2003		MSEs	0.0018	0.0148
n=75	MLEs	0.5148	0.5397	n=400	MLEs	0.5049	0.4983
	Biases	0.0148	0.0397		Biases	0.0049	0.0017
	MSEs	0.0081	0.0935		MSEs	0.0014	0.0099
n=100	MLEs	0.5099	0.5369	n=500	MLEs	0.5035	0.4996
	Biases	0.0099	0.0369		Biases	0.0035	0.0004
	MSEs	0.0067	0.0736		MSEs	0.0011	0.0082



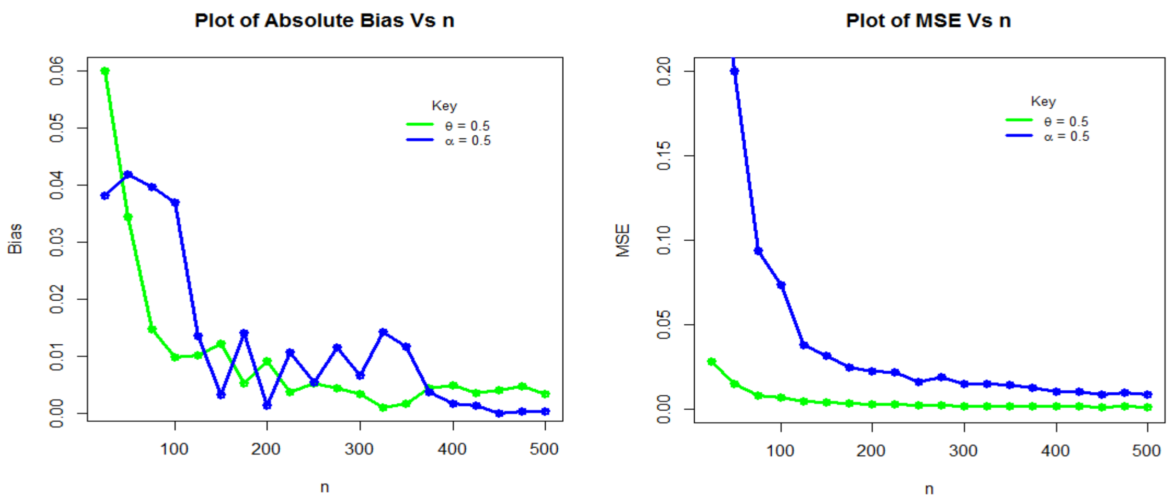


Figure 5: Plots of MLEs, Absolute Biases and MSEs of the EILD for $\theta = 0.5$ *and* $\alpha = 0.5$

Table 2: Simulation results for the EILD for $\theta = 2.5$ *and* $\alpha = 0.5$

n	Measures /Criteria	Parameters		N	Measures/ Criteria	Parameters	
		θ	α			θ	α
n=25	MLEs	2.7013	0.6233	n=200	MLEs	2.5180	0.5150
	Biases	0.2013	0.1233		Biases	0.0180	0.0150
	MSEs	0.6767	0.4867		MSEs	0.0676	0.0175
n=50	MLEs	2.6066	0.5710	n=300	MLEs	0.5262	0.5013
	Biases	0.1066	0.0710		Biases	0.0262	0.0013
	MSEs	0.3713	0.1569		MSEs	0.0469	0.0121
n=75	MLEs	2.5684	0.5383	n=400	MLEs	2.5169	0.5028
	Biases	0.0684	0.0383		Biases	0.0169	0.0028
	MSEs	0.2195	0.0684		MSEs	0.0337	0.0082
n=100	MLEs	2.5763	0.5099	n=500	MLEs	2.5162	0.5012
	Biases	0.0763	0.0099		Biases	0.0162	0.0012
	MSEs	0.1563	0.0356		MSEs	0.0269	0.0067

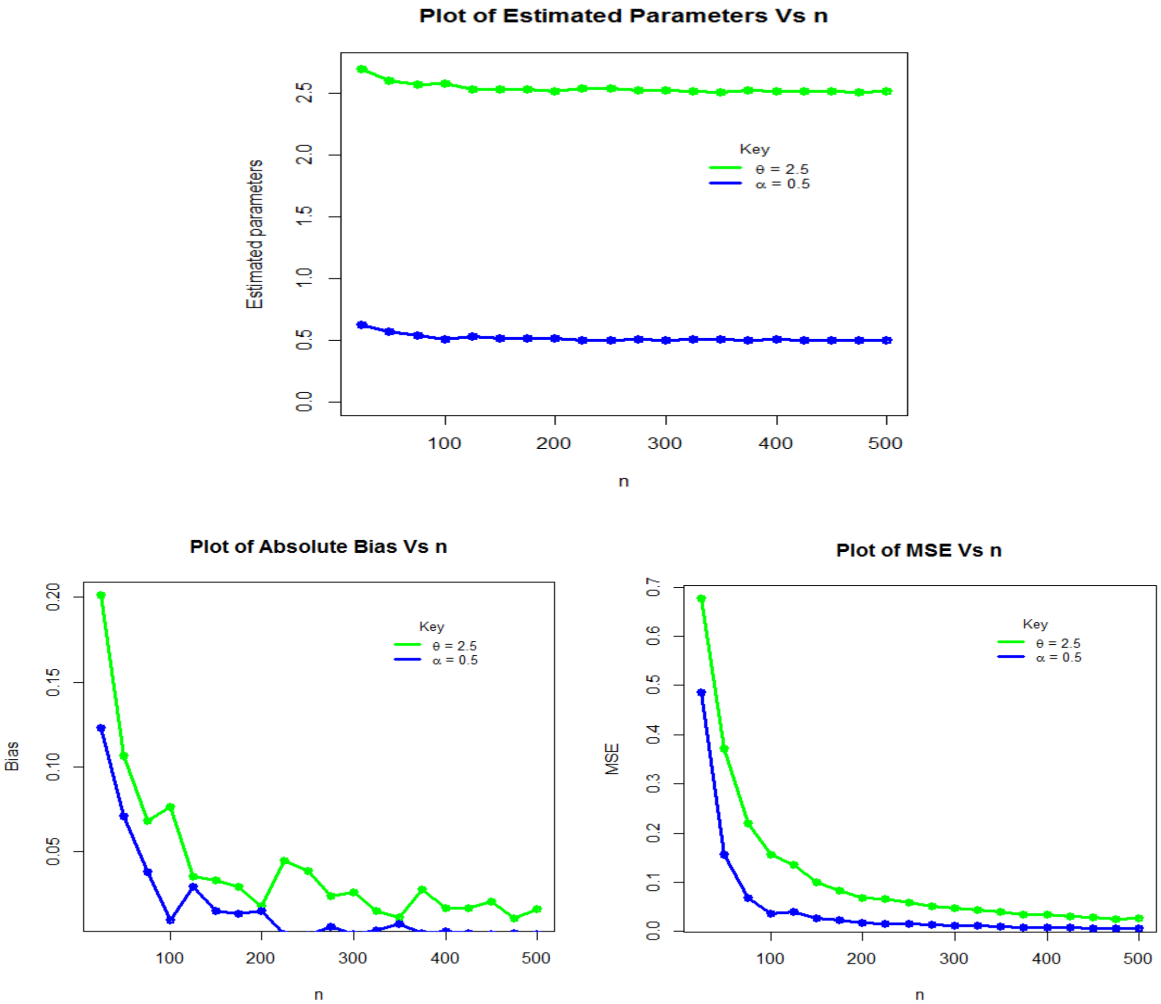


Figure 6: Plots of MLEs, Absolute Biases and MSEs of the EILD model for $\theta = 2.5$ and $\alpha = 0.5$

Table 3: Simulation results for the EILD for $\theta = 0.5\ and\ \alpha = 2.5$

n	Measures /Criteria	Parameters		N	Measures/ Criteria	Parameters	
		θ	α			θ	α
n=25	MLEs	0.7048	2.1812	n=200	MLEs	0.5148	2.7677
	Biases	0.2048	0.3188		Biases	0.0148	0.2677
	MSEs	0.1390	3.1735		MSEs	0.0153	1.6678
n=50	MLEs	0.6007	2.4817	n=300	MLEs	0.5158	2.6434
	Biases	0.1007	0.0183		Biases	0.0158	0.1434
	MSEs	0.0569	2.5852		MSEs	0.0109	1.2177
n=75	MLEs	0.5694	2.5658	n=400	MLEs	0.5086	2.6309
	Biases	0.0694	0.0658		Biases	0.0086	0.1309
	MSEs	0.0409	2.2804		MSEs	0.0074	0.8962
n=100	MLEs	0.5549	2.5642	n=500	MLEs	0.5080	2.6114
	Biases	0.0549	0.0642		Biases	0.0080	0.1114
	MSEs	0.0283	2.0422		MSEs	0.0067	0.7739

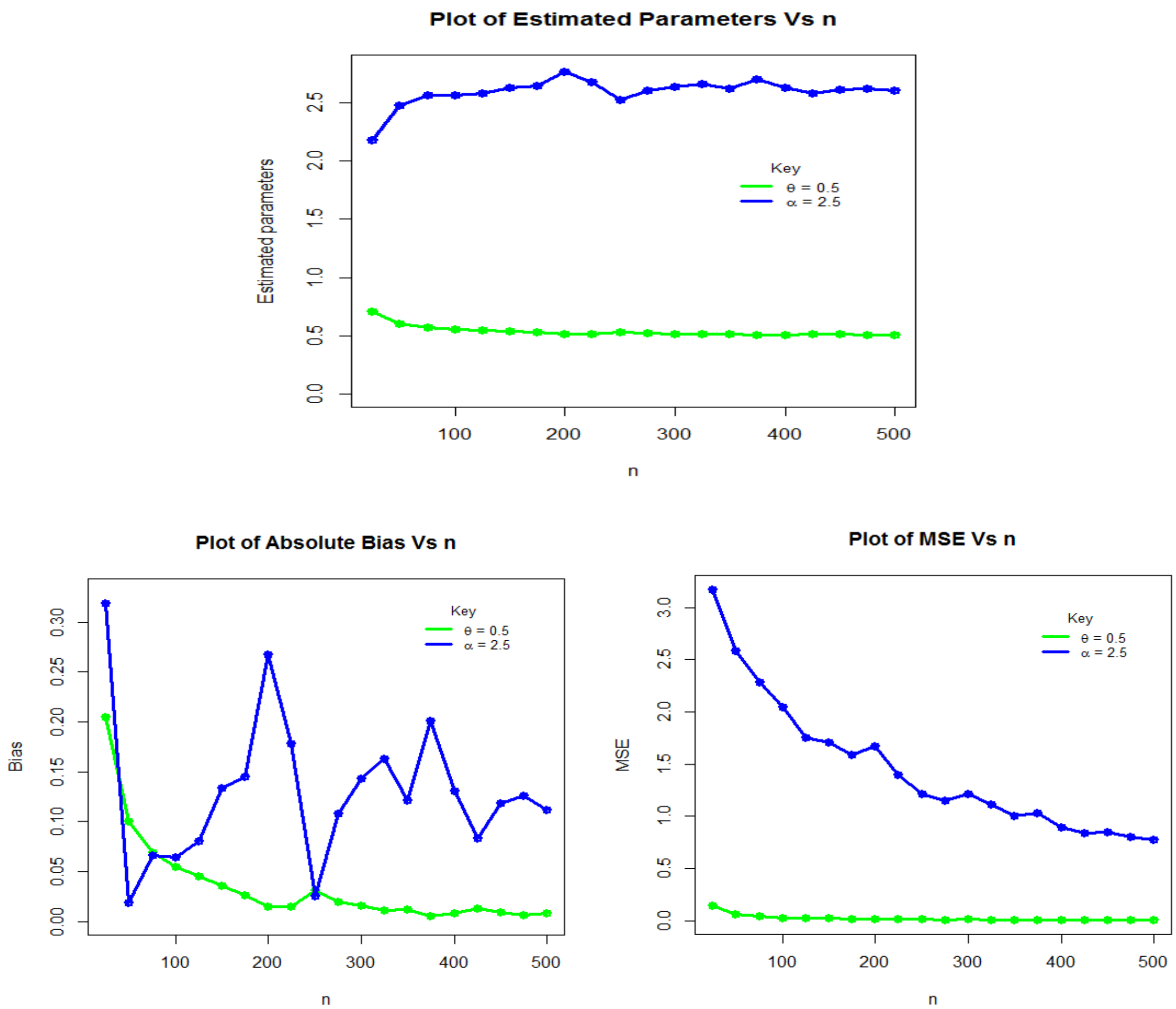


Figure 7: Plots of MLEs, Absolute Biases and MSEs of the EILD model for $\theta=0.5$ *and* $\alpha=2.5$

From the results of the simulation study of EILD presented in Tables 1–3 and Figures 5–7, we can see the averages of the MLEs (Mean), their biases (Absolute Bias) and mean square errors (MSEs) for the parameters of the EILD. Based on the values from the tables, it is clear that the average estimates tend to be closer to the true parameters when sample size increases and the biases and mean square errors all decrease as sample size increases which is in agreement with first-order asymptotic theory.

8. Applications of the proposed Model

This section of chapter four presents two datasets, their descriptive statistics and applications to some selected generalized probability distributions. It has compared the adequacy of the EILD to that of five other extended models. The models are: Lomax-inverse Lindley distribution (LOMINLIND) by Ieren *et al.*, (2019), odd Lomax-inverse exponential distribution (OLOMINExD) by Ieren *et al*., (2021), odd Lindley-inverse exponential distribution (OLINExD) by Ieren and Abdullahi (2020), Lomax inverse exponential distribution (LomINExD) by Abdulkadir *et al*. (2020) as well as the conventional inverse Lindley distribution (INLIND).

For us to assess the models listed above, we made use of some criteria: the AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion), BIC (Bayesian Information Criterion) and HQIC (Hannan Quin information criterion). The model with the lowest values of these statistics would be chosen as the best model to fit the data.

Data Set: This data represents the survival times of a group of patients suffering from head and neck cancer diseases and treated using a combination of radiotherapy and chemotherapy (RT+CT) (Efron (1988), Shanker *et al*. (2015), Oguntunde *et al*. (2017)) and Ieren *et al*. (2020). The observations are as follows:12.20, 23.56, 23.74, 25.87, 31.98, 37, 41.35, 47.38, 55.46, 58.36, 63.47, 68.46, 78.26, 74.47, 81.43, 84, 92, 94, 110, 112, 119, 127, 130, 133, 140, 146, 155, 159, 173, 179, 194, 195, 209, 249, 281, 319, 339, 432, 469, 519, 633, 725, 817, 1776. The summary is given as follows:

Table 4: Descriptive Statistics for dataset I.

parameters	**N**	**Minimum**	Q_1	**Median**	Q_3	**Mean**	**Maximum**	**Variance**	**Skewness**	**Kurtosis**
Values	44	12.20	67.21	128.5	219.0	223.48	1776.00	93286.4	3.38382	13.5596

Using the descriptive statistics in table 4 above, we observed that the first data (dataset I) is skewed to the right or positively skewed with a very large variance.

Table 5: Maximum Likelihood Parameter Estimates for dataset I

Distribution	Parameter Estimates			
EILD	$\hat{\theta}$=0.04167121	$\hat{\alpha}$=9.49748916	-	-
OLomINExD	$\hat{\theta}$=8.4647625	$\hat{\alpha}$=0.8465264	$\hat{\beta}$=8.5616942	-
LomILnD	$\hat{\theta}$=9.874074	$\hat{\alpha}$=3.535500	$\hat{\beta}$=8.494930	-
LomINExD	$\hat{\theta}$=9.939772	$\hat{\alpha}$=4.511308	$\hat{\beta}$=9.343647	-
OLINExD	$\hat{\theta}$=4.978665	$\hat{\alpha}$=1.523596	-	-
INLnD	$\hat{\theta}$=7.051431	-	-	-

Table 6: The statistics ℓ, AIC, CAIC, BIC and HQIC for dataset I

Distribution	$\hat{\ell}$	*AIC*	*CAIC*	*BIC*	*HQIC*	Ranks
EILD	281.3532	566.7063	566.999	570.2747	**568.0297**	1st
OLomINExD	285.6976	577.3953	577.9953	582.7478	**579.3803**	2nd
LomINLnD	308.9311	623.8623	624.4623	629.2149	**625.8473**	3rd
LomINExD	307.2215	620.443	621.043	625.7956	**622.428**	4th
OLINExD	2906.274	5816.548	5816.84	5820.116	**5817.871**	5th
INLnD	349.9063	701.8125	701.9078	703.5967	**702.4742**	6th

Table 7: **The A^*, W^*, K-S statistic and P-values based on the dataset used**.

Distribution	A^*	W^*	K-S	P-Value (K-S)	Ranks
EILD	0.8075271	0.1398597	0.1239	0.4717	1st
OLomINExD	0.1198959	0.01760287	0.18268	0.09317	2nd
LomINLnD	0.380369	0.06120365	0.33865	5.264e-05	3rd
LomINExD	0.3737685	0.06010676	0.36785	7.574e-06	4th
OLINExD	2.825167	0.439813	0.97178	2.2e-16	5th
INLnD	0.2080145	0.03300336	0.74617	2.2e-16	6th

The following figure presents a histogram and estimated densities and cdfs of the fitted models to the dataset.

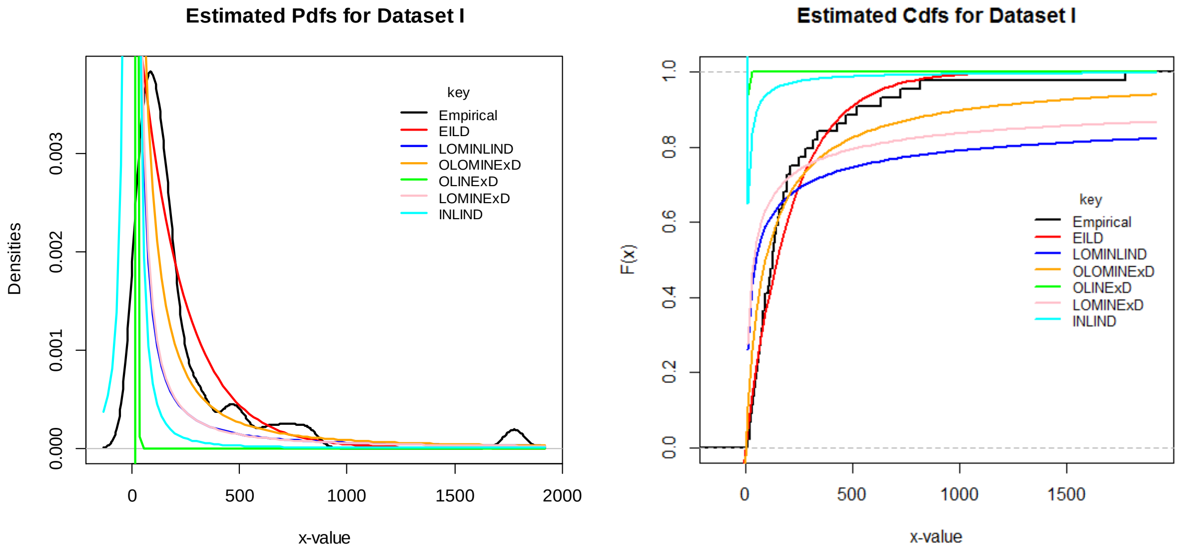


Figure 8: Plots of the estimated densities and cdfs of the fitted distributions to the dataset I.

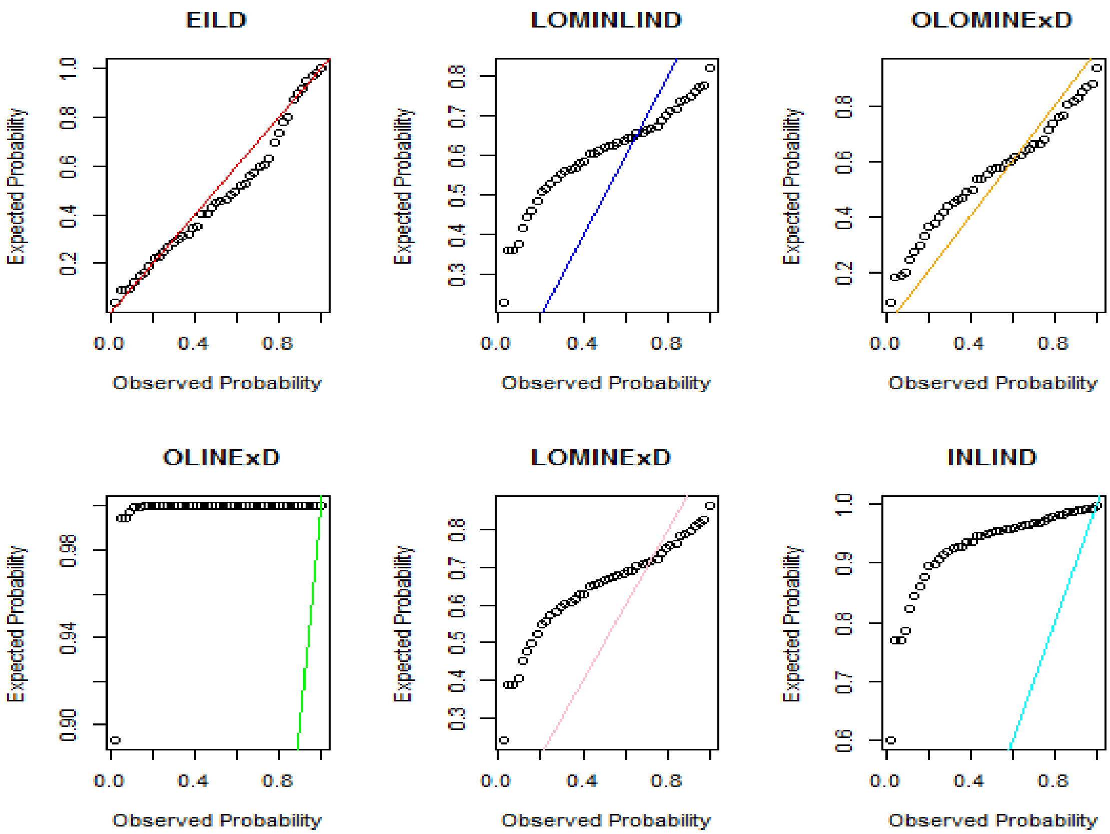


Figure 9: Probability plots for the six fitted distributions based on dataset I.

Tables 5 present the parameter estimates of the fitted distributions based on dataset I respectively and table 6 list the values of AIC, CAIC, BIC and HQIC for the fitted distributions based on dataset I respectively. The values of AIC, CAIC, BIC and HQIC in table 6 are smaller for the EILD compared to the other five distributions and this result indicates that the Exponential-inverse Lindley distribution (EILD) is better than the Lomax-inverse Lindley distribution (LOMINLIND), odd Lomax-inverse exponential distribution (OLOMINExD), odd Lindley-inverse exponential distribution (OLINExD), Lomax inverse exponential distribution (LomINExD) as well as the conventional inverse Lindley distribution (INLIND).

Also, Tables 7 present the values of A*, W*, and their associated p-values for the fitted distributions based on dataset I respectively. The values of A*, W*, and their associated p-values in table 7 show that the proposed distribution (EILD) is better than the Lomax-inverse Lindley distribution (LOMINLIND), odd Lomax-inverse exponential distribution (OLOMINExD), odd Lindley-inverse exponential distribution (OLINExD), Lomax inverse exponential distribution (LomINExD) as well as the conventional inverse Lindley distribution (INLIND).

These results confirm the fact that the exponential-G family from the Weibull-G family of distributions has the capacity to produce distributions with greater flexibility as compared to the other probability distributions. This study has proven that the additional shape parameter in the exponential-G family is responsible for additional skewness and flexibility in the generalized continuous probability distributions just as previously reported by Alzaatreh *et al*., (2013), Ieren and Yahaya (2017) and Oguntunde *et al*. (2015), e.t.c.

7. SUMMARY AND CONCLUSION

In this research work, we proposed a new extension of the inverse Lindley distribution called exponential-inverse Lindley distribution. Some mathematical and statistical properties of the proposed distribution have been studied appropriately. The derivations of some expressions for its moments, moment generating function, characteristics function, survival function, hazard function and ordered statistics has been done appropriately. Some plots of the distribution revealed that it is a flexible and skewed distribution. The implications of the plots for the survival function indicate that the exponential-inverse Lindley distribution (EILD) could be used to model time or age-dependent events, where survival rate decreases with time or age. From the results of the simulation study of EILD, it has been shown that the average estimates tend to be closer to the true parameters when sample size increases and the biases and mean square errors all decrease as sample size increases which is in agreement with first-order asymptotic theory.

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