On a Corrective Decile-based Confidence Interval Estimator of Mean for Normal and Skewed Distributions

ABSTRACT

This study addresses an issue with the decile t-confidence interval (dt-CI), which fails to achieve the desired coverage probabilities for large samples or skewed distributions (Mokhtar, Yusof & Sapiri, 2024). The article proposes a new corrective decile t-confidence interval (cdt-CI) that resolves these issues by modifying the decile standard deviation. Simulations using normal, chi-squared, log-normal, and gamma distributions show that the cdt-CI outperforms existing methods, particularly for skewed data, in terms of coverage probability and robustness. The real-life data sets analyzed in this study also support the conclusions of the simulation study.

***Keywords**: Confidence interval estimate, corrective decile t confidence interval, bootstrap approaches, coverage probability, simulation*

1. INTRODUCTION

The confidence interval (CI) estimator for any unknown population parameter is an essential statistical tool that provides a safeguard against estimation uncertainty. Many classical estimation methods assume that the data come from a normal distribution. For instance, the z-confidence interval (z-CI) and Student's t-confidence interval (t-CI) are derived under the assumption of normality. The t-CI, in particular, is applicable even when normality is not met, especially for large sample sizes. It is well known that the z-CI and t-CI are the most efficient methods when the data distribution is normal.

In real-life situations, however, the assumption of normality may not be met. As such, the Student's t-CI or other methods that rely on normality cannot be applied. Additionally, the sample size in many practical or experimental studies may be small enough to invalidate the use of the Student's t-CI. The limitations of the Student's t technique have been noted in the existing literature (Boos & Hughes-Oliver; 2000, David, 1998; Desharnais et al., 2015; Wilcox, 2021). However, prior studies also suggest that the coverage probability of the t-CI approaches the expected nominal confidence level even when the sample size is small or the underlying distribution is asymmetric, while the CI length remains relatively larger compared to other CI methods (e.g., Boos & Hughes-Oliver, 2000; Shi & Kibria, 2007; Wang, 2001; Zhou & Dinh, 2005).

Recently, Mokhtar, Yusof, and Sapiri (2024) compared the estimated coverage probabilities of the t-CI, percentile bootstrap CI (pb-CI), bootstrap-t CI (bt-CI), and the newly proposed decile t-CI (dt-CI) for simulated data from normal and skewed distributions. They suggest that the performance of these CIs varies significantly with respect to sample size and the type of skewness. Specifically, they note that the dt-CI fails to maintain the expected coverage probability for large sample sizes and skewed distributions.

In this paper, a new corrective decile t confidence interval (cdt-CI) is proposed by modifying the decile standard deviation utilized in the construction of the previous dt-CI. The newly proposed cdt-CI is expected to improve coverage probability and maintain robustness across sample sizes and skewness levels. The performance of cdt-CI has been evaluated and compared with the dt-CI and other underlying methods through real-life examples and simulations from normal and skewed distributions with varying degrees of skewness.

The organization of the remaining paper is as follows: Section 2 provides a literature review, focusing on CI estimators relevant to this study. Section 3 presents a new modified CI estimator. Section 4 discusses a simulation study, where samples are generated from normal and skewed distributions. Section 5 examines several real-life data sets. Finally, Section 6 offers concluding remarks.

2. Literature review

Let $X = (X_1 X_2, \cdots, X_n)$ be a sample from a population with an unknown population mean μ and unknown standard deviation σ. Let $\bar{X}$ be the sample mean. We wish to estimate the population mean μ via a confidence interval to ensure a desired level of certainty in the estimation. For the relevance of the study with the previous study, this section provides a brief description of the t-CI, pb-CI, bt-CI, and dt-CI methods.

2.1 The t-CI of mean μ

In real-life situations, where the population standard deviation σ is unknown, it is estimated by the sample standard deviation given by

$$s = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}}$$

Then, due to Student (1908), a classic 100(1−α)% t-CI for μ is given by

$$\left[\bar{X} - t_{\alpha/2} \times s/\sqrt{n}, \bar{X} + t_{\alpha/2} \times s/\sqrt{n}\right] \quad (1)$$

where $t_{\alpha/2}$ is the upper $(\alpha/2)$th of Student's t distribution with (n-1) degrees of freedom. Because t-CI approach is not robust against normality (Abu-Shawiesh & Saghir, 2019; Boos & Hughes-Oliver, 2000; Shi & Kibria, 2007; Bickel, 1965; Casella & Berger, 2024), it is often a common practice to employ bootstrap confidence interval ([Efron, 1979, 1987; Flowers-Cano et *al.,* 2018).

2.2 The pb-CI of mean μ

The percentile bootstrap CI (pb-CI) is the simplest choice due to the simplicity of the computational steps (Abu-Shawiesh, Sinsomboonthong & Kibria, 2022; Pek, Wong & Wong, 2017; Islam & Shapla, 2018).

Given a sample $X = (X_1 X_2, \cdots, X_n)$ of size n, the bth bootstrap sample is a sample of size n, drawn from X, with replacement, denoted by $X_b^* = (X_{b,1}^*, X_{b,2}^*, \cdots, X_{b,n}^*)$, $b = 1,2, \cdots, B$. The mean of the bth bootstrap sample is $\bar{X}_b^* = \frac{\sum_{j=1}^{n} X_{b,j}^*}{n}$, $b = 1,2, \cdots, B$. Given B bootstrap samples, a 100(1−α)% pb-CI for mean μ is the interval of the form

$$\left[(\bar{X}_b^*)_{\alpha/2}, (\bar{X}_b^*)_{1-\alpha/2}\right] \quad (2)$$

where $(\bar{X}_b^*)_{\alpha/2}$ and $(\bar{X}_b^*)_{1-\alpha/2}$ refer to (α/2)th and (1-α/2)th percentiles of B ordered bootstrap sample means $\bar{X}_b^*$. One can construct a pb-CI by following the algorithm below:

(i) Generate B bootstrap samples, compute and store their means, $\bar{X}_b^*$, $b = 1,2, \cdots, B$.
(ii) Find the (α/2)th and (1-α/2)th percentiles of B ordered bootstrap sample means $\bar{X}_b^*$.
(iii) A 100(1−α)% pb-CI for mean μ is the interval $\left[(\bar{X}_b^*)_{\alpha/2}, (\bar{X}_b^*)_{1-\alpha/2}\right]$. For example, if R=1000, a 95% CI μ is the interval: [25th largest value of $\bar{X}_b^*$, 975th largest value of $\bar{X}_b^*$]

The previous study suggests that pb-CI performs well in terms of estimated coverage probability and can also be inconsistent (Sinsomboonthong, Abu-Shawiesh & Kibria, 2020).

2.3 The bt-CI of mean μ

An alternative to pb-CI, one may use bootstrap t confidence interval (bt-CI) of μ (Berrar, 2019; Efron & Tibshirani, 1994, Zhao et al., 2021), which depends on the bootstrap studentized t-score. Using the notations of section 2.2, the bth bootstrap studentized t-score is calculated as follows

$$T_b^* = \frac{\bar{X}_b^* - \bar{X}}{SE(\bar{X}_b^*)} \tag{3}$$

where

T_b^* is the studentized score for bth bootstrap

$SE(\bar{X}_b^*) = \sqrt{\frac{\sum_{b=1}^{B}\left(\bar{X}_b^* - \overline{\overline{X^*}}\right)^2}{B-1}}$ is the standard error of bootstrap means $\bar{X}_b^*$

$\overline{\overline{X^*}} = \frac{\sum_{b=1}^{B}\bar{X}_b^*}{B}$ is the mean of B bootstrap means

Under above notations, a $100(1-\alpha)\%$ bt-CI is given by

$$\left[\bar{X} - \hat{t}_{1-\alpha/2} SE(\bar{X}_b^*), \bar{X} - \hat{t}_{\alpha/2} SE(\bar{X}_b^*)\right] \tag{4}$$

where $\hat{t}_{1-\alpha/2}$ is the $(1-\alpha/2)$th percentile of T_b^* over all B bootstrap samples.

The bt-CI provides better coverage probability compared to bp-CI, and needs to be compared with cdt-CI.

2.4 The dt-CI of mean μ

When data exhibits skewness, recent articles investigated CI of μ by implementing decile mean (DM) and decile standard deviation (Abu-Shawiesh, Sinsomboonthong & Kibria, 2022; Mokhtar, Yusof & Sapiri, 2024).

Let $D_j, j = 1,2,\cdots,9$ be 9 deciles of the sample $X_1, X_2, \cdots, X_n$. The decile mean (DM) and decile standard deviation (Siraj-Ud-Doulah, 2018) are defined by

$$DM = \frac{\sum_{i=1}^{9} D_i}{9} \text{ and } SD_{DM} = \sqrt{DM\ of\ (X_i - DM)^2} = \sqrt{\frac{\sum_{j=1}^{9} D_j(X - DM)^2}{9}} \tag{5}$$

where $D_j(X-DM)^2, j = 1,2,\cdots,9$ is the jth decile of $(X - DM)^2$ given the sample X. The $100(1-\alpha)\%$ decile t confidence interval (dt- CI) estimator of μ is defined by

$$\left[DM - t_{\alpha/2}\frac{SD_{DM}}{\sqrt{n}}, DM + t_{\alpha/2}\frac{SD_{DM}}{\sqrt{n}}\right] \tag{6}$$

where $t_{\alpha/2}$ is the upper $(\alpha/2)$th of Student's t distribution with (n-1) degrees of freedom.

The dt-CI fails to maintain the desired coverage probabilities of 0.95 for 95% CI and it gets worse when the sample size gets larger or bootstrap replication gets higher (Mokhtar, Yusof & Sapiri, 2024).

The reason for its worst performance may be due to the fact that $\frac{SD_{DM}}{\sqrt{n}}$ gets too small for large n which results in a notable narrower CI to capture the unknown mean μ. Indeed, the problem originates from the fact that SD_{DM} is computed on the basis of only 9 deciles, but the standard error of DM is calculated by dividing SD_{DM} by $\sqrt{n}$, which might have caused the problem, particularly for large n.

To address and resolve the noted problem of dt-CI estimator, we propose a corrective decile t confidence interval (cdt-CI) in section 3.

3. Methodology

In order to seek answer to what went wrong with the dt-CI in Mokhtar, Yusof & Sapiri (2024), we propose to construct a new corrective decile t confidence interval (cdt-CI) of estimator of μ, which uses the corrected decile standard deviation (CDSD) computed by using the equation

$$CDSD = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \mathrm{DM})^2}{n-1}} \tag{7}$$

Then, a $100(1-\alpha)\%$ corrective decile t-confidence interval (cdt-CI) of μ is constructed using equation

$$\left[\bar{X} - t_{\alpha/2} \times CDSD/\sqrt{n}, \bar{X} + t_{\alpha/2} \times CDSD/\sqrt{n}\right] \tag{8}$$

Compared to the dt-CI constructed in Section 2.4, the cdt-CI uses the correct form of the decile standard deviation, which aligns with the definition of sample standard deviation given by $s = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}}$ and the corresponding standard error calculated as $s/\sqrt{n}$.

In contrast, the dt-CI uses equation (5) where the denominator in the standard deviation computation is 9 (instead of 9-1 or n-1 for a sample of size n). While for large n, the effect of n and n-1 is negligible, the difference between 9 and n-1 remains significant for large n, especially for skewed distributions. Because of the specified difference in the standard deviation formula, the difference in the performance of the dt-CI and cdt-CI is expected to be significant in terms of the estimated coverage probability. We hypothesize that cdt-CI will provide better coverage probability compared to dt-CI, which requires a simulation study or real-life examples to justify.

4. Simulation study

When evaluation of various alternative methods is impossible or challenging theoretically, it is customary to use simulation to justify the usefulness of a given method compared to other alternative methods. As such, many researchers opt for simulation study to assess CIs (Abu-Shawiesh, Sinsomboonthong & Kibria, 2022; Mokhtar, Yusof & Sapiri, 2024, Islam & Shapla, 2018; Shi & Kibria, 2007).

In this simulation study, we generate data from four different models as defined below:

(M1) Generate samples from a positively skewed gamma distribution $G(\theta_1, \theta_2)$ with density function specified by

$$f(x) = \frac{x^{\theta_1 - 1}\exp(-\frac{x}{\theta_2})}{\theta_2^{\theta_1}\Gamma(\theta_1)}; x > 0, \theta_1, \theta_2 > 0 \qquad (9)$$

where θ_1 is a shape parameter, θ_2 is a scale parameter and the skewness is equal to $2/\sqrt{\theta_1}$. Since the mean of this distribution is $\mu = \theta_1\theta_2$, by choosing $\theta_2 = \frac{1}{\theta_1}$ and θ_1 values at 16, 4, 1, 1/4, we get mean fixed at $\mu = 1$ and skewness values at 0.5, 1, 2, 4, respectively. Our objective is to evaluate sensitivity of various CI methods with respect to varying skewness chosen arbitrarily to make the comparison notable.

(M2) Generate samples from a normal distribution with the density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0 \qquad (10)$$

This distribution is symmetric (i.e., skewness=0) with location parameter μ and scale parameter σ. For this simulation, we consider $\mu = 0$ and $\sigma = 1$, arbitrarily without the loss of generality.

(M3) A skewed Chi-squared distribution with degrees of freedom $df = k$ so as to have the mean $\mu = k$ and skewness equal to $\sqrt{8/k}$ with the density function given by

$$f(x, k) = \frac{x^{k/2-1}e^{-x/2}}{\Gamma(k/2)2^{k-1}}, x > 0 \qquad (11)$$

By choosing k=128, 32, 8 and 2, we allow skewness to be 0.25, 0.5, 1 and 2, while mean remains the same as the value of k.

(M4) Lognormal distribution with the density function given by

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}}e^{-\left(\frac{logx-\mu}{2\sigma^2}\right)}, x > 0, \sigma > 0, -\infty < \mu < \infty \qquad (12)$$

where μ and σ are log-location and log-scale parameters with $skewness = e^{\sigma^2+2}\sqrt{e^{\sigma^2-1}}$ and $mean = e^{\mu+\frac{\sigma^2}{2}}$.

By choosing $\mu = 1$ and $\sigma^2 = \frac{1}{10}, \frac{1}{4}, \frac{1}{2}$ and 1, we allow skewness values to be 1, 1.75, 2.94 and 6.18, mean to be 2.86, 3.08, 3.49 and 4.48.

In all simulations, the Monte Carlo size is chosen to be 5,000, and bootstrap replication B is chosen to be 1000, following the standard practice (Efron, 1979, 1987; Efron & Tibshirani,1994). Following (Mokhtar, Yusof & Sapiri, 2024) the bootstrap replications of B=250 and 500 were used in simulation, where no significant difference in results were noted. Therefore, the results of bootstrap replication of $B = 250$ or $B = 500$ have not been reported to avoid redundancy in reported results.

The results of simulation have been reported in Tables 2-5 for sample size varying between 10 and 400, at 10, 15, 20, 25, 30, 50, 100, 200 and 400 following the standard practice (Abu-Shawiesh, Sinsomboonthong & Kibria, 2022; Mokhtar, Yusof & Sapiri, 2024, Islam & Shapla, 2018; Shi & Kibria, 2007).

For simplicity of presentation, the skewness coefficients of four models of population distributions considered in the simulation study are made available in Table 1 below:

Table 1. The parameters of normal and chi-square distribution

Models	Parameters	Skewness
M1: $N(\mu, \sigma^2)$	$\mu = 1, \sigma^2 = 1$	0
M2: χ_k^2	k=128, 32, 8, 2	0.25, 0.5, 1, 2
M3:LN$(1, \sigma^2)$	$\sigma^2 = \frac{1}{10}, \frac{1}{4}, \frac{1}{2}, 1$	1,1.75, 2.94, 6.18
M4: Gamm(θ_1, θ_2)	$\theta_1 = 16, 4, 1, 1/4,\ \theta_2 = \frac{1}{\theta_1}$	0.5, 1, 2, 4

Evaluation criteria of simulations

The evaluation criteria for the goodness of any underlying confidence interval method are the estimated coverage probability and the average length of the corresponding confidence interval estimator (Abu-Shawiesh, Sinsomboonthong & Kibria, 2022; Mokhtar, Yusof & Sapiri, 2024, Moslim, Zubairi & Hassan, 2019; Islam & Shapla, 2018; Shi & Kibria, 2007; Waguespack, Krishnamoorthy & Lee, 2020).

Given that this study considers the Monte Carlo size of M=5000, the estimated coverage probability (cp) and the average length of any specified CI are estimated using 5000 samples as follows:

$$\text{Est. coverage probability of any CI method} = \frac{\#\text{ of CIs capturing } \mu}{5000}$$

$$\text{Est. average length of any CI method} = \frac{\sum_{i=1}^{5000}(UCL_i - LCL_i)}{5000}$$

In other words, the estimated coverage probability is the proportion of 5000 CIs capturing the mean μ by any underlying CI estimator. The average length is the average of lengths of all 5000 CIs, where length of an underlying CI method is the difference of lower confidence limit (LCL) from the upper confidence limit (UCL).

To estimate the coverage probability and the estimated CI length while doing the simulation, fix sample size and skewness value by selecting appropriate model parameters specified in Table 1 and then follow the steps below:

(1a) Take M=5000 samples from the desired model (e.g., any of models M1–M4), each of size n. For each of these samples, compute t-CI, dt-CI, and cdt-CI, and store them.

(1b) For bootstrap CIs (pb-CI or bt-CI), for each sample in (1a), generate B=1000 bootstrap samples and compute the pb-CI and bt-CI. This results in 5000 pb-CIs and bt-CIs, which are then stored.

(2a) The proportions of 5000 t-CIs, dt-CIs, and cdt-CIs in (1a) capturing μ are the estimated coverage probabilities for the corresponding CIs.

(2b) The proportions of 5000 pb-CIs and bt-CIs in (1b) capturing μ are the estimated coverage probabilities for the corresponding bootstrap CIs.

(3) For each of the t-CIs, dt-CIs, and cdt-CIs in (1a) or pb-CIs and bt-CIs in (1b), compute the CI length. The average of 5000 CI lengths is the estimated average CI length of the corresponding CI.

The estimated coverage probability of a 95% CI is expected to be close to 0.95. Therefore, the performance of an underlying CI is satisfactory if its coverage probability is close to 0.95 or near ($1-\alpha$) over all M Monte Carlo samples.

As a computational tool, in all computation and simulation, the statistical software R (R Core Team, 2024) has been utilized in this article.

Simulation results

The results of all simulations for varying sample sizes and skewness values are reported in Tables 2-5.

Let us first examine the results of simulations from skewed distributions, as reported in Tables 2-4. The results from two columns, labeled "dt-CI" and "cdt-CI," are presented in bold. When comparing the results from these two columns, it is clearly evident that the cdt-CI estimator outperforms the dt-CI estimator in terms of estimated coverage probability. For example, for the gamma distribution with skewness = 0.5, when the sample size n increases from 10 to 400, the estimated coverage probability increases from 0.83 to 0.95 for cdt-CI, while it decreases from 0.90 to 0.83 for dt-CI. When skewness = 4, and n increases from 10 to 400, the estimated coverage probability increases from 0.81 to 0.94 for cdt-CI, while it decreases from 0.59 to 0.00 for dt-CI. The performance trends for the cdt-CI and dt-CI estimators in the log-normal (Table 3) and chi-squared distributions (Table 4) are similar to the trend observed for the gamma distribution. The coverage performance of other CI estimators improves as the sample size increases.

The bt-CI estimator retains robustness against increasing skewness, as does the cdt-CI estimator. However, bt-CI maintains higher coverage compared to cdt-CI, while cdt-CI has a smaller width compared to bt-CI. For example, for the log-normal distribution with skewness = 6.18 and n increases from 10 to 400 (Table 3), the estimated coverage probability increases from 0.91 to 0.95 for bt-CI, while it increases from 0.84 to 0.94 for cdt-CI. Conversely, for the same simulation conditions, the estimated width decreases from 13.37 to 1.2 for bt-CI, while it decreases from 6.78 to 1.15 for cdt-CI. Therefore, length-wise, cdt-CI outperforms bt-CI, and dt-CI performs best with the lowest width, in all simulation cases. In performance, bt-CI and cdt-CI are followed by the t-CI or pb-CI in terms of coverage probability. The dt-CI performs poorly compared to all the other methods.

For simulation from symmetric normal distribution reported in Table 5, the bt-CI remains to be the best performer, or at least as good as the t-CI, followed by the dt-CI (particularly for small samples), pb-CI (which performs better for large samples), and cdt-CI. For example, for increasing n from 10 to 400, the coverage probability ranges from 0.95 to 0.96 for bt-CI mostly with the desired expectation of 0.95, from 0.94 to 0.96 for t-CI with no definite trend, from 0.91 to 0.96 mostly decreasingly for dt-CI, from 0.89 to 0.95 increasingly for pb-CI, and from 0.89 to 0.95 increasingly for cdt-CI. However, as compared all methods using the width of CI estimators, cdt-CI appears to the best with width ranging in the interval (0.20, 1.15) decreasingly with the increase of n, followed by pb-CI in (0.20, 1.15), dt-CI in (0.20,1.40), t-CI in (0.20, 1.40), and bt-CI in (0.20, 1.51). Overall, for normal distribution, the dt-CI performs reasonably well, better than the cdt-CI, in terms of coverage probability, but in terms of width cdt-CI is better performer than dt-CI. This simulation is in contradiction with Table 2, column 6, of Mokhtar, Yusof & Sapiri, (2024) where dt-CI appears to be very inconsistent and unexpectable even for normal distribution.

Table 2. Simulated coverage probability and length of CI for gamma distribution varying sample size and skewness

Skewness		Est. coverage probability of CI methods					Est. average length of CI methods				
	n	t-CI	pb-CI	bt-CI	dt-CI	cdt-CI	t-CI	pb-CI	bt-CI	dt-CI	cdt-CI
0.5	10	0.95	0.91	0.95	**0.90**	**0.95**	0.35	0.28	0.38	0.29	0.35
	15	0.95	0.92	0.95	**0.89**	**0.95**	0.27	0.24	0.28	0.23	0.27
	20	0.95	0.93	0.95	**0.89**	**0.95**	0.23	0.21	0.24	0.19	0.23
	25	0.94	0.93	0.94	**0.88**	**0.94**	0.20	0.19	0.21	0.17	0.20
	30	0.94	0.93	0.94	**0.89**	**0.95**	**0.18**	**0.17**	**0.19**	**0.16**	**0.18**
	50	0.95	0.94	0.95	**0.89**	**0.95**	**0.14**	**0.14**	**0.14**	**0.12**	**0.14**
	100	0.95	0.95	0.95	**0.88**	**0.95**	**0.10**	**0.10**	**0.10**	**0.09**	**0.10**
	200	0.95	0.95	0.95	**0.86**	**0.95**	**0.07**	**0.07**	**0.07**	**0.06**	**0.07**
	400	0.94	0.94	0.94	**0.83**	**0.95**	**0.05**	**0.05**	**0.05**	**0.05**	**0.05**
1	10	0.94	0.90	0.96	**0.87**	**0.94**	0.69	0.57	0.79	0.56	0.69
	15	0.93	0.90	0.95	**0.86**	**0.93**	0.54	0.48	0.58	0.44	0.54
	20	0.94	0.92	0.95	**0.86**	**0.94**	0.46	0.42	0.48	0.37	0.46
	25	0.95	0.93	0.95	**0.85**	**0.95**	0.41	0.38	0.42	0.33	0.41
	30	0.94	0.93	0.95	**0.86**	**0.95**	0.37	0.35	0.38	0.30	0.37
	50	0.95	0.94	0.95	**0.84**	**0.95**	0.28	0.27	0.29	0.23	0.28
	100	0.94	0.94	0.94	**0.80**	**0.94**	0.20	0.19	0.20	0.16	0.20
	200	0.95	0.95	0.96	**0.73**	**0.95**	0.14	0.14	0.14	0.11	0.14
	400	0.95	0.94	0.95	**0.63**	**0.95**	0.10	0.10	0.10	0.10	0.08
2	10	0.89	0.85	0.93	**0.77**	**0.89**	1.32	1.07	1.91	0.96	1.34
	15	0.92	0.89	0.95	**0.76**	**0.92**	1.05	0.93	1.33	0.76	1.06
	20	0.93	0.92	0.95	**0.75**	**0.93**	0.90	0.82	1.07	0.65	0.91
	25	0.92	0.91	0.95	**0.72**	**0.92**	0.80	0.74	0.91	0.57	0.80
	30	0.93	0.92	0.95	**0.73**	**0.93**	0.73	0.69	0.82	0.52	0.74
	50	0.94	0.93	0.95	**0.68**	**0.94**	0.56	0.54	0.60	0.40	0.57
	100	0.94	0.94	0.95	**0.56**	**0.95**	0.39	0.39	0.41	0.28	0.40
	200	0.96	0.95	0.96	**0.37**	**0.96**	0.28	0.28	0.28	0.20	0.28
	400	0.95	0.95	0.95	**0.17**	**0.95**	0.20	0.20	0.20	0.20	0.14
4	10	0.80	0.78	0.93	**0.59**	**0.81**	2.33	1.83	12.00	1.38	2.38
	15	0.83	0.82	0.94	**0.54**	**0.84**	1.94	1.66	5.08	1.06	1.98
	20	0.86	0.85	0.95	**0.49**	**0.86**	1.68	1.50	3.14	0.88	1.71
	25	0.87	0.87	0.95	**0.48**	**0.88**	1.51	1.38	2.54	0.78	1.53
	30	0.87	0.87	0.94	**0.45**	**0.88**	1.39	1.30	2.12	0.71	1.42
	50	0.90	0.91	0.95	**0.35**	**0.91**	1.07	1.03	1.39	0.53	1.09
	100	0.92	0.92	0.94	**0.19**	**0.92**	0.77	0.76	0.88	0.37	0.79
	200	0.93	0.94	0.95	**0.05**	**0.94**	0.55	0.55	0.59	0.27	0.56
	400	0.94	0.94	0.95	**0.00**	**0.94**	0.39	0.39	0.40	0.40	0.19

Table 3. Simulated coverage probability and length of CI for log-normal distribution with varying sample size and skewness

Skewness		Est. coverage probability of CI methods					Est. average length of CI methods				
	n	t-CI	pb-CI	bt-CI	dt-CI	cdt-CI	t-CI	pb-CI	bt-CI	dt-CI	cdt-CI
1	10	0.93	0.89	0.94	**0.87**	**0.93**	1.27	1.04	1.46	1.01	1.28
	15	0.95	0.92	0.95	**0.87**	**0.95**	0.99	0.88	1.07	0.80	1.00
	20	0.94	0.92	0.94	**0.86**	**0.94**	0.85	0.78	0.90	0.69	0.85
	25	0.94	0.92	0.95	**0.86**	**0.94**	0.75	0.70	0.79	0.61	0.76
	30	0.94	0.92	0.95	**0.86**	**0.94**	0.68	0.64	0.70	0.55	0.68
	50	0.95	0.94	0.95	**0.85**	**0.95**	0.52	0.50	0.53	0.43	0.52
	100	0.95	0.94	0.95	**0.81**	**0.95**	0.37	0.36	0.37	0.30	0.37
	200	0.95	0.95	0.95	**0.76**	**0.95**	0.26	0.26	0.26	0.21	0.26
	400	0.96	0.96	0.96	**0.64**	**0.96**	0.18	0.18	0.18	0.15	0.18
1.75	10	0.93	0.88	0.95	**0.84**	**0.93**	2.20	1.80	2.78	1.67	2.22
	15	0.92	0.89	0.94	**0.82**	**0.92**	1.71	1.51	1.98	1.30	1.73
	20	0.93	0.92	0.95	**0.81**	**0.94**	1.48	1.35	1.66	1.11	1.49
	25	0.93	0.92	0.94	**0.80**	**0.93**	1.32	1.23	1.45	0.99	1.32
	30	0.93	0.92	0.95	**0.79**	**0.93**	1.18	1.12	1.28	0.89	1.19
	50	0.94	0.93	0.94	**0.75**	**0.94**	0.92	0.88	0.96	0.68	0.92
	100	0.94	0.94	0.95	**0.68**	**0.94**	0.64	0.64	0.66	0.48	0.65
	200	0.95	0.95	0.95	**0.54**	**0.95**	0.45	0.45	0.46	0.34	0.46
	400	0.94	0.94	0.95	**0.34**	**0.95**	0.32	0.32	0.32	0.24	0.32
2.94	10	0.89	0.86	0.93	**0.79**	**0.90**	3.60	2.91	5.30	2.53	3.64
	15	0.90	0.88	0.93	**0.75**	**0.90**	2.85	2.49	3.76	1.95	2.88
	20	0.90	0.89	0.93	**0.72**	**0.90**	2.45	2.22	3.03	1.64	2.47
	25	0.91	0.90	0.93	**0.73**	**0.91**	2.19	2.03	2.61	1.47	2.21
	30	0.92	0.91	0.94	**0.70**	**0.92**	1.99	1.87	2.32	1.32	2.01
	50	0.93	0.92	0.95	**0.62**	**0.93**	1.54	1.49	1.71	1.01	1.56
	100	0.94	0.94	0.95	**0.49**	**0.94**	1.09	1.08	1.16	0.71	1.10
	200	0.95	0.95	0.95	**0.29**	**0.95**	0.77	0.77	0.80	0.50	0.78
	400	0.95	0.96	0.96	**0.09**	**0.95**	0.55	0.55	0.56	0.35	0.55
6.18	10	0.84	0.80	0.91	**0.68**	**0.84**	6.66	5.31	13.37	4.25	6.78
	15	0.86	0.84	0.92	**0.64**	**0.86**	5.42	4.68	9.23	3.20	5.50
	20	0.87	0.86	0.92	**0.58**	**0.87**	4.71	4.22	7.44	2.64	4.78
	25	0.88	0.87	0.93	**0.56**	**0.88**	4.30	3.94	6.47	2.34	4.37
	30	0.89	0.88	0.93	**0.54**	**0.89**	3.86	3.60	5.40	2.11	3.92
	50	0.89	0.89	0.93	**0.44**	**0.90**	3.04	2.91	3.85	1.61	3.08
	100	0.92	0.93	0.95	**0.26**	**0.93**	2.21	2.17	2.58	1.12	2.24
	200	0.93	0.93	0.94	**0.09**	**0.93**	1.59	1.57	1.74	0.79	1.60
	400	0.94	0.94	0.94	**0.01**	**0.94**	1.13	1.13	1.2	0.56	1.15

Table 4. Simulated coverage probability and length of CI for chi-squared distribution with varying sample size and skewness

Skewness		Est. coverage probability of CI methods					Est. average length of CI methods				
	n	t-CI	pb-CI	bt-CI	dt-CI	cdt-CI	t-CI	pb-CI	bt-CI	dt-CI	cdt-CI
0.25	10	0.95	0.90	0.96	**0.90**	**0.95**	22.33	18.30	24.20	18.63	22.38
	15	0.95	0.93	0.96	**0.91**	**0.95**	17.42	15.39	18.01	14.78	17.45
	20	0.96	0.93	0.95	**0.90**	**0.96**	14.76	13.48	15.08	12.59	14.77
	25	0.95	0.93	0.95	**0.90**	**0.95**	13.07	12.19	13.28	11.25	13.08
	30	0.95	0.93	0.95	**0.90**	**0.95**	11.85	11.20	12.01	10.24	11.87
	50	0.94	0.94	0.94	**0.89**	**0.94**	9.05	8.77	9.11	7.89	9.05
	100	0.94	0.94	0.95	**0.89**	**0.94**	6.32	6.24	6.36	5.55	6.33
	200	0.94	0.94	0.94	**0.90**	**0.94**	4.46	4.44	4.48	3.93	4.46
	400	0.95	0.95	0.95	**0.89**	**0.95**	3.14	3.14	3.16	2.77	3.15
0.50	10	0.94	0.90	0.95	**0.89**	**0.94**	11.07	9.08	12.17	9.15	11.11
	15	0.94	0.92	0.94	**0.89**	**0.94**	8.73	7.70	9.13	7.32	8.75
	20	0.95	0.92	0.95	**0.89**	**0.95**	7.39	6.76	7.61	6.24	7.40
	25	0.94	0.92	0.94	**0.88**	**0.94**	6.53	6.09	6.68	5.55	6.54
	30	0.96	0.94	0.96	**0.90**	**0.96**	5.92	5.59	6.03	5.07	5.93
	50	0.95	0.95	0.96	**0.90**	**0.95**	4.53	4.39	4.58	3.89	4.53
	100	0.94	0.94	0.94	**0.88**	**0.94**	3.16	3.12	3.18	2.73	3.16
	200	0.95	0.95	0.95	**0.87**	**0.95**	2.23	2.22	2.24	1.94	2.23
	400	0.95	0.95	0.95	**0.84**	**0.95**	1.57	1.57	1.58	1.57	1.37
1	10	0.93	0.89	0.94	**0.86**	**0.93**	5.52	4.52	6.44	4.41	5.55
	15	0.94	0.92	0.95	**0.86**	**0.95**	4.31	3.80	4.66	3.49	4.32
	20	0.94	0.92	0.94	**0.85**	**0.94**	3.66	3.35	3.88	2.97	3.68
	25	0.94	0.92	0.94	**0.85**	**0.94**	3.23	3.01	3.37	2.63	3.24
	30	0.94	0.93	0.95	**0.85**	**0.94**	2.94	2.78	3.05	2.40	2.95
	50	0.94	0.94	0.95	**0.84**	**0.94**	2.26	2.18	2.31	1.85	2.26
	100	0.96	0.95	0.95	**0.80**	**0.96**	1.58	1.56	1.60	1.30	1.58
	200	0.95	0.95	0.95	**0.74**	**0.95**	1.11	1.11	1.12	0.92	1.12
	400	0.94	0.94	0.94	**0.61**	**0.94**	0.79	0.79	0.79	0.79	0.65
2	10	0.91	0.88	0.95	**0.79**	**0.92**	2.66	2.16	3.85	1.93	2.69
	15	0.92	0.90	0.95	**0.75**	**0.92**	2.10	1.84	2.64	1.51	2.12
	20	0.93	0.91	0.95	**0.76**	**0.93**	1.79	1.63	2.11	1.29	1.81
	25	0.93	0.92	0.95	**0.73**	**0.93**	1.60	1.49	1.84	1.14	1.61
	30	0.93	0.92	0.95	**0.70**	**0.93**	1.45	1.36	1.63	1.02	1.46
	50	0.94	0.94	0.95	**0.67**	**0.94**	1.12	1.08	1.20	0.79	1.13
	100	0.94	0.94	0.95	**0.54**	**0.94**	0.79	0.77	0.82	0.56	0.79
	200	0.95	0.94	0.95	**0.39**	**0.95**	0.56	0.55	0.57	0.39	0.56
	400	0.94	0.94	0.94	**0.16**	**0.94**	0.39	0.39	0.40	0.40	0.28

Table 5. Simulated coverage probability and length of CI for normal distribution with varying sample size

	Est. coverage probability of CI methods					Est. average length of CI methods				
n	t-CI	pb-CI	bt-CI	dt-CI	cdt-CI	t-CI	pb-CI	bt-CI	dt-CI	cdt-CI
10	**0.94**	0.89	**0.95**	**0.94**	**0.89**	1.40	1.15	1.51	**1.40**	**1.15**
15	**0.94**	0.92	**0.95**	**0.94**	**0.89**	1.09	0.96	1.12	**1.09**	**0.93**
20	**0.96**	0.94	**0.95**	**0.96**	**0.91**	0.92	0.84	0.94	**0.92**	**0.79**
25	**0.95**	0.93	**0.95**	**0.95**	**0.91**	0.82	0.76	0.83	**0.82**	**0.71**
30	**0.95**	0.94	**0.95**	**0.95**	**0.91**	0.74	0.70	0.75	**0.74**	**0.64**
50	**0.96**	0.95	**0.96**	**0.96**	**0.92**	0.57	0.55	0.57	**0.57**	**0.50**
100	**0.95**	0.95	**0.95**	**0.95**	**0.92**	0.40	0.39	0.40	**0.40**	**0.35**
200	**0.95**	0.94	**0.95**	**0.94**	**0.91**	0.28	0.28	0.28	**0.28**	**0.25**
400	**0.95**	0.95	**0.95**	**0.91**	**0.95**	0.20	0.20	0.20	**0.20**	**0.20**

5. Examples and applications with real-life data

In this section, we construct a 95% CI for the mean μ using data from real-life situations, with one dataset following a normal distribution and others exhibiting either positive or negative skewed distributions. Since we cannot observe the coverage probability for real-life data, we report the CI and the length of the corresponding CI estimators instead.

Example 1

In this example, we utilize relative poverty (%) in Malaysia for selected years from 1970 to 2022 as has been reported in data.gov.my (2024a).

19.5, 19.7, 19.0, 20.0, 19.9, 17.4, 19.3, 19.2, 15.6, 15.9, 16.9, 16.2, 16.6

The data skewness value of -0.27, mean 18.1 and median 19.0 (mean<median) along with the histogram and boxplot in Figure 1, all suggest that the distribution of relative poverty is negatively skewed. The Shapiro-Wilk test of normality (W= 0.86128, p-value = 0.04005) suggests that at significance level α = 5%, the relative poverty population distribution fails to be normally distributed. Now, let us have a look at 95% CIs for various underlying methods reported in Table 6.

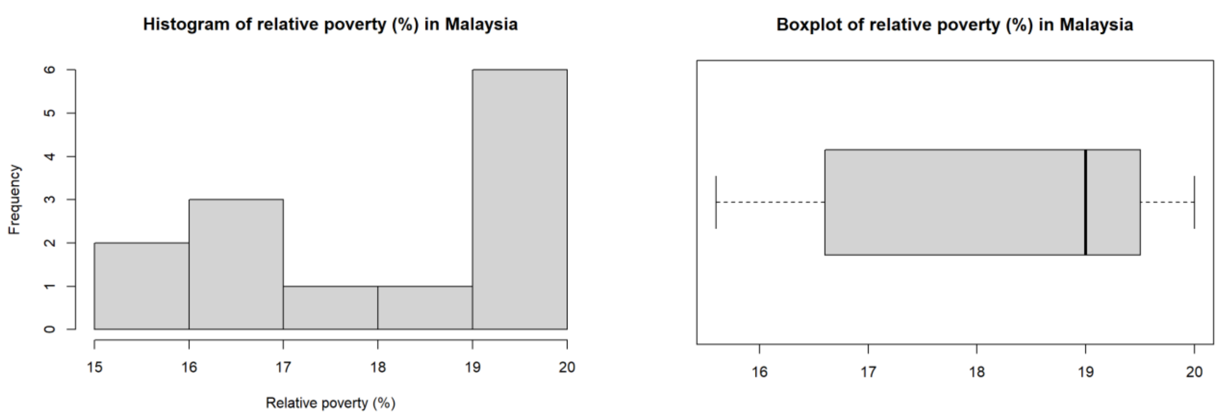

Figure 1. Histogram and boxplot of relative poverty in Malaysia

Table 6. 95% CIs and corresponding length for relative poverty data in Example 1

Methods	CI estimate	Length
t-CI	[17.08, 19.11]	2.03
pb-CI	[17.26, 18.88]	1.62
bt-CI	[17.08, 18.98]	1.90
dt-CI	[17.22, 19.10]	1.88
cdt-CI	[17.08, 19.11]	2.03

The results of Table 6 reveal that the pb-CI has the smallest length (length of 1.62), followed by dt-CI (length of 1.88), bt-CI (length of 1.90) and jointly t-CI and cdt-CI (both having length 2.03). Therefore, with length consideration, pb-CI or dt-CI are preferable to others. However, it may not be safe to make such conclusion on the basis of single sample given the noted performance of pb-CI or dt-CI through the simulations.

Example 2

In this example, we consider quarterly unemployment rate in Kedah state of Malaysia between 2017 and 2Q 2024 reported in data.gov.my (2024b):

2.5	3.1	3.1	2.8	3.7	4.0	3.8	3.3	2.8	2.1
3.0	3.0	3.0	3.2	4.1	3.6	3.8	3.0	2.6	1.9
2.7	2.5	3.0	3.1	4.4	3.9	3.4	3.3	2.3	1.7

The data skewness value of -0.0997, mean 3.09 and median 3.05 (mean appears to be close to the median) along with the histogram and boxplot in Figure 2, might suggest the population to be less skewed. The Shapiro-Wilk test of normality (W= 0.98394, p-value = 0.9178) suggests that at significance level $\alpha = 5\%$, the quarterly unemployment rate in Kedah state might have been normally distributed. As such, let focus our attention to 95% CIs for various underlying methods reported in Table 5.

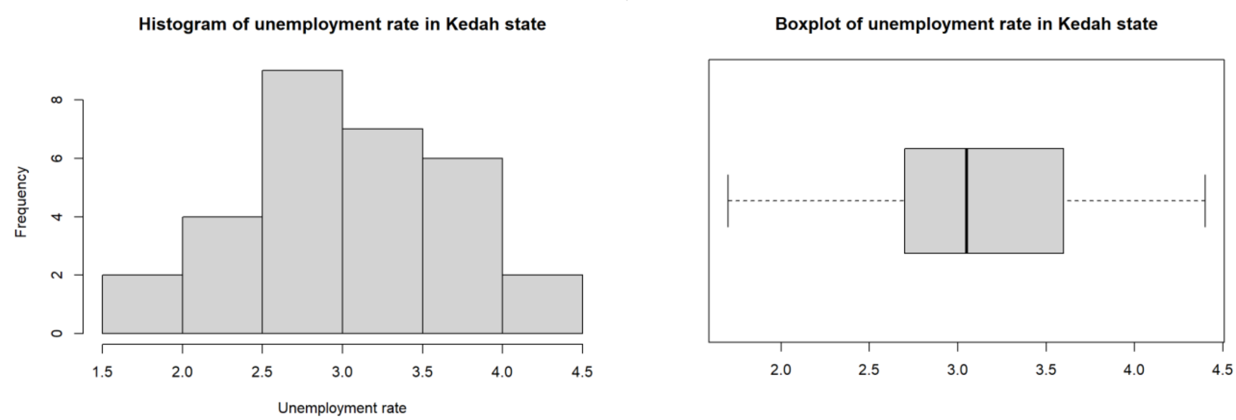

Figure 2. Histogram and boxplot of unemployment rate in Kedah state of Malaysia

Table 7. 95% CIs and length for various CIs for Kedah state unemployment rate data of Example 2

Methods	CI estimate	Length
t-CI	[2.85, 3.33]	0.48
pb-CI	[2.88, 3.33]	0.45
bt-CI	[2.84, 3.33]	0.49
dt-CI	[2.88, 3.30]	0.42
cdt-CI	[2.85, 3.33]	0.48

The results in Table 7 reveal that the dt-CI has the smallest length (0.42), followed by the pb-CI (0.45), with the t-CI and cdt-CI both having a length of 0.48, and the bt-CI having the largest length. Unlike Example 1, where the sample size is small, Example 2 uses a larger sample size (n=30), which leads to a switch in the performance of dt-CI and pb-CI.

Example 3

In this example, we consider the quarterly unemployment rates for all 16 states of Malaysia between 2017 and Q2 2024, with each state having 30 observations, resulting in a total sample size of n=16×30=480, as reported in data.gov.my (2024b).
The data has a skewness of 1.23, with the mean (3.6) being greater than the median (3.3). The histogram and boxplot in Figure 3 clearly suggest that the data distribution is positively skewed, with the boxplot indicating possible outliers on the right tail. The Shapiro-Wilk test of normality (W = 0.90738, p-value < 0.00001) provides strong evidence that the distribution of the quarterly unemployment rate is not normally distributed.

Given the large sample size (n=480), a t-test can be used to assess whether the population mean is indeed 3.6. The results of the t-test strongly suggest that the population distribution of unemployment could have a mean of 3.6, with a p-value of 0.9783. Estimates of the 95% confidence intervals for various methods are reported in Table 8.

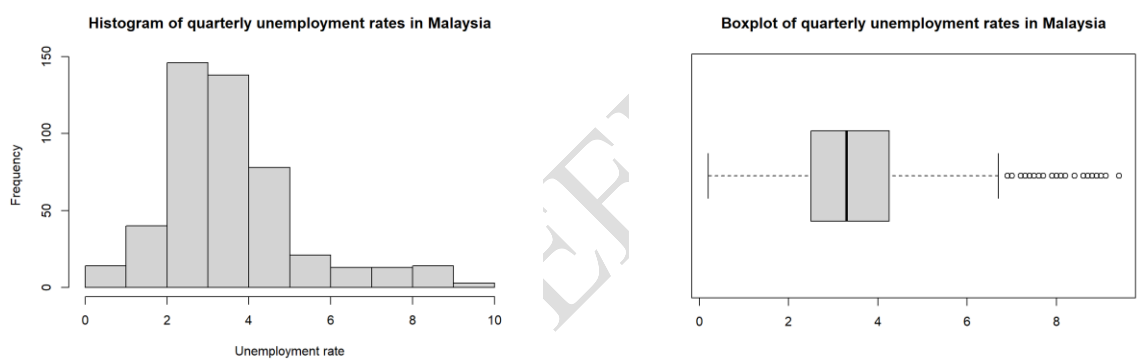

Figure 3. Histogram and boxplot of quarterly unemployment in Malaysia

Table 8. 95% CIs and length for quarterly unemployment rates in Malaysia for Example 3

Methods	CI estimate	Length
t-CI	[3.45, 3.75]	0.30
pb-CI	[3.46, 3.73]	0.27
bt-CI	[3.47, 3.74]	0.27
dt-CI	[3.31, 3.54]	0.23
cdt-CI	[3.45, 3.75]	0.30

As seen in other examples, the dt-CI has the smallest length (0.23), followed by the pb-CI and bt-CI, both with a length of 0.27. However, it is important to note that the dt-CI, [3.31,3.54], does not capture the hypothetical mean of 3.6, which aligns with the simulation results showing the poor performance of the dt-CI under conditions of large sample size and higher skewness. In contrast, the t-CI and the proposed cdt-CI both capture the hypothetical mean of 3.6, as do the other CI methods, except for the dt-CI. The consistent performance of the cdt-CI in terms of coverage probability is well supported, even though it comes at the cost of a relatively larger length.

Example 4

This example has been revisited from Example 2 (Mokhtar, Yusof & Sapiri, 2024), to reinvestigate why pb-CI reported in Table 8 differs to a greater extent form other underlying CI.

43.4, 24, 1.8, 0, 0.1, 170.1, 0.4, 150, 31.5, 5.2, 35.7, 27.3, 5, 64.3, 70, 94, 61.9, 9.1, 38.8 and 14.8.

The data has a skewness of 1.45, and the fact that the mean (42.4) is greater than the median (29.4), along with the histogram and boxplot in Figure 2, all suggest that the data is positively skewed. Upon revisiting this data, the 95% CIs for the various methods are presented in Table 9.

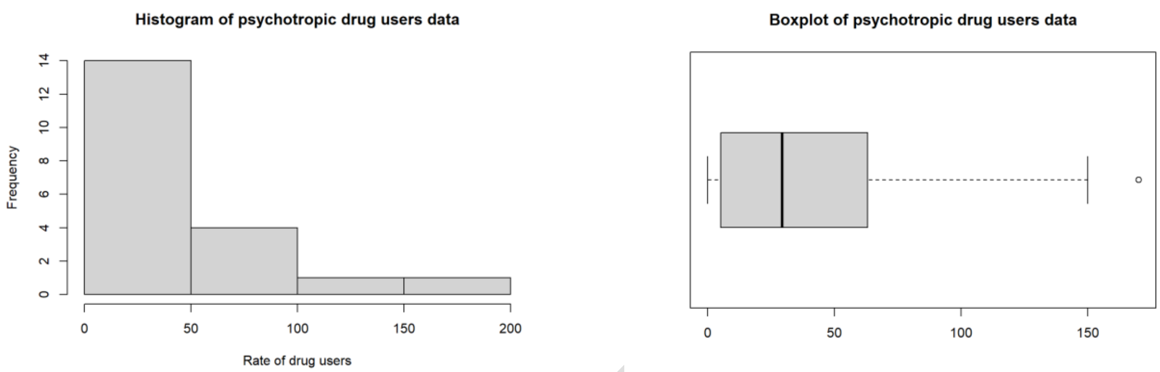

Figure 4. Boxplot and histogram of psychotropic drug user's data of Example 4

Table 9. 95% CIs and corresponding length for data in Example 4

Methods	CI estimate	Length
t-CI	[19.70, 65.04]	45.34
pb-CI	[22.59, 63.88]	41.29
bt-CI	[24.31, 81.09]	56.78
dt-CI	[19.34, 50.28]	30.94
cdt-CI	[19.41, 65.33]	45.92

From the results reported in Table 9, it is noted that the length of the dt-CI is the smallest (30.94), followed by the pb-CI (41.29), t-CI (45.34), cdt-CI (45.92), and bt-CI (56.78). These results are generally consistent with those in Table 8 (Mokhtar, Yusof & Sapiri, 2024), except for the pb-CI. In prior paper, the pb-CI was reported to be [0.05, 160.55], with a length of 160.5, which differed substantially from the other CIs reported in Table 8, possibly indicating an error.

6. Conclusions

This study proposes a new corrective decile-based confidence interval (CI) for the mean μ, by modifying the decile standard deviation formula previously proposed and studied by Mokhtar, Yusof, and Sapiri (2024). The newly proposed corrective decile t-CI (cdt-CI) outperforms the prior decile t-CI (dt-CI) in terms of estimated coverage probability, particularly for data with higher skewness. Through simulations with normal and skewed distributions, varying sample sizes, and different skewness levels, the study concludes that the dt-CI fails to achieve the expected coverage probability of 0.95 for a 95% CI as the sample size increases and skewness levels rise, especially for skewed distributions. The notable difference in performance is likely due to the fact that the cdt-CI uses a corrective decile standard deviation formula, which aligns with the classical sample standard deviation definition presented in Section 3. Overall, as noted in this study, for skewed distributions (e.g., the results for gamma, log-normal, and chi-squared distributions reported in Tables 2-4), the coverage probability of all CI estimators improves as the sample size increases, except for dt-CI, which struggles significantly to maintain the coverage probability. For higher skewness values, the bt-CI consistently performs the best, followed by the cdt-CI and t-CI, with dt-CI performing the worst. While dt-CI fails in terms of coverage probability, it performs best in terms of estimated length. For normal distribution (e.g., Table 5), bt-CI remains the best, or as good as the t-CI, followed by dt-CI, pb-CI, and cdt-CI. Interestingly, while dt-CI performs better than cdt-CI in terms of coverage probability for normal distributions, the length of the cdt-CI is narrower than that of the dt-CI across all sample sizes, making the cdt-CI more robust than the dt-CI (Table 5). While for normal data, choosing between t-CI, bt-CI, and dt-CI does not make a significant difference for small samples, for skewed distributions, the choice must be made between bt-CI and cdt-CI. Being a non-bootstrap sample, however, cdt-CI should be preferred in practice due to its simplicity. This study, therefore, recommends using the cdt-CI estimator when dealing with data exhibiting skewness.

Disclaimer (Artificial intelligence)

Authors hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

Acknowledgement

The authors are thankful to the reviewers and the editor for providing useful comments and suggestions which has improved the presentation of the article to a great extent.

Competing Interests

Authors have declared that no competing interests exist.

REFERENCES

1. Abu-Shawiesh, M. O. A. & Saghir, A. (2019). Robust confidence intervals for the population mean: Alternatives to the Student-t confidence interval. *Journal of Modern Applied Statistical Methods*, 18(1), 1–21. https://doi.org/10.22237/JMASM/1556669160

2. Abu-Shawiesh, M. O. A., Sinsomboonthong, J., & Kibria, B. M. G. (2022). A modified robust confidence interval for the population mean of distribution based on deciles. *Statistics in Transition New Series*, *23*(1), 109–128. https://doi.org/10.21307/stattrans-2022-007

3. Berrar, D. (2019). *Introduction to the Non-Parametric Bootstrap*. *April*, 0–15. https://doi.org/10.1016/B978-0-12-809633-8

4. Bickel, P. J. (1965). On some robust estimates of location. *Annals of Mathematical Statistics*, 36, 847–858.

5. Boos, D. D. & Hughes-Oliver, J. M. (2000). How Large Does n Have to be for Z and t Intervals? *American Statistician*, *54*(2), 121–128. https://doi.org/10.1080/00031305.2000.10474524

6. Casella, G. & Berger, R. (2024). *Statistical Inference*, 2nd ed., Chapman and Hall/CRC.

7. data.gov.my (2024a). Poverty rates in Malaysia from 1970 to 2022, Government of Malaysia, https://data.gov.my/data-catalogue/hh_poverty

8. data.gov.my (2024b). Quarterly Principal Labour Force Statistics by State, Department of Statistics Malaysia, Government of Malaysia, https://data.gov.my/data-catalogue/lfs_qtr_state

9. David, H. A. (1998). Early sample measures of variability. *Statistical Science*, *13*(4), 368–377. https://doi.org/10.1214/ss/1028905831

10. Desharnais, B., Camirand-Lemyre, F., Mireault, P. & Skinner, C. D. (2015). Determination of confidence intervals in non-normal data: Application of the bootstrap to cocaine concentration in femoral blood. *Journal of Analytical Toxicology*, *39*(2), 113–117. https://doi.org/10.1093/jat/bku127

11. Efron, B. (1979). Bootstrap method: another look at jack-knife. *The Annals of Statistics*, 7(1), 1-26, DOI: 10.1214/aos/1176344552

12. Efron, B. (1987). Better bootstrap confidence intervals (with discussion), *Journal of the American Statistical Association*, 82, 171–200.

13. Efron, B. & Tibshirani, R. J. (1994). *An Introduction to the Bootstrap*. Chapman and Hall/CRC, https://doi.org/10.1201/9780429246593

14. Flowers-Cano, R. S., Ortiz-Gómez, R., León-Jiménez, J. E., Rivera, R. L., & Cruz, L. A. P. (2018). Comparison of bootstrap confidence intervals using Monte Carlo simulations. *Water (Switzerland)*, *10*(2), 1–21. https://doi.org/10.3390/w10020166

15. Islam, K., & Shapla, T. J. (2018). On Performance of Confidence Interval Estimate of Mean for Skewed Populations: Evidence from Examples and Simulations. *Mathematical Theory and Modeling*, *8*(3), 41–51.

16. Mokhtar, S. F., Yusof, Z. M. & Sapiri, H. [2024]. Confidence Interval Estimating the Mean of Normal Distribution and Skewed Distribution, *Malaysian Journal of Fundamental and Applied Sciences*, 20, 1124-1135.

17. Moslim, N. H., Zubairi, Y. Z., Hussin, A. G., Hassan, S. F., & Mokhtar, N. A. (2019). A comparison of asymptotic and bootstrapping approaches in constructing confidence intervals of the concentration parameter in von Mises distribution. *Sains Malaysiana*, 48(5), 1151–1156.

18. Pek, J., Wong, A. C. M., & Wong, O. C. Y. (2017). Confidence Intervals for the Mean of Non-Normal Distribution: Transform or Not to Transform. *Open Journal of Statistics*, *07*(03), 405–421. https://doi.org/10.4236/ojs.2017.73029

19. R Core Team (2024). R: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing, Vienna, Austria. https://www.R-project.org/

20. Shi, W. & Kibria, B. M. G. (2007). On some confidence intervals for estimating the mean of a skewed population. *International Journal of Mathematical Education in Science and Technology*, *38*(3), 412–421. https://doi.org/10.1080/00207390601116086

21. Sinsomboonthong, J., Abu-Shawiesh, M. O. A., & Kibria, B. M. G. (2020). Performance of robust confidence intervals for estimating population mean under both non-normality and in presence of outliers. *Advances in Science, Technology and Engineering Systems*, *5*(3), 442–449. https://doi.org/10.25046/aj050355

22. Siraj-Ud-Doulah, M. (2018). Alternative Measures of Standard Deviation Coefficient of Variation and Standard Error. *International Journal of Statistics and Applications*, 8(6): 309-315.

23. Student. (1908). The probable error of a mean, *Biometrika, 6(1),* 1-25.

24. Wang, F.-K. (2001). Non-Normal Data. *Quality and Reliability Engineering International*, *267*(259), 257–267. https://doi.org/10.1002/qre400

25. Wilcox, R. R. (2021). A note on computing a confidence interval for the mean. *Communications in Statistics - Simulation and Computation*, *53*(1), 164–166. https://doi.org/10.1080/03610918.2021.2011926

26. Zhou, X. H., & Dinh, P. (2005). Nonparametric confidence intervals for the one- And two-sample problems. *Biostatistics*, *6*(2), 187–200. https://doi.org/10.1093/biostatistics/kxi002

27. Zhao, S., Yang, Z., Musa, S. S., Ran, J., Chong, M. K. C., Javanbakht, M., He, D., & Wang, M. H. (2021). Attach importance of the bootstrap t-test against Student's t-test in clinical epidemiology: A demonstrative comparison using COVID-19 as an example. *Epidemiology and Infection*, *149*. https://doi.org/10.1017/S0950268821001047

28. Waguespack, D., Krishnamoorthy, K., & Lee, M. (2020). Tests and Confidence Intervals for the Mean of a Zero-Inflated Poisson Distribution. *American Journal of Mathematical and Management Sciences*, *39*(4), 383–390. https://doi.org/10.1080/01966324.2020.1777914