AN EXTEND INVERSE LINDLEY DISTRIBUTION; MODEL, PROPERTIES AND APPLICATIONS.

ABSTRACT

This research work proposed a new extension of the inverse Lindley distribution called exponential-inverse Lindley distribution. The study derived some Statistical and mathematical properties of the distribution such as its ordinary moments, moment generating and characteristics function. It also considered the survival and hazard functions and the distribution of ordered statistics. Some plots of the distribution revealed that it is a flexible and skewed distribution. The implications of the plots for the survival function indicate that the exponential-inverse Lindley distribution could be used to model time or age-dependent events, where survival rate decreases with time. The research also conducted a simulation study to check the consistency of the model parameters using maximum likelihood estimation. From the results of the simulation study, it was revealed that the average estimates tend to be closer to the true parameters when sample size increases and the biases and mean square errors all decrease as sample size increases which is in agreement with the theory of estimation

KEYWORDS: Inverse Lindley distribution, Exponential-inverse Lindley Distribution, Statistical Properties, Parameters Estimation, Method of Likelihood Estimation

1. INTRODUCTION

Prof. D.V. Lindley investigated a probability distribution in context of fiducial statistic as a counter example of Bayesian theory which was later called “Lindley distribution” (Lindley (1958)). The fundamental properties of the Lindley distribution with application to waiting time data was discussed by Ghitany *et al*. (2008). Since then, many researchers have studied this distribution, for instance, Mazucheli and Achcar (2011) worked on the Lindley distribution applied to competing risks lifetime data. Krishna and Kumar (2011) estimated the parameter of Lindley distribution with progressive Type-II censoring scheme and also showed that it may be better lifetime model than exponential, lognormal and gamma distributions in some real life situations. Singh and Gupta (2012) used the distribution under load sharing system models. Al-Mutairi *et al*. (2013) developed an inferential procedure of the stress-strength parameter when both stress and strength variables follow Lindley distribution and discovered that the Lindley distribution is useful when the data show increasing failure rate. This particular property makes the use of Lindley distribution in lifetime data analysis more frequent than the exponential distribution. Despite the important properties and various applications of the Lindley distribution in many disciplines, its applicability may be restricted to non-monotone hazard rate data (see Sharma *et al*. (2014)). To solve the above mentioned problem therefore, several extensions of the Lindley distribution have been proposed in the literature and some of the recent generalizations are summarized as follows: Zakerzadeh and Dolati (2009) and Nadarajah*et al.* (2011) introduced a three parameters extension of the Lindley distribution called "a generalized Lindley distribution". Ghitany *al*. (2013) proposed two parameter generalizations of the Lindley distribution, called the power Lindley distribution which was generated using the power transformations to the Lindley distribution. Merovci (2013) investigated transmuted Lindley distribution and Merovci and Elbatal (2014) studied the transmuted Lindley-geometric distribution. The beta-Lindley distribution was also introduced by Merovci and Sharma (2014). Elbatal and Elgarhy (2013) studied the statistical and mathematical properties of Kumaraswamy Quasi-Lindley distribution and Kumaraswamy Lindley distribution was proposed and discussed by Akmakyapan and Kadlar (2014). The Exponentiated power Lindley distribution has been introduced by Ashour and Eltehiwy (2015).

The inverse Lindley distribution is a two component mixture of inverse exponential distribution and special case of inverse gamma distribution. From the introduction above, it can be seen that authors mainly focused on the Lindley distribution and little has been said about the inverse

Lindley distribution. Sharma *et al*. (2015) discussed the properties of inverse Lindley distribution with application to stress strength reliability analysis. Sharma *et al*. (2016) introduced a two parameter extension of inverse Lindley distribution (generalized inverse Lindley distribution). Also, Alkarni (2015) proposed a three parameter inverse Lindley distribution (extended inverse Lindley distribution) with application to maximum flood level data.

The probability density function (*pdf*) and the cumulative distribution function (*cdf*) of an exponential random variable *X* are respectively given by;

$$g(x)=\theta\exp\{-\theta x\} \qquad (1)$$

$$G(x)=1-\exp\{-\theta x\} \qquad (2)$$

Where $\theta>0$ is the exponential parameter and $x>0$ is the random variable.

The cumulative distribution function (c.d.f) and probability density function (pdf) of the Inverse Lindley distribution (ILD) are defined as:

$$G(x)=\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}} \qquad (3)$$

And

$$g(x)=\frac{\theta^2}{\theta+1}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}} \qquad (4)$$

respectively, for $x>0$ and $\theta>0$ where θ is the scale parameter of ILD.

2. The Exponential- Inverse Lindley Distribution (EILD).

According to Bouguignon *et al.*, (2014), the cumulative distribution function (cdf) and the probability density function (pdf) (for $x > 0$) of the Exponential generator (Exp-G family) of distributions with an additional shape parameter ($\alpha > 0$) are defined by:

$$F(x) = 1 - \exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]\right\} \tag{3.1}$$

and

$$f(x) = \frac{\alpha g(x)}{\left[1-G(x)\right]^2}\exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]\right\} \tag{3.2}$$

respectively, where $g(x)$ and $G(x)$ represent the pdf and the cdf of the continuous distribution to be modified, which in this case stands for the pdf and cdf of the Frechet distribution respectively.

The probability density function (p.d.f) and cumulative distribution function (cdf) of the Inverse Lindley distribution (ILD) are defined as:

$$g(x) = \frac{\theta^2}{\theta+1}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}} \tag{3.3}$$

and

$$G(x) = \left(1 + \frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}} \tag{3.4}$$

respectively, where $x > 0$ and $\theta > 0$ is the scale parameter of ILD.

Using equation (3.3) and (3.4) in (3.1) and (3.2) and simplifying, we obtain the cdf and pdf of the Exponential-Inverse Lindley distribution (EILD) as follows:

$$F(x)=1-\exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]\right\}$$

$$F(x)=1-\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\} \qquad (3.5)$$

And

$$f(x)=\frac{\alpha g(x)}{\left[1-G(x)\right]^{2}}\exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]\right\}$$

$$f(x)=\frac{\alpha\frac{\theta^{2}}{\theta+1}\left(\frac{1+x}{x^{3}}\right)e^{-\frac{\theta}{x}}}{\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^{2}}\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}$$

$$f(x)=\frac{\alpha\theta^{2}\left(\frac{1+x}{x^{3}}\right)e^{-\frac{\theta}{x}}}{(\theta+1)\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^{2}}\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\} \qquad (3.6)$$

respectively.

For $x>0$, $\alpha,\theta>0$ where $\alpha>0$ is a shape parameter and $\theta>0$ is a scale parameter. Hence equation (3.5) and (3.6) are the cdf and pdf of the Exponential-Inverse Lindley distribution (EILD).

Graphical representation of the PDF and CDF EILD

Given some values for the parameters α and β and θ, we provide some possible shapes for the *pdf* and the *cdf* of the Exponential-Inverse Lindley distribution (EILD) as shown in figure 1 and 2 below:

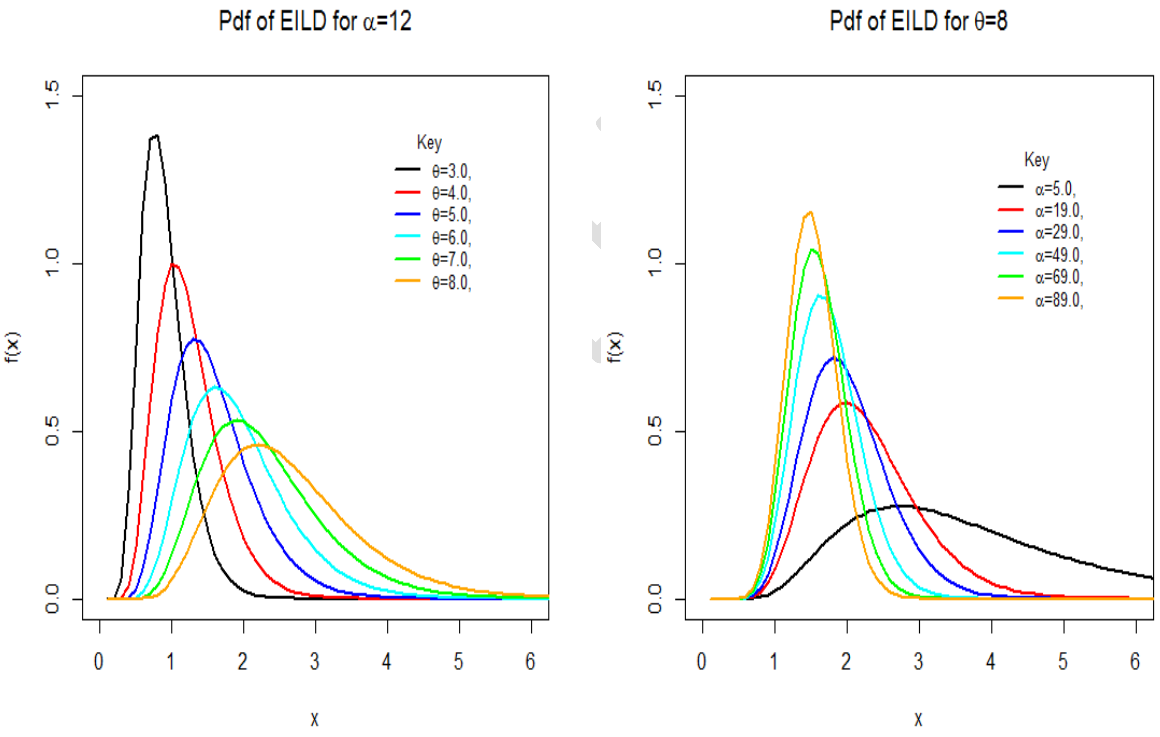


Figure 1: PDF of the EILD for different parameter values.

Figure **1** indicates that the Exponential-Inverse Lindley distribution (EILD) is a skewed or flexible distribution depending on the parameter values. This means that the distribution can be appropriate for datasets with different shapes.

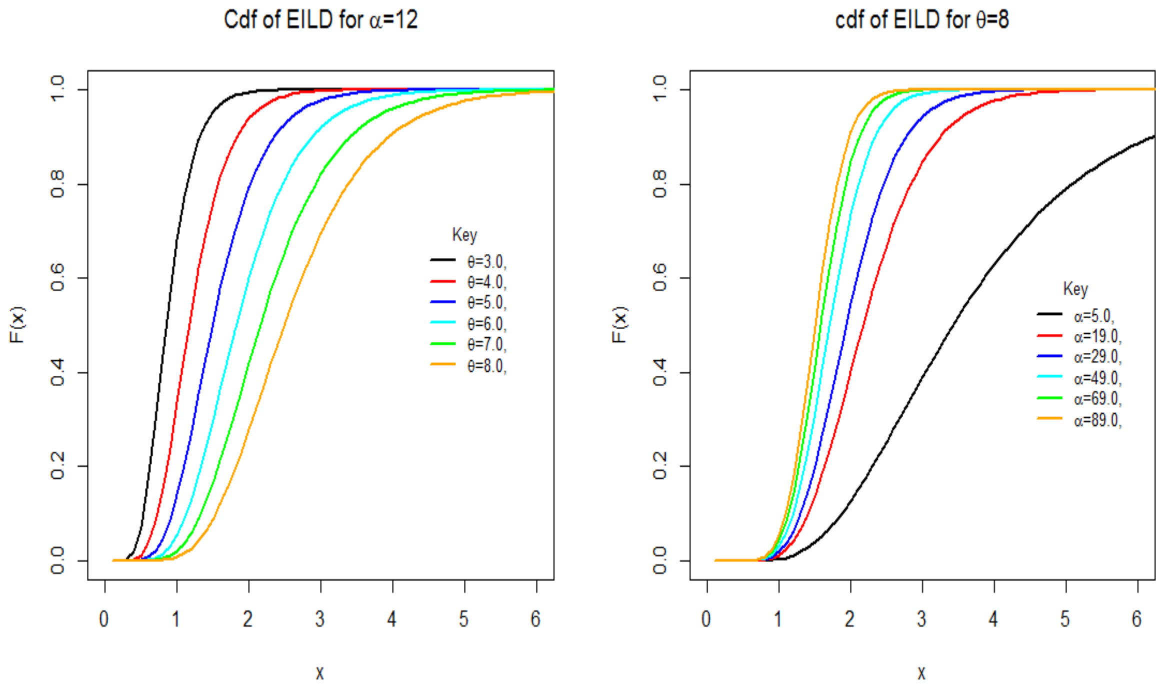


Fig. 2: CDF of the EILD for different parameter values.

From the above *cdf* plot, the *cdf* increases when *X* increases, and approaches 1 when *X* becomes large, as expected.

3. Some Properties of the Proposed Distribution

3.1 Moments

Let X denote a continuous random variable, the n^{th} moment of X is given by;

$$\mu_n' = E[X^n] = \int_0^\infty x^n f(x)dx$$

Using the pdf of the Exponential-Inverse Lindley distribution (EILD) as given in equation (3.6).

$$\mu_n' = E[X^n] = \int_0^1 x^n f(x)dx \tag{3.7}$$

Recall,

$$f(x)=\frac{\alpha\theta^{2}\left(\frac{1+x}{x^{3}}\right)e^{-\frac{\theta}{x}}}{(\theta+1)\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^{2}}\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\} \tag{3.8}$$

Before substituting (3.8) in (3.7), we perform the expansion, simplification and linear representation of the pdf as follows:

First, by using power series expansion on the last term in the pdf above, we have:

$$\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}=\sum_{k=0}^{\infty}\frac{(-1)^{k}\alpha^{k}}{k!}\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]^{k} \tag{3.9}$$

By using the result in equation 3.9 and simplifying, equation 3.8 becomes

$$f(x)=\frac{\alpha\theta^{2}\left(\frac{1+x}{x^{3}}\right)e^{-\frac{\theta}{x}}}{(\theta+1)\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^{2}}\sum_{k=0}^{\infty}\frac{(-1)^{k}\alpha^{k}}{k!}\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]^{k}$$

$$f(x)=\sum_{k=0}^{\infty}\frac{(-1)^{k}\alpha^{k+1}\theta^{2}}{(\theta+1)k!}\frac{\left(\frac{1+x}{x^{3}}\right)e^{-\frac{\theta}{x}}\left[\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^{k}}{\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^{k+2}}$$

$$f(x)=\sum_{k=0}^{\infty}\frac{(-1)^k\alpha^{k+1}\theta^2}{(\theta+1)k!}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}\left[\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^k\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^{-(k+2)}$$

(3.10)

Using binomial expansion on the last term in equation 3.10 gives

$$f(x)=\sum_{k=0}^{\infty}\frac{(-1)^k\alpha^{k+1}\theta^2}{(\theta+1)k!}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}\left[\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^k\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^{-(k+2)}$$

$$\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^{-(k+2)}=\sum_{l=0}^{\infty}(-1)^l\binom{-k-2}{l}\left[\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^l \qquad (3.11)$$

By substituting the result of equation 3.11 in 3.10, we obtain

$$f(x)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{-k-2}{l}\frac{(-1)^{k+l}\alpha^{k+1}\theta^2}{(\theta+1)k!}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}\left[\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^{k+l}$$

$$f(x)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{-k-2}{l}\frac{(-1)^{k+l}\alpha^{k+1}\theta^2}{(\theta+1)k!}\left(\frac{1+x}{x^3}\right)e^{-(k+l+1)\frac{\theta}{x}}\left[\left(1+\frac{\theta}{(\theta+1)x}\right)\right]^{k+l} \qquad (3.12)$$

Again using binomial expansion on the last term in (3.12) gives:

$$\left(1+\frac{\theta}{(\theta+1)x}\right)^{k+l}=\sum_{m=0}^{\infty}\binom{k+l}{m}\left(\frac{\theta}{\theta+1}\right)^m x^{-m} \qquad (3.13)$$

Making use of the result (3.13) in equation (3.12) and simplifying, we obtain:

$$f(x)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{k+l}{m}\left(\frac{\theta}{\theta+1}\right)^m\binom{-k-2}{l}\frac{(-1)^{k+l}\alpha^{k+1}\theta^2}{(\theta+1)k!}\left(\frac{1+x}{x^3}\right)x^{-m}e^{-(k+l+1)\frac{\theta}{x}}$$

$$f(x)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{k+l}{m}\left(\frac{\theta}{\theta+1}\right)^m\binom{-k-2}{l}\frac{(-1)^{k+l}\alpha^{k+1}\theta^2}{(\theta+1)k!}(1+x)x^{-m-3}e^{-(k+l+1)\frac{\theta}{x}} \qquad (3.14)$$

Now, let $W_{klm}=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{k+l}{m}\left(\frac{\theta}{\theta+1}\right)^{m}\binom{-k-2}{l}\frac{(-1)^{k+l}\alpha^{k+1}\theta^{2}}{(\theta+1)k!}$ be a constant, which implies that the pdf in equation (3.14) can also be written in its simple and linear form as:

$$f(x)=W_{klm}(1+x)x^{-m-3}e^{-(k+l+1)\frac{\theta}{x}} \tag{3.15}$$

Now, using the simplified of the pdf of the EILD in equation (3.15), the nth ordinary moment of the EILD is derived as follows:

$$\mu_n^{'}=E\left(X^n\right)=\int_0^{\infty}x^n f(x)dx=W_{klm}\int_0^{\infty}x^n(1+x)x^{-m-3}e^{-(k+l+1)\frac{\theta}{x}}dx$$

$$\mu_n^{'}=E\left(X^n\right)=\int_0^{\infty}x^n f(x)dx=W_{klm}\int_0^{\infty}x^{n-m-3}(1+x)e^{-(k+l+1)\frac{\theta}{x}}dx \tag{3.16}$$

Making use of integration by substitution method in equation (3.16), we perform the following operations:

Let $u=\theta(k+l+1)x^{-1}\Rightarrow x=\left(\frac{u}{\theta(k+l+1)}\right)^{-1}$ such that $\frac{du}{dx}=-\theta(k+l+1)x^{-2}\Rightarrow dx=-\frac{du}{\theta(k+l+1)x^{-2}}$.

Substituting for x, u and dx in equation (3.16) and simplifying; we have:

$$\mu_n^{'}=E\left(X^n\right)=\int_0^{\infty}x^n f(x)dx=W_{klm}\int_0^{\infty}x^{n-m-3}(1+x)e^{-(k+l+1)\frac{\theta}{x}}dx$$

$$\mu_n^{'}=E\left(X^n\right)=\int_0^{\infty}x^n f(x)dx=W_{klm}(-1)\left(\frac{1}{\theta(k+l+1)}\right)\int_0^{\infty}x^{n-m-3-(-2)}(1+x)e^{-u}du$$

$$\mu_n^{'}=E\left(X^n\right)=W_{klm}(-1)\left(\frac{1}{\theta(k+l+1)}\right)\left[\int_0^{\infty}x^{n-m-1}e^{-u}du+\int_0^{\infty}x^{n-m}e^{-u}du\right]$$

$$\mu_n^{'}=E\left(X^n\right)=W_{klm}(-1)\left(\frac{1}{\theta(k+l+1)}\right)\left[\int_0^{\infty}\left(\frac{u}{\theta(k+l+1)}\right)^{-1(n-m-1)}e^{-u}du+\int_0^{\infty}\left(\frac{u}{\theta(k+l+1)}\right)^{-1(n-m)}e^{-u}du\right]$$

$$\mu_n^{'}=E\left(X^n\right)=W_{klm}(-1)\left(\frac{1}{\theta(k+l+1)}\right)^{m-n+1}\left[\left(\frac{1}{\theta(k+l+1)}\right)\int_0^{\infty}u^{m-n+1}e^{-u}du+\int_0^{\infty}u^{m-n}e^{-u}du\right]$$

$$\mu_n' = E\left(X^n\right) = W_{klm}(-1)\left(\tfrac{1}{\theta(k+l+1)}\right)^{m-n+1}\left\lfloor\left(\tfrac{1}{\theta(k+l+1)}\right)\int_0^\infty u^{m-n+1}e^{-u}du + \int_0^\infty u^{m-n}e^{-u}du\right\rfloor$$

$$\mu_n' = E\left(X^n\right) = W_{klm}(-1)\left(\tfrac{1}{\theta(k+l+1)}\right)^{m-n+1}\left\lfloor\left(\tfrac{1}{\theta(k+l+1)}\right)\int_0^\infty u^{m-n+2-1}e^{-u}du + \int_0^\infty u^{m-n+1-1}e^{-u}du\right\rfloor \quad (3.17)$$

Now, recall that $\int_0^\infty t^{k-1}e^{-t}dt = \Gamma(k)$ and that $\int_0^\infty t^{k}e^{-t}dt = \int_0^\infty t^{k+1-1}e^{-t}dt = \Gamma(k+1)$

Using the statement above, the *n*th ordinary moment of X for the EILD is obtained as:

$$\mu_n' = E\left(X^n\right) = W_{klm}(-1)\left(\tfrac{1}{\theta(k+l+1)}\right)^{m-n+1}\left\lfloor\left(\tfrac{1}{\theta(k+l+1)}\right)\Gamma(m-n+2) + \Gamma(m-n+1)\right\rfloor$$

$$\mu_n' = E\left(X^n\right) = W_{klm}(-1)\left\lfloor\tfrac{\Gamma(m-n+2)}{\left(\theta(k+l+1)\right)^{m-n+2}} + \tfrac{\Gamma(m-n+1)}{\left(\theta(k+l+1)\right)^{m-n+1}}\right\rfloor \quad (3.18)$$

Again recall that W_{klm} is a constant and making use of its value as defined previously, the expression for the nth ordinary moment of EILD becomes:

$$\mu_n' = E\left(X^n\right) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{k+l}{m}\left(\frac{\theta}{\theta+1}\right)^{m}\binom{-k-2}{l}\frac{(-1)^{k+l+1}\alpha^{k+1}\theta^{2}}{(\theta+1)k!}\left[\tfrac{\Gamma(m-n+2)}{\left(\theta(k+l+1)\right)^{m-n+2}} + \tfrac{\Gamma(m-n+1)}{\left(\theta(k+l+1)\right)^{m-n+1}}\right]$$

(3.19)

Using equation (3.19), the mean (μ_1'), variance (σ^2), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) can be calculated based on some well-known relationships.

3.2 Moment Generating Function

The moment generating function of a random variable X can be obtained as

$$M_x(t) = E\left[e^{tx}\right] = \int_{-\infty}^{\infty} e^{tx} f(x)dx \quad (3.20)$$

Recall that by power series expansion,

$$e^{tx} = \sum_{r=0}^{\infty}\frac{(tx)^r}{r!} = \sum_{r=0}^{\infty}\frac{t^r}{r!}x^r \quad (3.21)$$

Therefore, the moment generating function can also be expressed as:

$$M_x(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty}x^r f(x)dx=\sum_{r=0}^{\infty}\frac{t^r}{r!}E\left(X^r\right)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\left[\mu_r'\right]$$

Using the result in equation (3.21) and simplifying the integral in (3.20) therefore we have:

$$M_x(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{k+l}{m}\left(\frac{\theta}{\theta+1}\right)^m\binom{-k-2}{l}\frac{(-1)^{k+l+1}\alpha^{k+1}\theta^2}{(\theta+1)k!}\left[\frac{\Gamma(m-r+2)}{(\theta(k+l+1))^{m-r+2}}+\frac{\Gamma(m-r+1)}{(\theta(k+l+1))^{m-r+1}}\right]\right]$$

(3.22)

3.3 Characteristics Function

A representation for the characteristics function is given by

$$\phi_x(t)=E\left(e^{itx}\right)=\int_0^{\infty}e^{itx}f(x)dx \tag{3.23}$$

Recall that by power series expansion,

$$e^{itx}=\sum_{r=0}^{\infty}\frac{(itx)^r}{r!}=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}x^r \tag{3.24}$$

Hence, simple algebra and use of (3.24) above produces the following results:

$$\phi_x(t)=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\int_0^{\infty}x^r f(x)dx=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}E\left(X^r\right)=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\left[\mu_r'\right]$$

$$\phi_x(t)=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{k+l}{m}\left(\frac{\theta}{\theta+1}\right)^m\binom{-k-2}{l}\frac{(-1)^{k+l+1}\alpha^{k+1}\theta^2}{(\theta+1)k!}\left[\frac{\Gamma(m-r+2)}{(\theta(k+l+1))^{m-r+2}}+\frac{\Gamma(m-r+1)}{(\theta(k+l+1))^{m-r+1}}\right]\right]$$

(3.25)

3.4 Quantile Function

Let $Q(u) = F^{-1}(u)$ be the quantile function (qf) of $F(x)$ for $0 < u < 1$.

Taking $F(x)$ to be the *cdf* of the Exponential-inverse Lindley distribution (EILD) and inverting it as above will give us the quantile function as follows.

$$F(x)=1-\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}$$

Inverting *F(x) = u*

$$F(x)=1-\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}=u \qquad (3.26)$$

Simplifying equation (3.26) above, we obtain:

$$1-u=\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}$$

$$\ln(1-u)=-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]$$

$$-\frac{\ln(1-u)}{\alpha}=\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}$$

$$-\ln(1-u)\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]=\alpha\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}$$

$$-\ln(1-u)+\ln(1-u)\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}=\alpha\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}$$

$$\left[\ln(1-u)-\alpha\right]\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}=\ln(1-u)$$

$$\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}=\frac{\ln(1-u)}{\left[\ln(1-u)-\alpha\right]}$$

$$\frac{\theta x+x+\theta}{(\theta+1)x}\mathrm{e}^{-\frac{\theta}{x}}=\frac{\ln(1-u)}{\left[\ln(1-u)-\alpha\right]}$$

$$\frac{\theta x+x+\theta}{(\theta+1)x}\mathrm{e}^{-\frac{\theta}{x}}=\frac{\ln(1-u)}{\left[\ln(1-u)-\alpha\right]}$$

$$(\theta+1)\frac{\ln(1-u)}{\left[\ln(1-u)-\alpha\right]}\mathrm{e}^{-(\theta+1)}=\frac{\theta x+x+\theta}{x}\mathrm{e}^{-\frac{\theta x+x+\theta}{x}}$$

From the expression above, it can be seen that $-\frac{\theta+x+\theta x}{x}$ is the Lambert function of the real argument, $(\theta+1)\frac{\ln(1-u)}{\left[\ln(1-u)-\alpha\right]}\mathrm{e}^{-(\theta+1)}$ because the Lambert function is defined as: $w(x)\mathrm{e}^{w(x)}=x$

Recall that the Lambert function has two branches with a branching point located at $\left(-\mathrm{e}^{-1},1\right)$. The lower branch, $W_{-1}(x)$ is defined in the interval $\left\lfloor-\mathrm{e}^{-1},1\right\rfloor$ and has a negative singularity for $x\to 0^{-1}$. The upper branch, $W_0(x)$, is defined for $x\in\left\lfloor-\mathrm{e}^{-1},\infty\right\rfloor$. Hence, equation (26) can be written as:

$$W\left((\theta+1)\frac{\ln(1-u)}{\left[\ln(1-u)-\alpha\right]}\mathrm{e}^{-(\theta+1)}\right)=-\frac{\theta+x+\theta x}{x} \qquad (3.27)$$

Now for any $\theta>0$ and $u\in(0,1)$, it follows that $\frac{\theta+x+\theta x}{x}>1$ and $\left((\theta+1)\left(1-\mathrm{e}^{\beta-\beta(1-u)^{-\frac{1}{\alpha}}}\right)\mathrm{e}^{-(\theta+1)}\right)<0$. Therefore, considering the lower branch of the Lambert function, equation (3.27) can be presented as:

$$W_{-1}\left((\theta+1)\frac{\ln(1-u)}{\left[\ln(1-u)-\alpha\right]}\mathrm{e}^{-(\theta+1)}\right)=-\frac{\theta+x+\theta x}{x} \qquad (3.28)$$

Collecting like terms in equation (3.28) and simplifying the result, the quantile function of the Exponential-inverse Lindley distribution (EILD) is obtained as:

$$Q(u)=\left\{-1-\frac{1}{\theta}-\frac{1}{\theta}W_{-1}\left((\theta+1)\frac{\ln(1-u)}{\left[\ln(1-u)-\alpha\right]}e^{-(\theta+1)}\right)\right\}^{-1} \qquad (3.29)$$

where *u* is a uniform variate on the unit interval *(0,1)* and $W_{-1}(.)$ represents the negative branch of the Lambert function.

3.5 Skewness and Kurtosis

In this dissertation, the quantile based measures of skewness and kurtosis will employed due to non-existence of the classical measures in some cases. The Bowley's measure of skewness (Kennedy and Keeping, 1962.) based on quartiles is given by;

$$SK=\frac{Q\left(\frac{3}{4}\right)-2Q\left(\frac{1}{2}\right)+Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)} \qquad (3.30)$$

And the Moores' (1998) kurtosis is on octiles and is given by;

$$KT=\frac{Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)-Q\left(\frac{3}{8}\right)+\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{1}{4}\right)} \qquad (3.31)$$

4. Reliability Analysis of the Inverse Lindley Distribution

Survival function is the likelihood that a system or an individual will not fail after a given time. Mathematically, the survival function is given by:

$$S(x) = 1-F(x) \qquad (3.32)$$

Now, taking *F(x)* to be the *cdf* of the proposed Exponential-inverse Lindley distribution (EILD) and substituting, we have;

$$F(x)=1-\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}$$

$$S(x)=1-\left\{1-\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}\right\}$$

$$S(x)=\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\} \qquad (3.33)$$

Below is a plot of the survival function at chosen parameter values in figure **3**

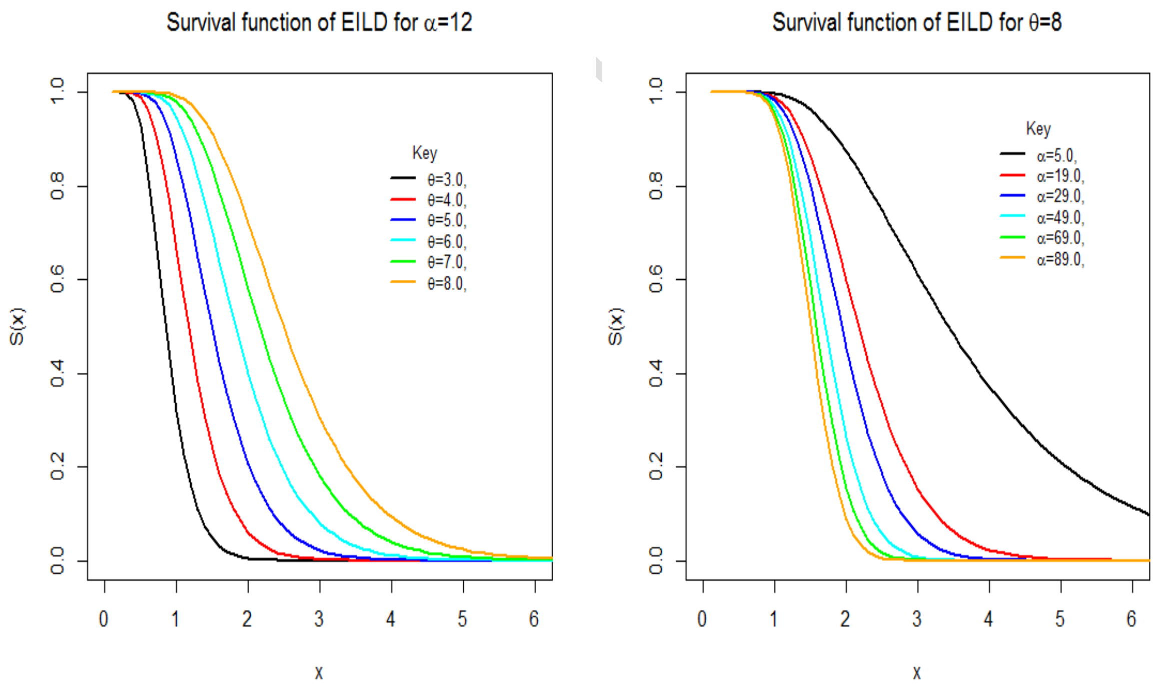


Figure 3: The survival function of the EILD.

The figure above revealed that the probability of survival for any random variable following a Exponential-inverse Lindley distribution (EILD) decreases with time, that is, as life gets older, probability of life decreases. This implies that the Exponential-inverse Lindley distribution (EILD) can be used to model random variables whose survival rate decreases as their age grows.

3.5 Hazard Function

Hazard function is the probability that a component will fail or die for an interval of time. The hazard function is defined as;

$$h(x)=\frac{f(x)}{1-F(x)}=\frac{f(x)}{S(x)} \tag{3.34}$$

Taking *f(x)* and *F(x)* to be the *pdf* and *cdf* of the proposed Exponential-inverse Lindley distribution (EILD) given as:

$$f(x)=\frac{\alpha\theta^{2}\left(\frac{1+x}{x^{3}}\right)e^{-\frac{\theta}{x}}}{(\theta+1)\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^{2}}\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}$$

and

$$F(x)=1-\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}$$

respectively.

Substituting for *f(x)* and *F(x)* and simplifying gives

$$h(x)=\frac{\dfrac{\alpha\theta^{2}\left(\dfrac{1+x}{x^{3}}\right)e^{-\frac{\theta}{x}}}{(\theta+1)\left[1-\left(1+\dfrac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^{2}}\exp\left\{-\alpha\left[\dfrac{\left(1+\dfrac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\dfrac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}}{1-\left\{1-\exp\left\{-\alpha\left[\dfrac{\left(1+\dfrac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\dfrac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}\right\}}$$

$$h(x)=\frac{\alpha\theta^{2}e^{-\frac{\theta}{x}}}{(\theta+1)}\left(\frac{1+x}{x^{3}}\right)\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^{-2} \tag{3.35}$$

The following is a plot of the hazard function at chosen parameter values in figure **4**

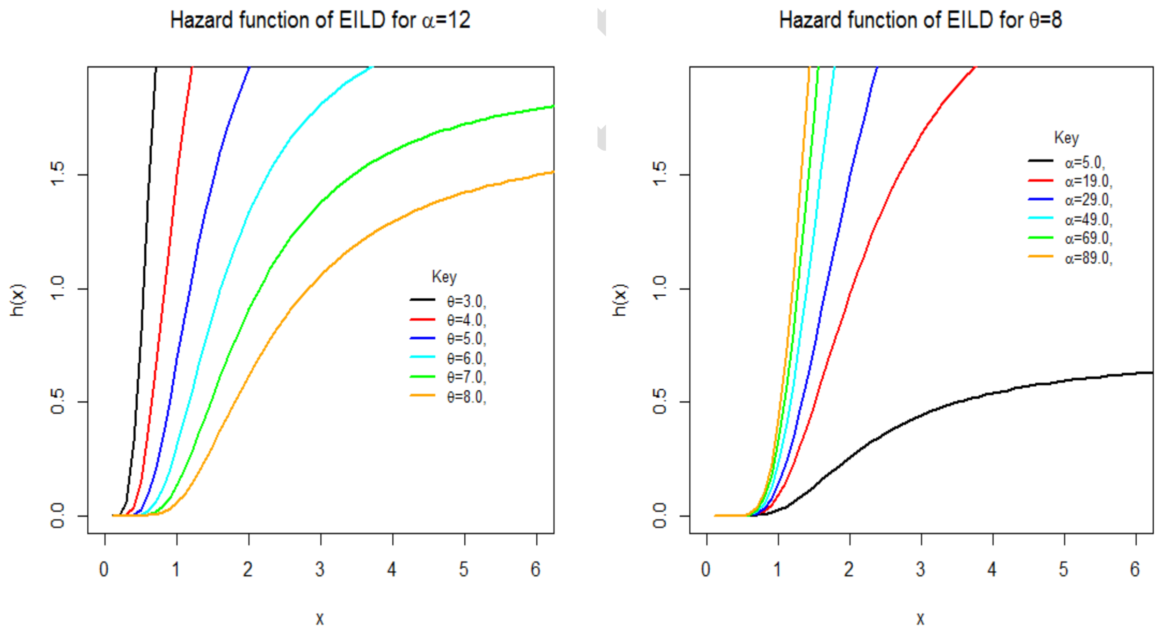


Figure 4: The hazard function of the EILD.

The figure above revealed that the probability of failure for any random variable following a Exponential-inverse Lindley distribution (EILD) increases with time, that is, as time goes on, probability of death increases. It also decreases slowly for some parameter values. This implies that the Exponential-inverse Lindley distribution (EILD) can be used to model random variables whose failure rate increases with time.

3.6 Order Statistics

Sample values such as the smallest, largest, or middle observation from a random sample provide important information. For example, the highest rainfall, flood or minimum temperature recorded during past years might be useful when planning for future emergencies. Suppose $X_1, X_2, ..., X_n$ is a random sample from a distribution with *pdf*, *f(x)*, and let $X_{1:n}, X_{2:n}, ..., X_{i:n}$ denote the corresponding order statistic obtained from this sample. The *pdf*, $f_{i:n}(x)$ of the i^{th} order statistic can be defined as;

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) F(x)^{i-1} \left[1 - F(x)\right]^{n-i} \tag{3.36}$$

Taking *f(x)* and *F(x)* to be the *pdf* and *cdf* of the Exponential-inverse Lindley distribution (EILD) respectively and using (3.5) and (3.6), the *pdf* of the i^{th} order statistics $X_{i:n}$ for the Exponential-inverse Lindley distribution (EILD) can be expressed from (3.36) as;

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} \left[\frac{\alpha(\theta+1)^{-1}\theta^2\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}}{\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^2} \exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}\right] \left[1-\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}\right]^{i+k-1} \tag{3.37}$$

Hence, the *pdf* of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the Exponential-inverse Lindley distribution (EILD) are given by;

$$f_{1n}(x)=n\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\left[\frac{\alpha(\theta+1)^{-1}\theta^2\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}}{\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^2}\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}\right]\left[1-\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}\right]^k \quad (3.38)$$

and

$$f_{nn}(x)=n\left[\frac{\alpha(\theta+1)^{-1}\theta^2\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}}{\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^2}\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}\right]\left[1-\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}\right]^{n-1} \quad (3.39)$$

respectively.

5. Estimation of Parameters of the Exponential-inverse Lindley distribution (EILD)

Let $X_1, X_2, \ldots\ldots, X_n$ be a sample of size 'n' independently and identically distributed random variables from the EIlD with unknown parameters, α *and* θ defined previously. The *pdf* of the EILD is given as:

$$f(x)=\frac{\alpha\theta^2\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}}{(\theta+1)\left[1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right]^2}\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}}\right]\right\}$$

The likelihood function is given by:

$$L(X|\alpha,\theta)\propto\left(\frac{\alpha\theta^2}{(\theta+1)}\right)^n\prod_{i=1}^{n}\left\{\frac{\left(\frac{1+x_i}{x_i^3}\right)e^{-\frac{\theta}{x}}}{\left[1-\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}\right]^2}\exp\left\{-\alpha\left[\frac{\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}}{1-\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}}\right]\right\}\right\} \qquad (3.40)$$

Let the log-likelihood function, $l=\log L\left(X/\alpha,\theta\right)$, therefore

$$l=n\log\alpha+2n\log\theta-n\log(\theta+1)+\sum_{i=1}^{n}\log\left(\frac{1+x_i}{x_i^3}\right)-\theta\sum_{i=1}^{n}x_i^{-1}-2\sum_{i=1}^{n}\log\left[1-\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}\right]-\alpha\sum_{i=1}^{n}\left[\frac{\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}}{1-\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}}\right]$$

(3.41)

Differentiating l partially with respect to α *and* θ respectively gives;

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}-\alpha\sum_{i=1}^{n}\left[\frac{\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}}{1-\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}}\right] \qquad (3.42)$$

$$\frac{\partial(\kappa)}{\partial\theta}=\frac{2n}{\theta}-\frac{n}{(\theta+1)}-\sum_{i=1}^{n}x_i^{-1}-\sum_{i=1}^{n}\left\{\frac{\left(\frac{1}{x}+\frac{\theta}{(\theta+1)^2x_i}\left(\frac{\theta+1-x_i}{x_i}\right)\right)\mathrm{e}^{-\frac{\theta}{x_i}}}{\left[1-\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}\right]}\right\}-\alpha\sum_{i=1}^{n}\left\{\frac{\left(\frac{1}{x}+\frac{\theta}{(\theta+1)^2x_i}\left(\frac{\theta+1-x_i}{x_i}\right)\right)\mathrm{e}^{-\frac{\theta}{x_i}}}{\left[1-\left(\left(1+\frac{\theta}{(\theta+1)x_i}\right)\mathrm{e}^{-\frac{\theta}{x_i}}\right)\right]^2}\right\} \qquad (3.43)$$

The solution of the non-linear system of equations; $\frac{\partial l}{\partial\alpha}=0$ *and* $\frac{\partial l}{\partial\theta}=0$ will give us the maximum likelihood estimates of parameters α *and* θ. However, the solution cannot be obtained analytically except numerically with the aid of suitable statistical software like R, SAS, Maple, e.t.c when data sets are available.

6. APPLICATION

4. Simulation study for Exponential-Inverse Lindley distribution (EILD)

We perform a Monte Carlo (MC) simulation study with the objective to assess the behavior of the MLEs of EILD via the optim() R-function with the argument method = "L-BFGS-B". It is used for maximizing the log-likelihood function of a probabilistic model. We consider 500 MC replicates under different sample sizes *n* = 25, 50. . . 500. These samples are obtained using the inverse CDF (also known as quantile function). The SS is conducted for three different combination of θ and α. These combination values are given by (i) $\theta = 0.5 \ and \ \alpha = 0.5$, (ii) $\theta = 2.5 \ and \ \alpha = 0.5$, and (iii) $\theta = 0.5 \ and \ \alpha = 2.5$.

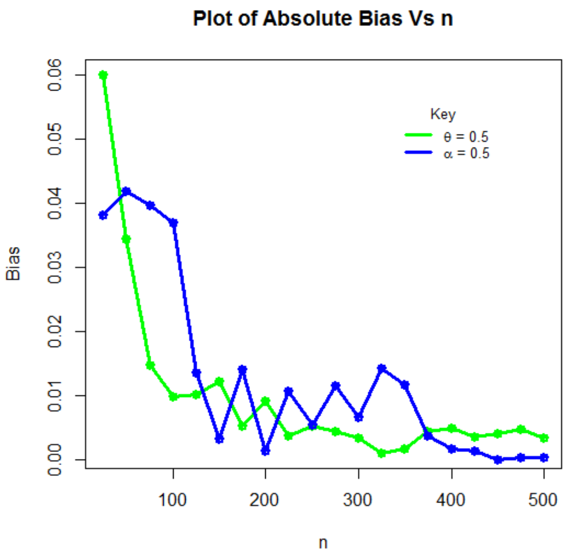
The judgment about the performances of $\hat{\theta}_{MLE}$ and $\hat{\alpha}_{MLE}$ is made by considering two evaluation criteria. These criteria are the Mean square error (MSE) and Bias.

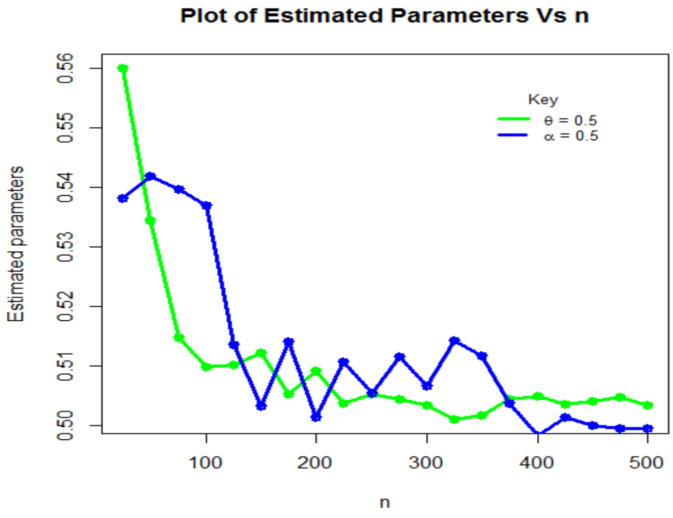
For every sample size, the average MLEs, mean square errors (MSE), Biases and Absolute biases were computed. The results obtained after performing the MC simulation are provided in Tables 1-3 and displayed graphically in Figures 5-7.

Table 1: Simulation results for the EILD for $\theta = 0.5 \ and \ \alpha = 0.5$

N	Measures /Criteria	Parameters		N	Measures/ Criteria	Parameters	
		θ	α			θ	α
n=25	MLEs	0.5600	0.5382	n=200	MLEs	0.5091	0.5014
	Biases	0.0600	0.0382		Biases	0.0091	0.0014
	MSEs	0.0279	0.3045		MSEs	0.0028	0.0222
n=50	MLEs	0.5344	0.5419	n=300	MLEs	0.5034	0.5067

	Biases	0.0344	0.0419		Biases	0.0034	0.0067
	MSEs	0.0148	0.2003		MSEs	0.0018	0.0148
n=75	MLEs	0.5148	0.5397	n=400	MLEs	0.5049	0.4983
	Biases	0.0148	0.0397		Biases	0.0049	0.0017
	MSEs	0.0081	0.0935		MSEs	0.0014	0.0099
n=100	MLEs	0.5099	0.5369	n=500	MLEs	0.5035	0.4996
	Biases	0.0099	0.0369		Biases	0.0035	0.0004
	MSEs	0.0067	0.0736		MSEs	0.0011	0.0082


Plot of Absolute Bias Vs n
Key
θ = 0.5
α = 0.5
Bias
0.00
0.01
0.02
0.03
0.04
0.05
0.06
100
200
300
400
500
n


Plot of Estimated Parameters Vs n
Key
θ = 0.5
α = 0.5
Estimated parameters
0.50
0.51
0.52
0.53
0.54
0.55
0.56
100
200
300
400
500
n

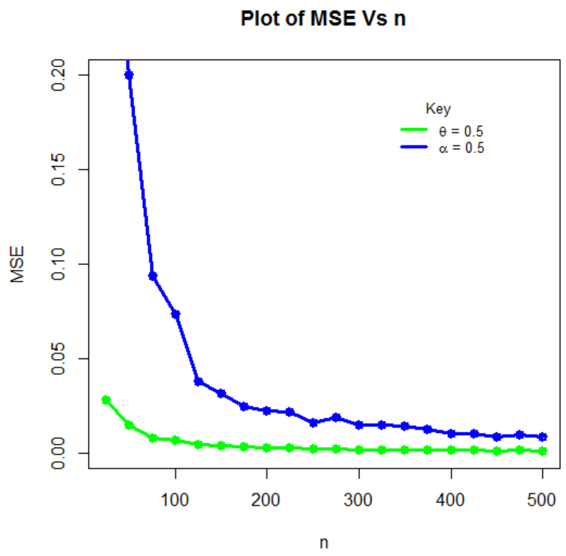


Fig. 5: Plots of MLEs, Absolute Biases and MSEs of the EILD for $\theta=0.5$ and $\alpha=0.5$

Table 2: Simulation results for the EILD for $\theta=2.5$ and $\alpha=0.5$

n	Measures /Criteria	Parameters		N	Measures/ Criteria	Parameters	
		θ	α			θ	α
n=25	MLEs	2.7013	0.6233	n=200	MLEs	2.5180	0.5150
	Biases	0.2013	0.1233		Biases	0.0180	0.0150
	MSEs	0.6767	0.4867		MSEs	0.0676	0.0175
n=50	MLEs	2.6066	0.5710	n=300	MLEs	0.5262	0.5013
	Biases	0.1066	0.0710		Biases	0.0262	0.0013
	MSEs	0.3713	0.1569		MSEs	0.0469	0.0121

n=75	MLEs	2.5684	0.5383	n=400	MLEs	2.5169	0.5028
	Biases	0.0684	0.0383		Biases	0.0169	0.0028
	MSEs	0.2195	0.0684		MSEs	0.0337	0.0082
n=100	MLEs	2.5763	0.5099	n=500	MLEs	2.5162	0.5012
	Biases	0.0763	0.0099		Biases	0.0162	0.0012
	MSEs	0.1563	0.0356		MSEs	0.0269	0.0067

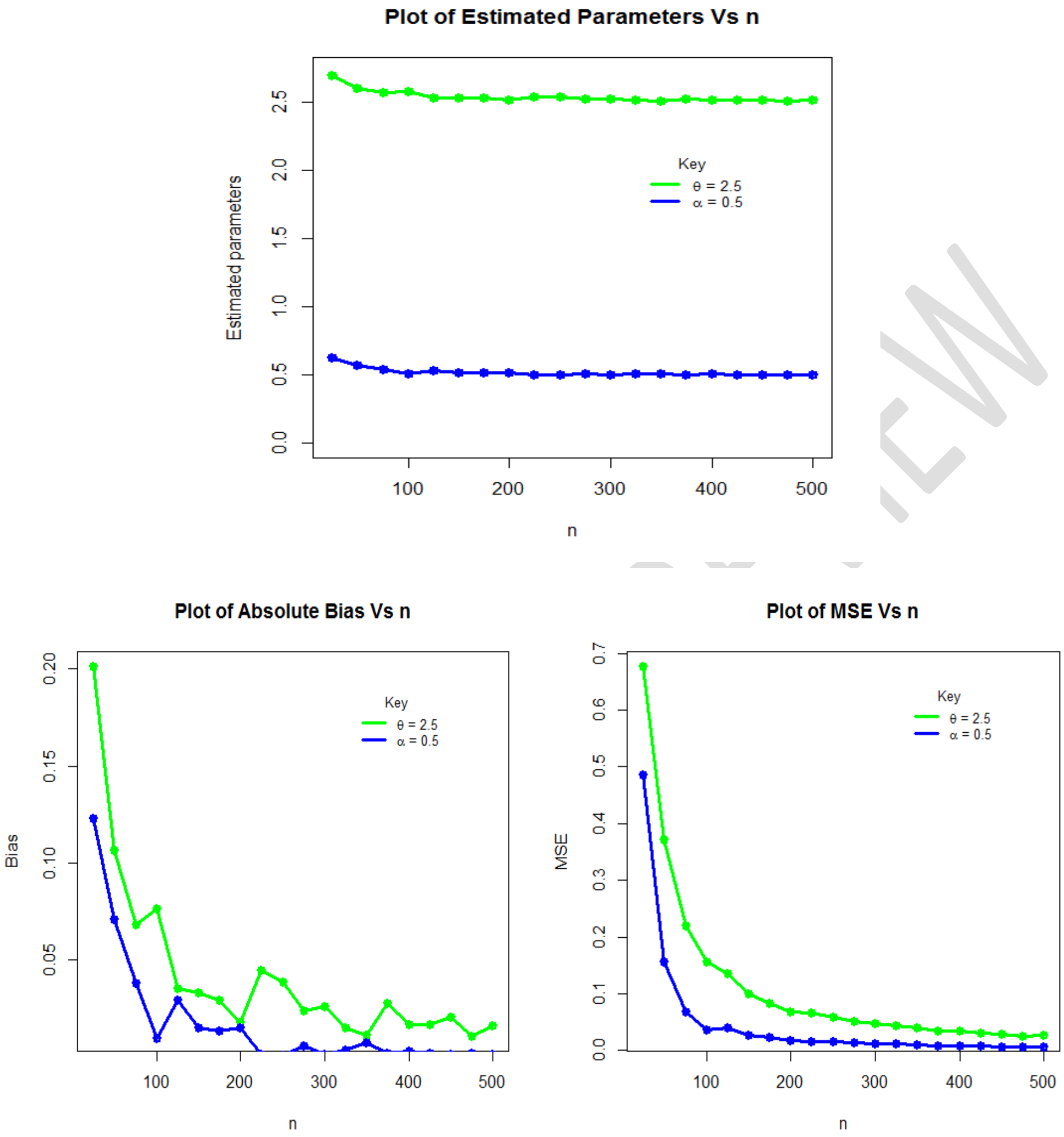


Fig. 6: Plots of MLEs, Absolute Biases and MSEs of the EILD model for $\theta = 2.5 \text{ and } \alpha = 0.5$

Table 3: Simulation results for the EILD for $\theta = 0.5 \ and \ \alpha = 2.5$

n	Measures /Criteria	Parameters		N	Measures/ Criteria	Parameters	
		θ	α			θ	α
n=25	MLEs	0.7048	2.1812	n=200	MLEs	0.5148	2.7677
	Biases	0.2048	0.3188		Biases	0.0148	0.2677
	MSEs	0.1390	3.1735		MSEs	0.0153	1.6678
n=50	MLEs	0.6007	2.4817	n=300	MLEs	0.5158	2.6434
	Biases	0.1007	0.0183		Biases	0.0158	0.1434
	MSEs	0.0569	2.5852		MSEs	0.0109	1.2177
n=75	MLEs	0.5694	2.5658	n=400	MLEs	0.5086	2.6309
	Biases	0.0694	0.0658		Biases	0.0086	0.1309
	MSEs	0.0409	2.2804		MSEs	0.0074	0.8962
n=100	MLEs	0.5549	2.5642	n=500	MLEs	0.5080	2.6114
	Biases	0.0549	0.0642		Biases	0.0080	0.1114
	MSEs	0.0283	2.0422		MSEs	0.0067	0.7739

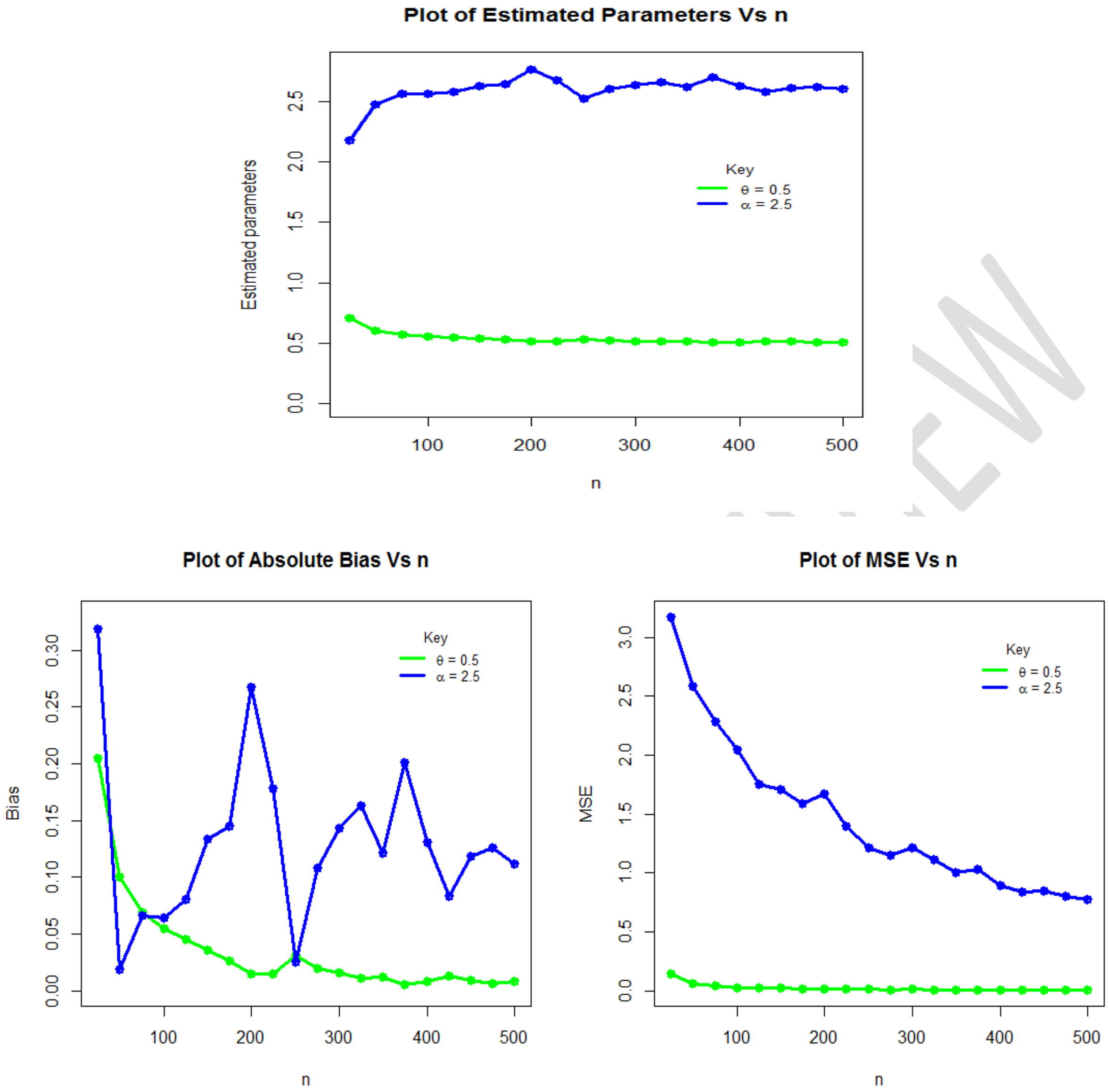


Fig. 7: Plots of MLEs, Absolute Biases and MSEs of the EILD model for $\theta=0.5 \text{ and } \alpha=2.5$

From the results of the simulation study of EILD presented in Tables 1-3 and Figures 5-7, we can see the averages of the MLEs (Mean), their biases (Absolute Bias) and mean square errors (MSEs) for the parameters of the EILD. Based on the values from the tables, it is clear that the average estimates tend to be closer to the true parameters when sample size increases and the biases and mean square errors all decrease as sample size increases which is in agreement with first-order asymptotic theory.

7. SUMMAR AND CONCLUSION

In this research work, we proposed a new extension of the inverse Lindley distribution called exponential-inverse Lindley distribution. Some mathematical and statistical properties of the proposed distribution have been studied appropriately. The derivations of some expressions for its moments, moment generating function, characteristics function, survival function, hazard function and ordered statistics has been done appropriately. Some plots of the distribution revealed that it is a flexible and skewed distribution. The implications of the plots for the survival function indicate that the exponential-inverse Lindley distribution (EILD) could be used to model time or age-dependent events, where survival rate decreases with time or age. From the results of the simulation study of EILD, it has been shown that the average estimates tend to be closer to the true parameters when sample size increases and the biases and mean square errors all decrease as sample size increases which is in agreement with first-order asymptotic theory.

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