An Efficient Numerical Approach for Fractional Diffusion Equations Using Shifted Legendre Operational Matrices and Multi-step Methods.

Abstract:
The paper deals with an efficient scheme to solve fractional diffusion equation including both time and spatial fractional derivative in Caputo sense. Firstly, the operational matrix of fractional derivative of shifted Legendre orthogonal polynomials is computed along with spectral Tau method and used to reduce the problem in the form system of fractional ordinary differential equation (FODE). Then the fractional linear multi-step method (FLMMS) permit to obtain the approximate solution. In order to show the accuracy and validity of the proposed method, two numerical examples are reported.

Keywords: fractional diffusion equation, operational matrix of shifted Legendre polynomials, spectral Tau method, fractional ordinary differential equation, Linear multi-step methods.

Mathematics Subject Classification: 34A05, 34A08, 42A10, 40A30, 65B10.

1 Introduction

The diffusion equation describe the density of specific entities in certain medium as function of time and space [1, 2]. The diffusion equation is obtained by considering the mass conservation principe and Fick's law of diffusion. The application of this equation can be found in physics, epidemiology, rumor propagation, chemistry, economy and many others. The understanding of the mechanics and dynamics of diffusion is crucial for developping prevention and control strategies. The fractional extension of ordinary models give improvents attributed to the memory and long effects presented in fractional [3, 4, 5, 6, 7, 8] formulation. In the present paper, we consider the problem:

$$\begin{cases} \dfrac{\partial^{\alpha} u\left(x,t\right)}{\partial t^{\alpha}} = a\dfrac{\partial^{\sigma} u\left(x,t\right)}{\partial x^{\sigma}} + K\left(x,t\right) \\ u\left(0,t\right) = p\left(t\right) \\ u\left(1,t\right) = q\left(t\right) \\ u\left(x,0\right) = g\left(x\right), \end{cases} \tag{1.1}$$

where $0 < \alpha \leq 1$; $0 < \sigma \leq 2$; $0 \leq x, t \leq 1$, a is the coefficient of diffusibility, $K(x,t)$ is the source given function, p, q and g are known functions. Analytical solution for fractional differential equations can be hard to find. Due to this, numerical methods are frequently applied. For the solution of (1.1), numerical techniques such as: homotopy perturbation method [9], Tau method [10, 11, 12, 13, 14, 15, 16], sinc-Legendre method, Legendre spatial methods [17, 18] and so on are employed [19, 20, 21]. These methods are very efficient but because of the high computational complexities they are hard to be applied to fractional partial differential equation. We need therefore an easy and efficient method to solve such equations. The approach in this study is to built the numerical method as a combination of operational matrix [11, 14, 22] method based on the fractional derivative of shifted Legendre polynomial [18] and the FLMMs Method [23, 24, 25, 26, 27]. The application of the orthogonal shifted Legendre operational matrix method and the spectral Tau method lead to system of FODE [28, 29] wich can be solved by FLMMs method. In the first step, the shifted Legendre operational matrix method is used to replace spatial fractional derivative part of the problem, and in the second step the spectral Tau method permit to obtain a system of FODE. Then FLMM method is used to solve the temporal fractional part of the problem.
The paper is organized as follows: In section 2, mathematical preliminaries used in this work are given. In section 3, shifted Legendre polynomial and the operational matrix are introduced. In section 4, application of the proposed method for solving space and fractional diffusion problem is presented. In section 5, the proposed method are used to demonstrate his reliability and his accuracy on two illustratives examples. In section 6, the concluded results are presented.

2 Preliminaries

In this section, we recall the definitions of fractional Riemann-Liouville integral and his derivative, the fractional derivative in Caputo sense, the fractional ordinary differential equation and the linear multi-step method for fractional equation [30, 31, 32].

2.1 The fractional derivatives and property

Definition 2.1 *: The Riemann–Liouville (RL) fractional integral of order $\alpha > 0$ and origin at to for a function $y(t) \in L^1[0,\pi]$ is defined as:*

$$J_{t_0}^{\alpha} y(\tau) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} y(\tau)\, d\tau. \tag{2.1}$$

Definition 2.2 *: The RL fractional derivative is defined as:*

$$\widehat{D}_{t_0}^{\alpha} y(\tau) := D^m J_{t_0}^{m-\alpha} y(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_{t_0}^{t} (t-\tau)^{m-\alpha-1} y(\tau)\, d\tau, \tag{2.2}$$

where $m = \lceil \alpha \rceil$ is the smallest integer greater or equal to α and D^m, $y^{(m)}$, $\frac{d^m}{dt^m}$ denotes the standard integer-order derivative.

Definition 2.3 *: For $y \in A^m([t_0, T])$ $\big($i.e., $y^{(m)}$ absolutely continuous$\big)$, the Caputo derivative is given by:*

$$D_{t_0}^{\alpha} y(\tau) := J_{t_0}^{m-\alpha} D^m y(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} (t-\tau)^{m-\alpha-1} y^{(m)}(\tau)\, d\tau. \tag{2.3}$$

If $t_0 = 0$, then $D_{t_0}^{\alpha} y(\tau) = D^{\alpha} y(t)$ and we have

$$D^{\alpha} y(t) = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{t} \frac{y^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, \quad m-1 < \alpha \leq m, \quad m \in \mathbb{N}. \tag{2.4}$$

For the Caputo derivative:

$$D^{\alpha}C = 0,\ (C\ is\ a\ constant), \tag{2.5}$$

$$D^{\alpha}t^{\beta} = \begin{cases} 0, & for \quad \beta \in \mathbb{N}_0 \ \ and\ \ \beta < \lceil \alpha \rceil,\ \mathbb{N}_0 = \{0, 1, \ldots\} \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, & for \quad \beta \in \mathbb{N}_0 \ and\ \ \beta \geq \lceil \alpha \rceil \ or\ \beta \notin \mathbb{N} \ \ and \ \ \beta > \lfloor \alpha \rfloor, \end{cases} \tag{2.6}$$

where $\lceil \alpha \rceil$ is the smallest integer greater than or equal to α and $\lfloor \alpha \rfloor$ is the largest integer less than or equal to α. The Caputo fractional differentiation is a linear, then we have:

$$D^{\alpha}(\lambda f(x) + \mu g(x)) = \lambda D^{\alpha} f(x) + \mu D^{\alpha} g(x), with \quad \lambda \quad and \quad \mu \quad constants. \tag{2.7}$$

We have also [5]

$$J^{\alpha}_{t_0} D^{\alpha}_{t_0} y(t) = y(t) - T_{m-1}[y; t_0](t), \tag{2.8}$$

where $T_{m-1}[y; t_0](t) = \sum_{k=0}^{m-1} \frac{(t-t_0)^k}{k!} y^{(k)}(t)$ is the polynomial.
The relation between Riemann-Liouville and the Caputo derivative is of the form

$$\widehat{D}^{\alpha}_{t_0} y(t) = D^{\alpha}_{t_0} y(t) + \sum_{k=0}^{m-1} \frac{(t-t_0)^k}{k!} y^k(t). \tag{2.9}$$

2.2 Fractional ordinary differential equation (FODE)

The most important application of the fractional Caputo's derivative is the fractional ordinary differential equation (FODE) [9] of the form

$$\begin{cases} D^{\alpha}_{t_0} y(t) = f(t, y(t)) \\ \ \ y(t_0) = y_0,\ y'(t_0) = y_0^{(1)},\ \ldots,\ y^{(m-1)}(t_0) = y_0^{(m-1)} \end{cases} \tag{2.10}$$

where $m-1 < \alpha \leq m,\ 0 \leq t \leq T,\ f(t, y(t))$ is continuous and $y_0^{(1)},\ \ldots,\ y^{(m-1)}$ are the value of derivative at t_0. The application to both sides of Equation (2.10) of the RL integration $J^{\alpha}_{t_0}$, together with Equation (2.8), leads to the reformulation of FDE in terms of the weakly-singular Volterra Integration Equation (VIE) [15, 16, 19]

$$y(t) = T_{m-1}[y; t_0](t) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau, y(\tau))\, d\tau. \tag{2.11}$$

The integral formulation (2.11) is surely useful since it allows exploiting theorical and numerical results already available for this class of VIEs in order to study and solve FODEs.

2.3 Convolution quadrature rules

One way to obtain solution of (2.10) is to approximate (2.11) by convolution quadrature [33, 34, 35, 36, 37] formulas, wich can be written in the general form:

$$y_n = \varphi_n + \sum_{j=0}^{n} c_{n-j} f_j, \quad f_j = f(t_j, y_j), \tag{2.12}$$

where φ_n and c_n are known coefficient $t_n = t_0 + nh$ is an assigned grid. With a constant step-size $h > 0$ and $y_n = y(t_n)$. The way in which the coefficients are derived depends on the specific method. In particular, the following two classes of convolution quadrature rules for FODEs are mainly studied in the literature: Product-Integration (PI) rules [39] and Fractional Linear Multi-Step methods (FLMMs) [40, 41].

2.3.1 Product-Integration (PI) rules

Given a grid $t_n = t_0 + nh$, with constant step-size $h > 0$, in PI rules, the solution of the VIE (2.11) at t_n is first written in a piece-wise way:

$$y(t_n) = T_{m-1}[y;t_0](t_n) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_n} (t_n - \tau)^{\alpha-1} f(\tau, y(\tau))\, d\tau, \tag{2.13}$$

where $f(\tau, y(\tau))$ can be approximated, in each subinterval $[t_j, t_{j+1}]$ by means of some interpolant polynomial. If $f(\tau, y(\tau))$ is approximated by $f(t_j, y_j)$, we obtain the explicit PI rectangular:

$$y_n = T_{m-1}[y;t_0](t_n) + h^\alpha \sum_{j=0}^{n-1} b_{n-j-1}^{(\alpha)} f(_j, y(t_j)). \tag{2.14}$$

If $f(\tau, y(\tau))$ is approximated by $f(t_{j+1}, y(t_{j+1}))$, we obtain the implicit PI rectangular:

$$y_n = T_{m-1}[y;t_0](t_n) + h^\alpha \sum_{j=1}^{n-1} b_{n-j}^{(\alpha)} f(_j, y(t_j)), \tag{2.15}$$

with $b_n^{(\alpha)} = \dfrac{((n+1)^\alpha - n^\alpha)}{\Gamma(\alpha+1)}$. If $f(\tau, y(\tau))$ is approximated by $f(t_{j+1}, y_{j+1}) + \dfrac{s - t_{j+1}}{h}(f(t_{j+1}, y_{j+1})) + f(t_j, y_j)$, we obtain implicit Trapezoidal rule [40]

$$y_n = T_{m-1}[y;t_0](t_n) + h^\alpha \left(\widehat{a}_n^{(\alpha)} f_0 + \sum_{j=1}^{n-1} a_{n-j}^{(\alpha)} f(t_j, y(t_j)) \right), \tag{2.16}$$

with

$$\widehat{a}_n^{(\alpha)} = \frac{(n-1)^{\alpha+1} - n^\alpha (n - \alpha - 1)}{\Gamma(\alpha+2)}, \quad a_n^{(\alpha)} = \begin{cases} \dfrac{1}{\Gamma(\alpha+2)}, & for \quad n = 0, \\ \dfrac{(n-1)^{\alpha+1} - 2n^{\alpha+1} + (n+1)^{\alpha+1}}{\Gamma(\alpha+2)}, & for \quad n = 1, 2, \ldots. \end{cases}$$

2.3.2 Fractional Linear Multi-Step Methods (FLMMs)

Another way to approximate FODE is to use the FLMMs Method. Given the initial value problem:

$$y'(t) = f(t), \; y(t_0) = y_0, \tag{2.17}$$

its solution can be approximated by means of an linear multi-step method (LMM) given by:

$$\sum_{j=0}^{k} \rho_j y_{n-j} = h \sum_{j=0}^{k} \sigma_j f(t_{n-j}),$$

where $\rho(z) = \rho_0 z^k + \rho_1 z^{k-1} + \cdots + \rho_k$ and $\sigma(z) = \sigma_0 z^k + \sigma_1 z^{k-1} + \cdots + \sigma_k$ are the first and second characteric polynomial of the LMM. Problem (2.17) can be rewritten in the integral from:

$$y(t) = y_0 + \int_{t_0}^{t} f(\tau)\, d\tau.$$

The solution $y(t)$ can be approximated by using LMMs reformulated in terms of convolution quadrature formulas:

$$y_n = h \sum_{j=0}^{n} \omega_{n-j} f(t_j), \; n \geq k$$

where the weights ω_n depend on the characteristic polynomials $\rho(z)$ and $\sigma(z)$, but not on h.

$$\delta(\zeta) = \sum_{n=0}^{\infty} \omega_n \zeta^n, \quad \delta(\zeta) = \frac{\rho(1/\zeta)}{\sigma(1/\zeta)}.$$

The idea underlying FLMMs is to derive convolution quadratures for the RL integral (2.1) with convolution weights given by the function:

$$F\left(\frac{\delta(\zeta)}{h}\right) = \left(\frac{\delta(\zeta)}{h}\right)^{-\alpha} = h^{\alpha}\left(\frac{\rho(1/\zeta)}{\sigma(1/\zeta)}\right), \tag{2.18}$$

where $F(s) = s^{-\alpha}$ the Laplace transform of the kernel $\dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ in (2.1). The assumptions that make possible this generalization of LMMs are that the generating function $\delta(\zeta)$ has no zeros in the closed unit disc $|\zeta| \le 1$, except for $\zeta = 1$, and $|\arg\delta(\zeta)| < \pi$ for $|\zeta| < 1$.
When an LMM is generalized to Equation (2.1) in the above Lubich sense, the resulting FLMM reads as:

$$hJ^{\alpha}_{t_0} f(t_n) = h^{\alpha} \sum_{j=0}^{n} \omega^{(\alpha)}_{n-j} f(t_j), \tag{2.19}$$

where the convolution quadratique weights $\omega^{(\alpha)}_n$ are obtained from:

$$\sum_{n=0}^{\infty} \omega^{(\alpha)}_n \zeta^n = \omega^{(\alpha)}(\zeta), \quad \omega^{(\alpha)}(\zeta) = (\delta(\zeta))^{-\alpha}.$$

Thus,to handle non-smooth functions (as happens in the solution of fractional-order problems), it is necessary to introduce a correction term:

$$hJ^{\alpha}_{t_0} f(t_n) = h^{\alpha} \sum_{j=0}^{n} \omega^{(\alpha)}_{n,j} f(t_j) + h^{\alpha} \sum_{j=0}^{n} \omega^{(\alpha)}_{n-j} f(t_j). \tag{2.20}$$

From the application of the discretized convolution quadrature rule (2.20) to integral Equation (2.11), we are able to derive FLMMs for the approximation of the solution of FDEs:

$$y_n = T_{m-1}[y; t_0](t_n) + h^{\alpha} \sum_{j=0}^{s} \omega^{(\alpha)}_{n,j} f(t_j, y_j) + h^{\alpha} \sum_{j=0}^{n} \omega^{(\alpha)}_{n-j} f(t_j, y_j). \tag{2.21}$$

The starting weights $\omega_{n,j}$ are chosen by imposing that the quadrature rule (2.20) is exact when applied to $f(t) = t^{\gamma}$, with γ assuming all the possible fractional values expected in the expansion of the true solution and, hence, by solving at each step the algebraic linear system:

$$\sum_{j=0}^{s} \omega^{(\alpha)}_{n,j} j^{\gamma} = -\sum_{j=0}^{s} \omega^{(\alpha)}_{n-j} j^{\gamma} + \frac{\Gamma(\gamma+1)}{\Gamma(1+\gamma+\alpha)} n^{\gamma+\alpha}. \tag{2.22}$$

One of the simplest FLMMs is obtained from the implicit Euler method (or BDF1). No starting weights are necessary in this case, and since the generating function is $\delta(\zeta) = 1 - \zeta$, we see that $\omega^{(\alpha)}_n$, $n = 0, 1, 2, \ldots$, are the coefficients of the generalized binomial series $(1-\zeta)^{-\alpha}$, namely:

$$\omega^{(\alpha)}_n = (-1)^n \binom{-\alpha}{n} = (-1)^n \frac{\Gamma(1-\alpha)}{\Gamma(n+1)\Gamma(-\alpha-n+1)},$$

which can be also evaluated by the recurrence $\omega_n^{(\alpha)} = \left(\frac{1-(1-\alpha)}{n}\right)\omega_{n-1}^{(\alpha)}$ with $\omega_0^{(\alpha)} = 1$. The corresponding method:

$$y_n = T_{m-1}\left[y; t_0\right](t_n) + h^{\alpha}\sum_{j=0}^{n}(-1)^n\binom{-\alpha}{n-j} f\left(t_j, y_j\right),$$

is commonly referred to as the Grünwald–Letnikov scheme [32].

3 Properties of shifted Legendre polynomials

The well-known Legendre polynomials [17] are defined on the interval $[-1,1]$ and can be determined with the aid of the following recurrence formulae:

$$P_{i+1}(t) = \frac{2i+1}{i+1} t P_i(t) - \frac{i}{i+1} P_{i-1}(t), \; i = 1, 2, 3, \ldots,$$

where $P_0(t) = 1$ and $P_1(t) = t$. In order to use these polynomials on the interval $x \in [0,1]$ we define the so-called shifted Legendre polynomials by introducing the change of variable $t = 2x-1$. Let the shifted Legendre polynomials $P_i(2x-1)$ be denoted by $G_i(x)$. Then $G_i(x)$ can be obtained as follows:

$$G_{i+1}(x) = \frac{(2i+1)(2x-1)}{i+1} G_i(x) - \frac{i}{i+1} G_{i-1}(x), \; i = 1, 2, 3, \ldots, \tag{3.1}$$

where $G_0(x) = 1$ and $G_1(x) = 2x-1$. The analytic form of the shifted Legendre polynomial $G_i(x)$ of degree i given by:

$$G_i(x) = \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)! x^k}{(i-k)! \left(k!\right)^2}. \tag{3.2}$$

Note that $G_i(0) = (-1)^i$ and $G_i(1) = 1$. The orthogonality condition is

$$\int_0^1 G_i(x) G_j(x) = \begin{cases} \frac{1}{2i+1}, & for \quad i = j, \\ 0, & for \quad i \neq j. \end{cases} \tag{3.3}$$

A function $z(x)$, square integrable in $[0,1]$, may be expressed in terms of shifted Legendre polynomials as

$$z(x) = \sum_{j=0}^{\infty} c_j G_j(x),$$

where the coefficients c_j are given by:

$$c_j = (2j+1)\int_0^1 z(x) G_j(x)\, dx, \; i = 1, 2, 3, \ldots.$$

In practice,only the first $(m+1)$-terms shifted Legendre polynomials are considered. Then we have

$$z(x) = \sum_{j=0}^{m} c_j G_j(x) = C^T \Phi(x),$$

where the shifted Legendre coefficient vector C and the shifted Legendre vector $\Phi(x)$ are given by

$$C^T = \left[c_0, c_1, \ldots, c_m\right],$$

$$\Phi(x) = [G_0(x), G_1(x), \ldots, G_m(x)]^T. \tag{3.4}$$

The derivative of the vector $\Phi(x)$ can be expressed by

$$\frac{d\Phi(x)}{dx} = D^{(1)}\Phi(x), \tag{3.5}$$

where $D^{(1)}$ is the $(m+1)\times(m+1)$ operational matrix of derivative given by

$$D^{(1)} = (d_{ij}) = \begin{cases} 2(2j+1), & for \quad j = i-k, \\ 0, & otherwise, \end{cases} \begin{cases} k = \quad 1,3,\cdots,m, & if\ m\ odd, \\ k = \quad 1,3,\cdots,m-1, & if\ m\ even, \end{cases}$$

for example for even m we have

$$D^{(1)} = 2\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 5 & 0 & \cdots & 2m-3 & 0 & 0 \\ 0 & 3 & 0 & 7 & \cdots & 0 & 2m-1 & 0 \end{bmatrix}$$

4 Generalized Legendre operational matrix to fractional calculus

By using Eq.(3.5),it is clear that

$$\frac{d^n\Phi(x)}{dx^n} = \left(D^{(1)}\right)^n \Phi(x), \tag{4.1}$$

where $n \in \mathbb{N}$ and the superscript, in $D^{(1)}$, denotes matrix powers. Thus

$$D^{(n)} = \left(D^{(1)}\right)^n, \ n = 1,2,3,\ldots. \tag{4.2}$$

Lemma 4.1 *Let $G_i(x)$ be a shifted Legendre polynomial then*

$$D^{\alpha}G_i(x) = 0, \ i = 0,1,2,\ldots,\lceil\alpha\rceil - 1, \ \alpha > 0. \tag{4.3}$$

Proof 4.1 *Using Eqs.(2.5) − (2.6) in Eq. (3.1) the lemma can be proved.*

In the following theorem we generalize the operational matrix of derivative of shifted Legendre polynomials given in (3.5) for fractional derivative.

Theorem 4.1 *Let $\Phi(x)$ be shifted Legendre vector defined in (3.4) and also suppose $\alpha > 0$ then*

$$D^{\alpha}\Phi(x) \simeq D^{(\alpha)}\Phi(x), \tag{4.4}$$

where $D^{(\alpha)}$ is the $(m+1)\times(m+1)$ operational matrix of fractional derivative of order α in the Caputo *sense and is defined as follows:*

$$D^{(\alpha)} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \delta_{\lceil\alpha\rceil,0,k} & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \delta_{\lceil\alpha\rceil,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \delta_{\lceil\alpha\rceil,m,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{i} \delta_{i,0,k} & \sum_{k=\lceil\alpha\rceil}^{i} \delta_{i,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{i} \delta_{i,m,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{m} \delta_{m,0,k} & \sum_{k=\lceil\alpha\rceil}^{m} \delta_{m,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{m} \delta_{m,m,k} \end{bmatrix},$$

where $\delta_{i,j,k}$ is given by:

$$\delta_{i,j,k} = (2j+1) \sum_{l=0}^{j} \frac{(-1)^{i+j+k+l} (i+k)! \, (j+l)!}{(i-k)!k!\Gamma(k-\alpha+1)\,(j-l)!\,(l!)^2\,(k+l-\alpha+1)}. \tag{4.5}$$

Note that in D^{α}, the first $\lceil\alpha\rceil$ rows, are all zero.

Proof 4.2 *Using Eqs.* (2.6) , (2.7) *and* (3.2) *we have:*

$$\begin{aligned} D^{\alpha} G_i(x) &= \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)!\,(k!)^2} D^{\alpha} x^k \\ &= \sum_{k=\lceil\alpha\rceil}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)!\,(k!)\,\Gamma(k-\alpha+1)} x^{k-\alpha}, \;\; i = \lceil\alpha\rceil, \cdots, m. \end{aligned} \tag{4.6}$$

Now, approximate $x^{k-\alpha}$ by $(m+1)$ terms of shifted Legendre series, we have

$$x^{k-\alpha} \simeq \sum_{j=0}^{m} b_{k,j} G_j(x), \tag{4.7}$$

where

$$\begin{aligned} b_{k,j} &= (2j+1) \int_0^1 x^{k-\alpha} G_j(x)\, dx = (2j+1) \sum_{l=0}^{j} (-1)^{j+l} \frac{(j+l)!}{(j-l)!\,(l!)^2} \int_0^1 x^{k+l-\alpha} dx \\ &= (2j+1) \sum_{l=0}^{j} (-1)^{j+l} \frac{(j+l)!}{(j-l)!\,(l!)^2\,(k+l-\alpha+1)}. \end{aligned} \tag{4.8}$$

Employing Eqs. (4.6) – (4.8) *we get*

$$\begin{aligned} D^{\alpha} G_i(x) &\simeq \sum_{k=\lceil\alpha\rceil}^{i} \sum_{j=0}^{m} \frac{(-1)^{i+k} (i+k)!}{(i-k)!\,(k!)\,\Gamma(k-\alpha+1)} b_{k,j} G_j(x) \\ &= \sum_{j=0}^{m} \left(\sum_{k=\lceil\alpha\rceil}^{i} \delta_{i,j,k} \right) G_j(x), \;\; i = \lceil\alpha\rceil, \ldots, m, \end{aligned} \tag{4.9}$$

where $\delta_{i,j,k}$ *is given in Eq.* (4.5)*. Rewrite Eq.* (4.9) *as a vector form we have*

$$D^{\alpha}G_i\left(x\right)\simeq\left[\sum_{k=\lceil\alpha\rceil}^{i}\delta_{i,0,k},\sum_{k=\lceil\alpha\rceil}^{i}\delta_{i,1,k},\ldots,\sum_{k=\lceil\alpha\rceil}^{i}\delta_{i,m,k}\right]\Phi\left(x\right),\ i=\lceil\alpha\rceil,\ldots,m. \tag{4.10}$$

Also according to Lemma 4.1*, we can write*

$$D^{\alpha}G_i\left(x\right)=\left[0,0,\ldots,0\right]\Phi\left(x\right),\ i=0,1,\ldots,\lceil\alpha\rceil-1. \tag{4.11}$$

A combination of Eqs.(4.10) *and* (4.11) *leads to the desired result.*

Remark 4.1 *If* $\alpha=n\in\mathbb{N}$*, then theorem* 4.1 *gives the same result as Eq.* (4.2)*.*

5 Applications of the operational matrix of fractional derivative

In this section, we apply the operational matrix of fractional derivative relative to shifted Legendre polynomial and the spectral Tau method in order to reduce the problem to a system of fractional ordinary differential equation.

5.1 Linear fractional diffusion differential equation with integer order in space

Consider the Linear diffusion fractional differential equation in time with non homogeneous Dirichlet boundary conditions

$$\begin{cases}\dfrac{\partial^{\alpha}u\left(x,t\right)}{\partial t^{\alpha}}=a\dfrac{\partial^{2}u\left(x,t\right)}{\partial x^{2}}+f\left(x,t\right)\\ u\left(0,t\right)=p\left(t\right)\\ u\left(1,t\right)=q\left(t\right)\\ u\left(x,0\right)=g\left(x\right),\end{cases} \tag{5.1}$$

where $0<\alpha\leq 1,\ 0\leq t\leq T,\ 0\leq x\leq 1$; p *and* q are given functions. We approximate $\dfrac{\partial^{2}u\left(x,t\right)}{\partial x^{2}}$, $f\left(x,t\right)$ *and* $g\left(x\right)$ as follows using operational matrix $D^{(2)}$

$$\frac{\partial^{2}u\left(x,t\right)}{\partial x^{2}}\simeq C^{T}\left(t\right)D^{(2)}\Phi\left(x\right), \tag{5.2}$$

$$f\left(x,t\right)\simeq F^{T}\left(t\right)\Phi\left(x\right),\quad F_i\left(t\right)=\left(2i+1\right)\int_0^1 f\left(x,t\right)G_i\left(x\right)dx,\ i=1,2,3,\ldots. \tag{5.3}$$

$$g\left(x\right)\simeq g^{T}\Phi\left(x\right),\quad g_i=\left(2i+1\right)\int_0^1 g\left(x\right)G_i\left(x\right)dx. \tag{5.4}$$

Then Eq. (5.1) becomes:

$$\begin{cases}D_t^{\alpha}C^{T}\left(t\right)\Phi\left(x\right)=aC^{T}\left(t\right)D^{(2)}\Phi\left(x\right)+F^{T}\left(t\right)\Phi\left(x\right)\\ C^{T}\left(t\right)\Phi\left(0\right)=p\left(t\right)\\ C^{T}\left(t\right)\Phi\left(1\right)=q\left(t\right)\\ C^{T}\left(0\right)\Phi\left(x\right)=g^{T}\Phi\left(x\right),\end{cases} \tag{5.5}$$

$$\text{where}\quad c_i\left(0\right)=g_i,\quad i=0,\ldots,N,\quad F\left(t\right)=\begin{bmatrix}F_1\left(t\right)\\ \vdots\\ F_N\left(t\right)\end{bmatrix},\quad C\left(0\right)=\begin{bmatrix}c_0\left(0\right)\\ \vdots\\ c_N\left(0\right)\end{bmatrix}.$$

From (5.5), let $R_m(x)$ defined as:

$$R_m(x) = \left[D_t^{\alpha} C^T(t) - aC^T(t) D^{(2)} - F(t)\right] \Phi(x) = 0, \ x \in [0,1], \tag{5.6}$$

application the spectral Tau method [10, 11], we have:

$$\int_0^1 R_m(x) G_j(x) \, dx = 0, \ j = 0, 1, \ldots, N-2. \tag{5.7}$$

By the properties of (3.3), we then have

$$\begin{cases} D_t^{\alpha} c_i(t) = ac_i(t) + F_i(t), \ i = 0, \ldots, N-2 \\ \sum_{i=0}^{N} (-1)^i c_i(t) = p(t) \\ \sum_{i=0}^{N} c_i(t) = q(t) \\ c(0) = g_i, \ i = 0, 1, \ldots, N. \end{cases} \tag{5.8}$$

The problem (5.8) can be written in the vectorial form as:

$$\begin{cases} D_t^{\alpha} C(t) = \wedge C(t) + F(t) + V(t) \\ C(0) = G \\ \sum_{i=0}^{N} (-1)^i c_i(t) = p(t) \\ \sum_{i=0}^{N} c_i(t) = q(t) \end{cases} \tag{5.9}$$

$$\text{where} \quad \wedge = \begin{bmatrix} a & 0 & \cdots & 0 & 0 \\ 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad V(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ p(t) \\ q(t) \end{bmatrix}.$$

The equation (5.9) can be as well of the form:

$$\begin{cases} D_t^{\alpha} C(t) = Q(t, C(t)) \\ C(0) = g, \end{cases} \tag{5.10}$$

where $Q(t, C(t)) = \wedge C(t) + F(t) + V(t)$, with $Q_i(t, C(t)) = DC_i(t) + F_i(t) + V_i(t)$, $i = 0 \ldots N$.

5.2 Linear fractional diffusion differential equation with fractional order in space

Consider the linear fractional diffusion equation in time and in space with non homogeneous Dirichlet boundary conditions

$$\begin{cases} \dfrac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = a \dfrac{\partial^{\sigma} u(x,t)}{\partial x^{\sigma}} + k(x,t) \\ u(0,t) = r(t) \\ u(1,t) = s(t) \\ u(x,0) = h(x), \end{cases} \tag{5.11}$$

where $0 < \alpha \leq 1,\ 1 < \sigma \leq 2,\ 0 \leq t \leq T,\ 0 \leq x \leq 1;$ r and s are two given functions. We approximate $h(x)$ *and* $k(x,t)$ as follows:

$$k(x,t) \simeq K^T(t)\,\Phi(x), \quad k_i(t) = (2i+1)\int_0^1 k(x)\,G_i(x)\,dx. \tag{5.12}$$

$$h(x) \simeq H^T\Phi(x), \quad H_i = (2i+1)\int_0^1 h(x)\,G_i(x)\,dx, \tag{5.13}$$

The Eq. (5.11) becomes:

$$\begin{cases} D_t^\alpha C^T(t)\,\Phi(x) = aC^T(t)\,D^{(\sigma)}\Phi(x) + K^T(t)\,\Phi(x) \\ C^T(t)\,\Phi(0) = r(t) \\ C^T(t)\,\Phi(1) = s(t) \\ C^T(0)\,\Phi(x) = H^T\Phi(x) \end{cases} \tag{5.14}$$

Where $C_i(0) = H_i,\ i = 0,\dots,N$. Define the residual function $R_m(x)$ as:

$$R_m(x) = \left[D_t^\alpha C^T(t) - aC^T(t)\,D^{(\sigma)} - K(t)\right]\Phi(x) = 0,\ x \in [0,1]. \tag{5.15}$$

The application of the Galerkin-Tau method give

$$\int_0^1 R_m(x)\,G_j(x)\,dx = 0, \quad j = 0,1,\dots,N-2. \tag{5.16}$$

6 Numerical experiment

We present numerical test using shifted Legendre operational method followed by multi-step method described in the previous section.

Example 6.1 *Consider the following linear one-dimensional fractional diffusion equation in time and integer in space:*

$$\begin{cases} \dfrac{\partial^{\frac{1}{2}} u(x,t)}{\partial t^{\frac{1}{2}}} - \dfrac{\partial^2 u(x,t)}{\partial x^2} = \dfrac{2t^{\frac{3}{2}}\sin(2\pi x)}{\Gamma\left(\frac{3}{2}\right)} + 4\pi^2\sin(2\pi x) \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 0,\ (x,t) \in [0,1]\times[0,1] \end{cases} \tag{6.1}$$

The exact solution of the problem (6.1) *is* $u(x,t) = 2t^2\sin(2\pi x)$. *By applying the technique described in section* 5.1 *with* $m = 3$, *we approximate solution as*

$$u_{app}(x,t) = c_0G_0(x) + c_1G_1(x) + c_2G_2(x) + c_3G_3(x) = C^T\Phi(x),$$

where c_0, c_1, c_2, c_3 *are found from*

$$\begin{cases} D_t^{\frac{1}{2}} c_0(t) = 0 \\ D_t^{\frac{1}{2}} c_1(t) = -\dfrac{12}{\pi}\left(\dfrac{t^{\frac{3}{2}}}{\sqrt{\pi}} + \pi^2 t^2\right) \\ c_0(0) = 0 \\ c_1(0) = 0 \end{cases}$$

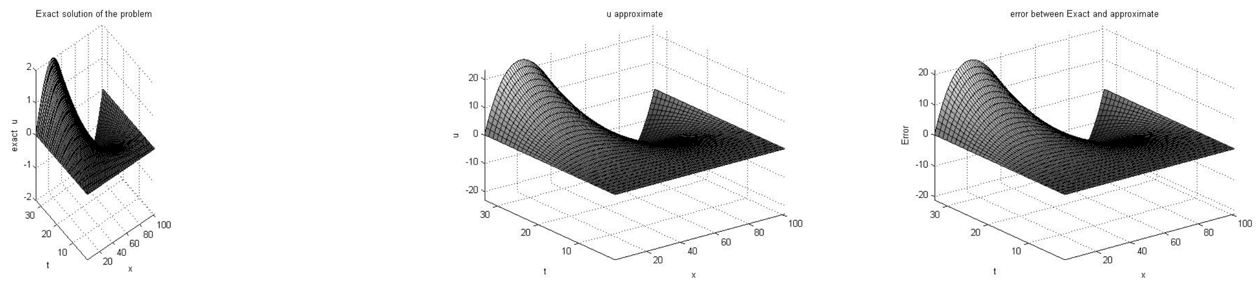
and the boundary condition

$$\begin{cases} c_0(t) - c_1(t) + c_2(t) - c_3(t) = 0 \\ c_0(t) + c_1(t) + c_2(t) + c_3(t) = 0, \end{cases}$$

using the operational matrix

$$D^{(2)} = 2\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 0 & 30 & 0 & 0 \end{bmatrix}, \quad F(t) = \begin{bmatrix} 0 \\ -\dfrac{12}{\pi}\left(\dfrac{t^{\frac{3}{2}}}{\sqrt{\pi}} + \pi^2 t^2\right) \\ 0 \\ 28\left(\dfrac{t^{\frac{3}{2}}}{\sqrt{\pi}} + \pi^2 t^2\right)\left(\dfrac{19}{\pi} - \dfrac{15}{\pi^3}\right) \end{bmatrix}.$$

Setting $N = 101$ *in space and* $\Delta t = \dfrac{1}{2^5}$ *for step time, Figure 1 displays the profil of exact and approximate solution and show the profil of error* $E = u - u_{app}$.



(a) Profil of the exact solution. (b) Profil of the approximate solution. (c) Profil of the error between exact and approximate solution.

Figure 1

The results show an excellent agreement between the exact and approximate solution, wich confirm the accuracy of the method.

Example 6.2 *Consider the following linear one-dimensional fractional diffusion equation in time and in space:*

$$\begin{cases} \dfrac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + \dfrac{\partial^{\sigma} u(x,t)}{\partial x^{\sigma}} = f(x,t), \ 0 \le x, t \le 1, \ 0 < \alpha \le 1, \ 1 < \sigma \le 2 \\ u(0,t) = t^2 + 1 \\ u(1,t) = 2\left(t^2+1\right) \\ u(x,0) = x^2 + 1 \end{cases}, \tag{6.2}$$

where $f(x,t) = \dfrac{2\Gamma(3)\, x^{2-\alpha}\left(t^2+1\right)}{\Gamma(3-\alpha)} + \dfrac{2\Gamma(3)\, t^{2-\sigma}\left(x^2+1\right)}{\Gamma(3-\sigma)}$ *and* $\alpha = \sigma = \dfrac{1}{2}$. *The exact solution of the problem* (6.2) *is* $u(x,t) = \left(x^2+1\right)\left(t^2+1\right)$. *By applying the technique described in section* 5.1 *with* $m = 2$, *we approximate solution as*

$$u_{app}(x,t) = c_0 G_0(x) + c_1 G_1(x) + c_2 G_2(x) = C^T \Phi(x).$$

where c_0, c_1, c_2 *are found by solving*

$$\begin{cases} D_t^{\frac{1}{2}} c_0(t) = \dfrac{8}{\sqrt{\pi}} \left(\dfrac{4}{15}\left(t^2+1\right) + \dfrac{1}{9} t^{\frac{3}{2}} \right) \\ c_0(0) = \dfrac{4}{3} \end{cases}$$

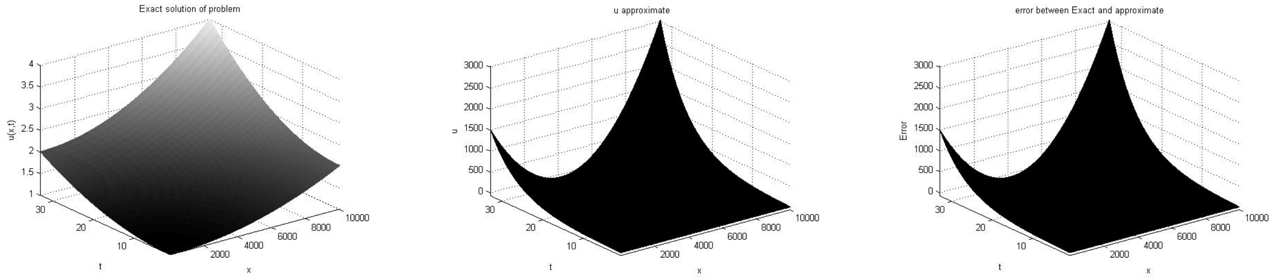
and the boundary conditon

$$\begin{cases} c_0(t) - c_1(t) + c_2(t) = t^2 + 1 \\ c_0(t) + c_1(t) + c_2(t) = 2\left(t^2+1\right), \end{cases}$$

using the operational matrix

$$D^{\left(\frac{1}{2}\right)} = \frac{8}{\sqrt{\pi}} \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{5} & \frac{-1}{21} \\ \frac{3}{5} & \frac{3}{7} & \frac{1}{3} \end{bmatrix}, \quad F(t) = \frac{8}{\sqrt{\pi}} \begin{bmatrix} \frac{4}{15}\left(t^2+1\right) + \frac{1}{9} t^{\frac{3}{2}} \\ \frac{12}{35}\left(t^2+1\right) + \frac{1}{3} t^{\frac{3}{2}} \\ \frac{4}{63}\left(t^2+1\right) + \frac{1}{9} t^{\frac{3}{2}} \end{bmatrix}.$$

As in the previous problem, by setting $N = 101$ *and* $\Delta t = \dfrac{1}{2^5}$ *the Figure 2 displays the profil exact and approximate solution and show the error* $E = u - u_{app}$.



(a) Profil of the exact solution.

(b) Profil of the approximate solution.

(c) Profil of the error between exact and approximate solution.

Figure 2

The results show an excellent agreement between the exact and approximate solution, wich confirm the accuracy of the method.

7 Conclusion

In this paper, we have presented a numerical approach for solving fractional linear diffusion equation one in time only and one in time and space, using the shifted Legendre operational matrix, the Tau method and LMMs methods. The operational matrix of fractional differentiation in Caputo sense $D^{(\sigma)}$ have been used via the spectral Tau method for reducing the space fractional diffusion equation into fractional ordinary differential equation (FODE) in time that can be solved by fractional linear multi-step method (FLMM). Two numerical examples reveal that the proposed method is very accurate and efficace for solving this problem when compared with exact solution.

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