On a New Decile-Mean Confidence Interval Estimator of Mean for Normal and Skewed Distributions

ABSTRACT

Various CI techniques—such as the Student's t confidence interval (t-CI), percentile bootstrap confidence interval (pb-CI), bootstrap t confidence interval (bt-CI), and decile mean t confidence interval (dt-CI)—have been reviewed and explored in a recent study (Mokhtar, Yusof & Sapiri, 2024). The efficiency of these CI methods has been evaluated through examples and simulations, focusing on the length of the CI and estimated coverage probability for data exhibiting both normality and skewness. The study suggests that the performance of these confidence intervals varies significantly with respect to sample size and the type of skewness. However, a significant estimation problem is observed with the proposed dt-CI method, with no explanation provided for why the problem persists in the studied examples and simulation results. In this paper, a new corrective decile t confidence interval (cdt-CI) is proposed, making a significant modification to the dt-CI (Mokhtar, Yusof & Sapiri, 2024). This approach addresses and resolves issues identified with the dt-CI when estimating the mean of data exhibiting both normality and skewness. Through real-life examples and simulations using normal, skewed chi-squared, and skewed log-normal distributions, it is demonstrated that the proposed method outperforms the dt-CI (Mokhtar, Yusof & Sapiri, 2024). Additionally, the simulation is extended by incorporating varying degrees of skewness in the gamma distribution to better assess the performance of the underlying CI methods and evaluate their sensitivity to changes in skewness and sample size. Except for the bt-CI, the new cdt-CI outperforms all other methods in normal models, and for skewed distributions, it is as good as or better than all other methods considered in this study. As such, we recommend the newly proposed CI method for constructing confidence intervals when dealing with normal or any skewed distribution.

***Keywords**: Confidence interval estimate, corrective decile t confidence interval, bootstrap approaches, coverage probability, simulation*

1. INTRODUCTION

Many classical estimation methods assume that the data from the underlying studies come from a normal distribution. For example, the z-confidence interval (z-CI) and Student's t confidence interval (t-CI) methods are derived under the assumption of normality of the population distribution. The t-CI, in particular, applies to situations where the sample size is large, even if normality is not met. It is well known that the z-ci and t-CI are the most efficient methods when the data distribution is normal. Many classical estimation methods assume that the data from the underlying studies come from a normal distribution. For example, the z-confidence interval (z-CI) and Student's t confidence interval (t-CI) methods are derived under the assumption of normality of the population distribution. The t-CI, in particular, applies to situations where the sample size is large, even if normality is not met. It is well known that the z-CI and t-CI are the most efficient methods when the data distribution is normal.

In real-life situations, however, the assumption of normality may not be met. As such, the Student's t-CI or other methods that rely on normality cannot be applied. Additionally, the sample size in many practical or experimental studies may be small enough to invalidate the use of the Student's t-CI. As a result, the limitations of the Student's t technique have been noted in the existing literature (Boos & Hughes-Oliver; 2000, David, 1998; Desharnais et al., 2015; Wilcox, 2021). Despite these drawbacks, including small sample sizes or non-normality of the data distribution, some prior studies suggest that the coverage probability of the t-CI approaches the expected nominal confidence level when sample sizes are small or the underlying distributions are asymmetric, while the CI length remains relatively larger compared to other CI methods (e.g., Boos & Hughes-Oliver, 2000; Shi & Kibria, 2007; Wang, 2001; Zhou & Dinh, 2005).

In this paper, we revisit some confidence interval estimation approaches such as the classic t-CI, percentile bootstrap confidence interval (pb-CI), bootstrap-t confidence interval (bt-CI), and their proposed decile t confidence interval (dt-CI) investigated in Mokhtar, Yusof & Sapiri (2024). The efficiency of these CI methods has been evaluated through examples and simulations. By focusing on the estimated coverage probability for data exhibiting both normality and skewness, it was suggested that the performance of these CIs varies significantly with respect to sample size and the type of skewness. In addition, a significant estimation problem was noted with their proposed dt-CI method, but no explanation was provided for why the problem persisted in the studied examples and simulation results.

In this paper, a new corrective decile t confidence interval (cdt-CI) is proposed, making a significant modification to the dt-CI (Mokhtar, Yusof & Sapiri, 2024). The newly proposed cdt-CI is expected to resolve some of the problems identified in the performance of the dt-CI when estimating the population mean of data exhibiting both normality and skewness, particularly as sample sizes increase. The performance of the newly proposed method has been evaluated and compared with the dt-CI and other underlying methods through real-life examples and simulations using samples from normal, skewed chi-squared, skewed log-normal, and skewed gamma distributions with varying degrees of skewness.

The results of the examples and simulations suggest that the newly proposed cdt-CI outperforms the dt-CI (Mokhtar, Yusof & Sapiri, 2024) when the data follow a normal distribution or exhibit skewness. This study also suggests that all underlying methods are sensitive to increasing skewness values and sample sizes, with the dt-CI performing poorly, while the new method either outperforms all other methods in normal models or is as good as or better than all other methods in the presence of skewness. A better understanding and exploration of underlying methods may lead to the recommendation of appropriate method for constructing confidence intervals when dealing with any skewed distribution.

The organization of the remaining paper is as follows. The literature review has been considered in section 2, along with subsections 2.1-2.4 to address some popular confidence interval estimators relevant to this study. A new modified confidence interval estimator has been proposed in section 3. A simulation study has been considered in section 4 by generating samples from normal and skewed distributions. A number of real-life examples have been studied section 5. A section with concluding remarks has been incorporated in section 6.

2. Literature review

Let $X_1 X_2, \cdots, X_n$ be a sample from a population with an unknown population mean μ pertaining to a skewed distribution. We wish to estimate the population mean μ via a confidence interval to ensure a desired level of certainty in the estimation. For the relevance of the study with Mokhtar, Yusof & Sapiri (2024), this section provides a brief description of the t-CI, pb-CI, bt-CI, and dt-CI methods.

2.1 The t-CI of mean μ

Under the assumption that the sample or the underlying data comes from a normal distribution with a known standard deviation σ, a $100(1-\alpha)\%$ confidence interval (CI) estimate of μ is given

$$\left[\bar{X} - z_{\alpha/2} \times \sigma/\sqrt{n}, \bar{X} + z_{\alpha/2} \times \sigma/\sqrt{n}\right] \qquad (1)$$

where $z_{\alpha/2}$ is the upper $(\alpha/2)$th percentile of standard normal distribution.

The confidence interval in (1) is referred to as z-ci. The z-ci is the most omnipresent in statistical literature. However, as is always, it is very unlikely that the population standard deviation σ is known, and as such it is estimated by the sample standard deviation given by

$$s = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{X})^2}{n-1}}$$

Then, Due to Student [9], a classic (1–α)100% t-CI for μ can be given by

$$\left[\bar{X} - t_{\alpha/2} \times s_2/\sqrt{n}, \bar{X} + t_{\alpha/2} \times s_2/\sqrt{n}\right] \quad (2)$$

where $t_{\alpha/2}$ is the upper $(\alpha/2)$th of Student's t distribution with (n-1) degrees of freedom. It is well known that t-CI approach is not robust against normality, distribution having extreme or outlying observation or even asymmetry in distributions (Abu-Shawiesh & Saghir, 2019; Boos & Hughes-Oliver, 2000; Shi & Kibria, 2007; Bickel, 1965; Casella & Berger, 2024).

In absence of normality or a data distribution with skewness, the most important alternative is to employ bootstrap confidence interval ([Efron, 1979, 1987; Flowers-Cano *et al.,* 2018).

2.2 The pb-CI of mean μ

The percentile bootstrap CI (pb-CI) is a particular choice among available bootstrap procedures due to the simplicity of the computational steps (Abu-Shawiesh, Sinsomboonthong & Kibria, 2022; Pek, Wong & Wong, 2017; Islam & Shapla, 2018). A bootstrap sample of a given sample of size n, is a sample of size n drawn from the given sample, with replacement. Given a reasonable set of bootstrap replications, one can construct a (1−α)100% pb-CI by following the algorithm below:
(i) generate a bootstrap sample, compute and store its mean.
(ii) Repeat step (i) *B* times (e.g., 200 times).
(iii) Then, a (1−α)100% pb-CI for the population mean is formed from the (α/2)100th and (1-α/2)100th percentiles of *B* bootstrap sample means.
Using above algorithm in mathematical expression, (1−α)100% pb-CI of the population mean μ is given by

$$\left[(\bar{X}_b^*)_{\alpha/2}, (\bar{X}_b^*)_{1-\alpha/2}\right] \quad (3)$$

where
$\bar{X}_b^*$ is the mean of bth bootstrap sample, $b = 1,2,\ldots,B$
$(\bar{X}_b^*)_{\alpha/2}$ is the $(\alpha/2)th$ percentile of $\bar{X}_b^*$

While performs well in terms of estimated coverage probability, the pb-CI method may often tend to be inconsistent (Sinsomboonthong, Abu-Shawiesh & Kibria, 2020).

2.3 The bt-CI of mean μ

An alternative to pb-CI, one may use bootstrap t confidence interval (bt-CI) of μ (Berrar, D., 2019; Efron & Tibshirani, 1994), which depends on the bootstrap studentized t-score. Given the sample $X_1 X_2, \cdots, X_n$ and the sample mean $\bar{X}$, the bootstrap studentized t-score is calculated using the following equation

$$T_b^* = \frac{\bar{X}_b^* - \bar{X}}{SE(\bar{X}_b^*)} \quad (4)$$

where
$\bar{X}_b^*$ is the bth bootstrap mean
T_b^* is the studentized score for bth bootstrap
$SE(\bar{X}_b^*) = \sqrt{\frac{\sum_{b=1}^{B}(\bar{X}_b^* - \bar{\bar{X}}_b^*)^2}{B-1}}$ is the standard error of bootstrap means $\bar{X}_b^*$
$\bar{\bar{X}}_b^* = \frac{\sum_{b=1}^{B}\bar{X}_b^*}{B}$ is the mean of bootstrap means

Under above notations, a $100(1-\alpha)\%$ bt-CI is given by

$$\left[\bar{X} - \hat{t}_{1-\alpha/2} SE(\bar{X}_b^*), \bar{X} - \hat{t}_{\alpha/2} SE(\bar{X}_b^*)\right] \quad (5)$$

where $\hat{t}_{1-\alpha/2}$ is the $(1-\alpha/2)$th percentile of T_b^* over all bootstrap samples.

Unlike the t-CI, the bt-CI is not symmetric, and is believed to outperform the t-CI for varying sample sizes (Zhao et al., 2021).

2.4 The dt-CI of mean μ

When the sample or underlying data comes from a skewed distribution, a recent article [Boos, D. D., & Hughes-Oliver, J. M. (2000) has proposed to construct CI of mean μ by implementing decile mean (DM) and decile standard deviation (Abu-Shawiesh, Sinsomboonthong & Kibria, 2022; Mokhtar, Yusof & Sapiri, 2024; Siraj-Ud-Doulah, 2018).

Let $D_j, j = 1,2,\cdots,9$ be 9 deciles of the sample $X_1, X_2, \cdots, X_n$.

The decile mean (DM) and decile standard deviation (SD_{DM}) are given by

$$DM = \frac{\sum_{i=1}^{9} D_i}{9}$$

$$SD_{DM} = \sqrt{DM\ of\ (X_i - DM)^2} = \sqrt{\frac{\sum_{j=1}^{9} D_j(X-DM)^2}{9}} \tag{6}$$

where $D_j(X - DM)^2, j = 1,2,\cdots,9$ is the jth decile of $(X - DM)^2$ given the sample X.

The $100(1-\alpha)\%$ decile t confidence interval (dt- ci) estimator of μ, due to Mokhtar, Yusof & Sapiri (2024) is given by

$$\left[DM - t_{\alpha/2}\frac{SD_{DM}}{\sqrt{n}}, DM + t_{\alpha/2}\frac{SD_{DM}}{\sqrt{n}}\right] \tag{7}$$

where $t_{\alpha/2}$ is the upper $(\alpha/2)$th of Student's t distribution with (n-1) degrees of freedom.

It is noted that dt-CI suffer significantly in achieving the desired coverage probabilities of 0.95 for 95% confidence interval. It gets worse when the sample size gets larger or bootstrap replication gets higher (Mokhtar, Yusof & Sapiri, 2024). The reason for its worst performance is due to the fact that $\frac{SD_{DM}}{\sqrt{n}}$ gets too small to capture the unknown mean μ. The problem originates from the fact that while SD_{DM} is computed on the basis of only 9 deciles. Dividing SD_{DM} by increasing value of $\sqrt{n}$ (for large n) may be impractical or does not make any sense. Indeed, the dt-CI severely fails to capture the unknown parameter μ, as was clearly noted in examples and simulation results of Mokhtar, Yusof & Sapiri (2024).

In order to resolve the problem of dt-CI estimator noted in Mokhtar, Yusof & Sapiri (2024), in the following section we propose a corrective decile t confidence interval (cdt-CI).

3. Methodology

As we seek answer to what went wrong with the dt-CI proposed in Mokhtar, Yusof & Sapiri (2024), we propose to construct a new corrective decile t confidence interval (cdt-CI) of estimator of μ, which uses the corrected decile standard deviation (CDSD) computed by using the equation

$$CDSD = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \mathrm{DM})^2}{n-1}} \tag{8}$$

Then, a $100(1-\alpha)\%$ corrective decile t-confidence interval (cdt-CI) of μ is constructed using equation

$$\left[\bar{X} - t_{\alpha/2} \times CDSD/\sqrt{n}, \bar{X} + t_{\alpha/2} \times CDSD/\sqrt{n}\right] \tag{9}$$

This method is expected to perform better than the dt-CI method by Mokhtar, Yusof & Sapiri (2024). The performance of cdt-CI needs to be justified via simulation and examples of real-life problems.

4. The Simulation Study

When evaluation of various alternative methods is impossible or challenging theoretically, it is customary to use simulation to justify the usefulness of a given method compared to other alternative methods. As such, many researchers opt for simulation study to assess CIs (Abu-Shawiesh, Sinsomboonthong & Kibria, 2022; Mokhtar, Yusof & Sapiri, 2024, Islam & Shapla, 2018; Shi & Kibria, 2007).

In this simulation study, we generate data from four different models as defined below:

(M1) A positively skewed gamma distribution $G(\theta_1, \theta_2)$ with density function specified by

$$f(x) = \frac{x^{\theta_1 - 1}\exp(-\frac{x}{\theta_2})}{\theta_2^{\theta_1}\Gamma(\theta_1)}; x > 0, \theta_1, \theta_2 > 0 \quad (10)$$

where θ_1 is a shape parameter, θ_2 is a scale parameter and the skewness is equal to $2/\sqrt{\theta_1}$.
Since the mean of this distribution is $\mu = \theta_1\theta_2$, by we choosing $\theta_2 = \frac{1}{\theta_1}$ and θ_1 values at 16, 4, 1, 1/4, we get mean fixed at $\mu = 1$ and skewness values at 0.5, 1, 2, 4, respectively. Our objective is to evaluate sensitivity of various CI methods with respect to varying skewness chosen arbitrarily to make the comparison notable.

(M2) A symmetric standard normal N(0,1) distribution
(M3) A skewed Chi-squared distribution with df=k so as to have the mean $\mu = k$ and skewness equal to $\sqrt{8/k)}$ with the density function given by

$$f(x,k) = \frac{x^{k/2-1}e^{-x/2}}{\Gamma(k/2)2^{k-1}}, x > 0$$

By choosing k=128, 32, 8 and 2, we allow skewness to be 0.25, 0.5, 1 and 2, while mean remains the same as the value of k.
(M4) Lognormal distribution with the density function given by

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}}e^{-\left(\frac{logx-\mu}{2\sigma^2}\right)}, x > 0, \sigma > 0, -\infty < \mu < \infty$$

with $skewness = e^{\sigma^2+2}\sqrt{e^{\sigma^2-1}}$ and $mean = e^{\mu+\frac{\sigma^2}{2}}$

By choosing $\mu = 1$ and $\sigma^2 = \frac{1}{10}, \frac{1}{4}, \frac{1}{2}$ and 1 we allow skewness values to be 1, 1.75, 2.94 and 6.18 mean=2.86, 3.08, 3.49 and 4.48.

In all simulations, the Monte Carlo size is chosen to be 5,000, and bootstrap replication B is chosen to be 1000. The results of simulation have been reported in Tables 2-5 for sample size varying between 10 and 400, at 10, 15, 20, 25, 30, 50, 100, 200 and 400 following the standard practice (Abu-Shawiesh, Sinsomboonthong & Kibria, 2022; Mokhtar, Yusof & Sapiri, 2024, Islam & Shapla, 2018; Shi & Kibria, 2007).

For simplicity of presentation, the skewness coefficients of four models of population distributions considered in the simulation study are made available in Table 1 below:

Table 1. The parameters of normal and chi-square distribution

Models	Parameters	Skewness
M1: $N(\mu, \sigma^2)$	$\mu = 1, \sigma^2 = 1$	0
M2: χ_k^2	k=128, 32, 8, 2	0.25, 0.5, 1, 2
M3:LN(1,σ^2)	$\sigma^2 = \frac{1}{10}, \frac{1}{4}, \frac{1}{2}, 1$	1,1.75, 2.94, 6.18
M4: Gamm(θ_1, θ_2)	$\theta_1 = 16, 4, 1, 1/4$, $\theta_2 = \frac{1}{\theta_1}$	0.5, 1, 2, 4

Evaluation criteria of simulation

The evaluation criteria for the goodness of any underlying confidence interval method are the estimated coverage probability and the average length of the corresponding confidence interval estimator (Abu-Shawiesh, Sinsomboonthong & Kibria, 2022; Mokhtar, Yusof & Sapiri, 2024, Moslim, Zubairi & Hassan, 2019; Omar & Abu, 2011; Islam & Shapla, 2018; Shi & Kibria, 2007; Waguespack, Krishnamoorthy & Lee, 2020). The process involves repeatedly drawing samples and constructing a confidence interval for each drawn sample. The proportion of confidence intervals across all repetitions that capture the mean μ is the estimated coverage probability.

In all simulations, the Monte Carlo size M=5000 and the bootstrap replication size B=1000 were considered. The significance level was set to α=0.05. Thus, the proportion of confidence intervals across all 5000 samples under each method that capture the mean μ is the estimated coverage probability.

The average length of the 5000 confidence intervals under each method is the estimated length.

Note that t-CI, dt-CI, and cdt-CI do not involve any bootstrap samples because they are estimated directly from the sample. However, Mokhtar, Yusof & Sapiri (2024) referred to dt-CI as Bootstrap-t Decile Mean, perhaps inappropriately.

The following algorithm can be used to estimate the coverage probability and the estimated length (est. length) of the associated confidence interval while performing the simulation:

Fix the sample size n and the skewness value for any desired model by selecting the corresponding model parameters.

(1a) Take M=5000 Monte Carlo samples from the desired model (e.g., any of models M1–M4), each of size n. For each of the M samples, compute t-CI, dt-CI, and cdt-CI, and store them.

(1b) For bootstrap confidence intervals (pb-CI or bt-CI), for each of the M samples in (1a), generate B=1000 bootstrap samples and compute the pb-CI and bt-CI. This results in M pb-CIs and bt-CIs, which are then stored.

(2a) The proportion of M t-CIs, dt-CIs, and cdt-CIs in (1a) that contain μ are the estimated coverage probabilities for the corresponding CIs.

(2b) The proportion of M pb-CIs and bt-CIs in (1b) that contain μ are the estimated coverage probabilities for the corresponding CIs.

(3) For each of the t-CIs, dt-CIs, and cdt-CIs in (1a) or pb-CIs and bt-CIs in (1b), compute the length of the confidence interval. The average of the M lengths across all M CIs is the estimated length of the corresponding CI.

Note that coverage probability of a 95% CI ($\alpha = 0.05$) is expected to be close to 0.95 across all Monte Carlo size of M samples (e.g., M=5000 in this study). Therefore, any CI method can be considered satisfactory if its coverage probability is close to 0.95 or near ($1-\alpha$) over all M Monte Carlo samples.

As a computational tool, in all computation and simulation, the statistical software R [26] has been utilized in this article.

Simulation Results

Let us have a look at the results of simulation for varying values of sample and skewness, with sample size varying at 10, 15, 20, 25, 30, 50, 100, 200 and 400 following standard practice (Abu-Shawiesh, Sinsomboonthong & Kibria, 2022; Mokhtar, Yusof & Sapiri, 2024; Islam & Shapla, 2018; Shi & Kibria, 2007) and standard practice, arbitrarily. The coverage probability of 95% CI ($\alpha = 0.05$) for various CI methods, along with average length of associated CIs over all Monte Carlo size of 5000 samples, have been reported in Tables 2-5.

First, let us examine the results in Table 2 from the simulation with the gamma distribution. When the skewness is 0.5, the estimated coverage probability increases as the sample size increases for all methods, except for dt-CI, which fails to show an increase in coverage probability with the increasing sample size—a behavior that is unexpected for any estimator. When the skewness is 4, the bt-CI appears to perform the best, followed by the cdt-CI and t-CI, respectively, with dt-CI performing the worst in terms of estimated coverage probability. While dt-CI shows significant underestimation in coverage probability, it performs best in terms of estimated length. Researchers must decide what matters most: coverage probability or length. However, it should be noted that the length of the CI should only be considered if the coverage probability meets expectations. A shorter length with underestimated coverage cannot be favored over the one with the attained expected coverage probability.

Table 2. Simulated coverage probability and length of CI for gamma distribution varying sample size and skewness

Skewness		Est. coverage probability of CI methods					Est. average length of CI methods				
	n	t-CI	pb-CI	bt-CI	dt-CI	cdt-CI	t-CI	pb-CI	bt-CI	dt-CI	cdt-CI
0.5	10	0.95	0.91	0.95	0.90	0.95	0.35	0.28	0.38	0.29	0.35
	15	0.95	0.92	0.95	0.89	0.95	0.27	0.24	0.28	0.23	0.27
	20	0.95	0.93	0.95	0.89	0.95	0.23	0.21	0.24	0.19	0.23
	25	0.94	0.93	0.94	0.88	0.94	0.20	0.19	0.21	0.17	0.20
	30	0.94	0.93	0.94	0.89	0.95	0.18	0.17	0.19	0.16	0.18
	50	0.95	0.94	0.95	0.89	0.95	0.14	0.14	0.14	0.12	0.14
	100	0.95	0.95	0.95	0.88	0.95	0.10	0.10	0.10	0.09	0.10
	200	0.95	0.95	0.95	0.86	0.95	0.07	0.07	0.07	0.06	0.07
	400	0.94	0.94	0.94	0.83	0.95	0.05	0.05	0.05	0.05	0.05
1	10	0.94	0.90	0.96	0.87	0.94	0.69	0.57	0.79	0.56	0.69
	15	0.93	0.90	0.95	0.86	0.93	0.54	0.48	0.58	0.44	0.54
	20	0.94	0.92	0.95	0.86	0.94	0.46	0.42	0.48	0.37	0.46
	25	0.95	0.93	0.95	0.85	0.95	0.41	0.38	0.42	0.33	0.41
	30	0.94	0.93	0.95	0.86	0.95	0.37	0.35	0.38	0.30	0.37
	50	0.95	0.94	0.95	0.84	0.95	0.28	0.27	0.29	0.23	0.28
	100	0.94	0.94	0.94	0.80	0.94	0.20	0.19	0.20	0.16	0.20
	200	0.95	0.95	0.96	0.73	0.95	0.14	0.14	0.14	0.11	0.14
	400	0.95	0.94	0.95	0.63	0.95	0.10	0.10	0.10	0.10	0.08
2	10	0.89	0.85	0.93	0.77	0.89	1.32	1.07	1.91	0.96	1.34
	15	0.92	0.89	0.95	0.76	0.92	1.05	0.93	1.33	0.76	1.06
	20	0.93	0.92	0.95	0.75	0.93	0.90	0.82	1.07	0.65	0.91
	25	0.92	0.91	0.95	0.72	0.92	0.80	0.74	0.91	0.57	0.80
	30	0.93	0.92	0.95	0.73	0.93	0.73	0.69	0.82	0.52	0.74
	50	0.94	0.93	0.95	0.68	0.94	0.56	0.54	0.60	0.40	0.57
	100	0.94	0.94	0.95	0.56	0.95	0.39	0.39	0.41	0.28	0.40
	200	0.96	0.95	0.96	0.37	0.96	0.28	0.28	0.28	0.20	0.28
	400	0.95	0.95	0.95	0.17	0.95	0.20	0.20	0.20	0.20	0.14
4	10	0.80	0.78	0.93	0.59	0.81	2.33	1.83	12.00	1.38	2.38
	15	0.83	0.82	0.94	0.54	0.84	1.94	1.66	5.08	1.06	1.98
	20	0.86	0.85	0.95	0.49	0.86	1.68	1.50	3.14	0.88	1.71
	25	0.87	0.87	0.95	0.48	0.88	1.51	1.38	2.54	0.78	1.53
	30	0.87	0.87	0.94	0.45	0.88	1.39	1.30	2.12	0.71	1.42
	50	0.90	0.91	0.95	0.35	0.91	1.07	1.03	1.39	0.53	1.09
	100	0.92	0.92	0.94	0.19	0.92	0.77	0.76	0.88	0.37	0.79
	200	0.93	0.94	0.95	0.05	0.94	0.55	0.55	0.59	0.27	0.56
	400	0.94	0.94	0.95	0.00	0.94	0.39	0.39	0.40	0.40	0.19

As we examine the results in Table 3, based on simulations from a log-normal distribution, it is evident that the bt-CI consistently performs the best, both as the sample size and skewness values increase. Similar to the case of the gamma distribution in Table 2, the dt-CI performs very poorly compared to all the other methods. The results in Table 3 clearly suggest that the bt-CI is the most robust method against increasing skewness, with the cdt-CI performing second best, followed by the t-CI or pb-CI in terms of coverage probability. In all cases where the cdt-CI outperforms the dt-CI, it can be concluded that the corrected decile standard deviation (CDSD) is more robust and efficient in both estimated average length and estimated coverage probability.

Table 3. Simulated coverage probability and length of CI for log-normal distribution with varying sample size and skewness

Skewness		Est. coverage probability of CI methods					Est. average length of CI methods				
	n	t-CI	pb-CI	bt-CI	dt-CI	cdt-CI	t-CI	pb-CI	bt-CI	dt-CI	cdt-CI
1	10	0.93	0.89	0.94	0.87	0.93	1.27	1.04	1.46	1.01	1.28
	15	0.95	0.92	0.95	0.87	0.95	0.99	0.88	1.07	0.80	1.00
	20	0.94	0.92	0.94	0.86	0.94	0.85	0.78	0.90	0.69	0.85
	25	0.94	0.92	0.95	0.86	0.94	0.75	0.70	0.79	0.61	0.76
	30	0.94	0.92	0.95	0.86	0.94	0.68	0.64	0.70	0.55	0.68
	50	0.95	0.94	0.95	0.85	0.95	0.52	0.50	0.53	0.43	0.52
	100	0.95	0.94	0.95	0.81	0.95	0.37	0.36	0.37	0.30	0.37
	200	0.95	0.95	0.95	0.76	0.95	0.26	0.26	0.26	0.21	0.26
	400	0.96	0.96	0.96	0.64	0.96	0.18	0.18	0.18	0.15	0.18
1.75	10	0.93	0.88	0.95	0.84	0.93	2.20	1.80	2.78	1.67	2.22
	15	0.92	0.89	0.94	0.82	0.92	1.71	1.51	1.98	1.30	1.73
	20	0.93	0.92	0.95	0.81	0.94	1.48	1.35	1.66	1.11	1.49
	25	0.93	0.92	0.94	0.80	0.93	1.32	1.23	1.45	0.99	1.32
	30	0.93	0.92	0.95	0.79	0.93	1.18	1.12	1.28	0.89	1.19
	50	0.94	0.93	0.94	0.75	0.94	0.92	0.88	0.96	0.68	0.92
	100	0.94	0.94	0.95	0.68	0.94	0.64	0.64	0.66	0.48	0.65
	200	0.95	0.95	0.95	0.54	0.95	0.45	0.45	0.46	0.34	0.46
	400	0.94	0.94	0.95	0.34	0.95	0.32	0.32	0.32	0.24	0.32
2.94	10	0.89	0.86	0.93	0.79	0.90	3.60	2.91	5.30	2.53	3.64
	15	0.90	0.88	0.93	0.75	0.90	2.85	2.49	3.76	1.95	2.88
	20	0.90	0.89	0.93	0.72	0.90	2.45	2.22	3.03	1.64	2.47
	25	0.91	0.90	0.93	0.73	0.91	2.19	2.03	2.61	1.47	2.21
	30	0.92	0.91	0.94	0.70	0.92	1.99	1.87	2.32	1.32	2.01
	50	0.93	0.92	0.95	0.62	0.93	1.54	1.49	1.71	1.01	1.56
	100	0.94	0.94	0.95	0.49	0.94	1.09	1.08	1.16	0.71	1.10
	200	0.95	0.95	0.95	0.29	0.95	0.77	0.77	0.80	0.50	0.78
	400	0.95	0.96	0.96	0.09	0.95	0.55	0.55	0.56	0.35	0.55
6.18	10	0.84	0.80	0.91	0.68	0.84	6.66	5.31	13.37	4.25	6.78
	15	0.86	0.84	0.92	0.64	0.86	5.42	4.68	9.23	3.20	5.50
	20	0.87	0.86	0.92	0.58	0.87	4.71	4.22	7.44	2.64	4.78
	25	0.88	0.87	0.93	0.56	0.88	4.30	3.94	6.47	2.34	4.37
	30	0.89	0.88	0.93	0.54	0.89	3.86	3.60	5.40	2.11	3.92
	50	0.89	0.89	0.93	0.44	0.90	3.04	2.91	3.85	1.61	3.08
	100	0.92	0.93	0.95	0.26	0.93	2.21	2.17	2.58	1.12	2.24
	200	0.93	0.93	0.94	0.09	0.93	1.59	1.57	1.74	0.79	1.60
	400	0.94	0.94	0.94	0.01	0.94	1.13	1.13	1.2	0.56	1.15

As we examine the results in Table 4, based on simulations from a chi-squared distribution, it is evident that the bt-CI is again the best performer, both as the sample size and skewness values increase. The performance of dt-CI is highly underestimated. There is a clear and suggestive indication that bt-CI remains the most robust against increasing skewness and sample size. The performance of cdt-CI and t-CI is very comparable, followed by pb-CI given skewed data distributions.

Table 4. Simulated coverage probability and length of CI for chi-squared distribution with varying sample size and skewness

Skewness		Est. coverage probability of CI methods					Est. average length of CI methods				
	n	t-CI	pb-CI	bt-CI	dt-CI	cdt-CI	t-CI	pb-CI	bt-CI	dt-CI	cdt-CI
0.25	10	0.95	0.90	0.96	0.90	0.95	22.33	18.30	24.20	18.63	22.38
	15	0.95	0.93	0.96	0.91	0.95	17.42	15.39	18.01	14.78	17.45
	20	0.96	0.93	0.95	0.90	0.96	14.76	13.48	15.08	12.59	14.77
	25	0.95	0.93	0.95	0.90	0.95	13.07	12.19	13.28	11.25	13.08
	30	0.95	0.93	0.95	0.90	0.95	11.85	11.20	12.01	10.24	11.87
	50	0.94	0.94	0.94	0.89	0.94	9.05	8.77	9.11	7.89	9.05
	100	0.94	0.94	0.95	0.89	0.94	6.32	6.24	6.36	5.55	6.33
	200	0.94	0.94	0.94	0.90	0.94	4.46	4.44	4.48	3.93	4.46
	400	0.95	0.95	0.95	0.89	0.95	3.14	3.14	3.16	2.77	3.15
0.50	10	0.94	0.90	0.95	0.89	0.94	11.07	9.08	12.17	9.15	11.11
	15	0.94	0.92	0.94	0.89	0.94	8.73	7.70	9.13	7.32	8.75
	20	0.95	0.92	0.95	0.89	0.95	7.39	6.76	7.61	6.24	7.40
	25	0.94	0.92	0.94	0.88	0.94	6.53	6.09	6.68	5.55	6.54
	30	0.96	0.94	0.96	0.90	0.96	5.92	5.59	6.03	5.07	5.93
	50	0.95	0.95	0.96	0.90	0.95	4.53	4.39	4.58	3.89	4.53
	100	0.94	0.94	0.94	0.88	0.94	3.16	3.12	3.18	2.73	3.16
	200	0.95	0.95	0.95	0.87	0.95	2.23	2.22	2.24	1.94	2.23
	400	0.95	0.95	0.95	0.84	0.95	1.57	1.57	1.58	1.57	1.37
1	10	0.93	0.89	0.94	0.86	0.93	5.52	4.52	6.44	4.41	5.55
	15	0.94	0.92	0.95	0.86	0.95	4.31	3.80	4.66	3.49	4.32
	20	0.94	0.92	0.94	0.85	0.94	3.66	3.35	3.88	2.97	3.68
	25	0.94	0.92	0.94	0.85	0.94	3.23	3.01	3.37	2.63	3.24
	30	0.94	0.93	0.95	0.85	0.94	2.94	2.78	3.05	2.40	2.95
	50	0.94	0.94	0.95	0.84	0.94	2.26	2.18	2.31	1.85	2.26
	100	0.96	0.95	0.95	0.80	0.96	1.58	1.56	1.60	1.30	1.58
	200	0.95	0.95	0.95	0.74	0.95	1.11	1.11	1.12	0.92	1.12
	400	0.94	0.94	0.94	0.61	0.94	0.79	0.79	0.79	0.79	0.65
2	10	0.91	0.88	0.95	0.79	0.92	2.66	2.16	3.85	1.93	2.69
	15	0.92	0.90	0.95	0.75	0.92	2.10	1.84	2.64	1.51	2.12
	20	0.93	0.91	0.95	0.76	0.93	1.79	1.63	2.11	1.29	1.81
	25	0.93	0.92	0.95	0.73	0.93	1.60	1.49	1.84	1.14	1.61
	30	0.93	0.92	0.95	0.70	0.93	1.45	1.36	1.63	1.02	1.46
	50	0.94	0.94	0.95	0.67	0.94	1.12	1.08	1.20	0.79	1.13
	100	0.94	0.94	0.95	0.54	0.94	0.79	0.77	0.82	0.56	0.79
	200	0.95	0.94	0.95	0.39	0.95	0.56	0.55	0.57	0.39	0.56
	400	0.94	0.94	0.94	0.16	0.94	0.39	0.39	0.4	0.4	0.28

For simulation results from normal distribution in Table 5, the t-CI, bt-CI and dt-CI seem very close to each other and comparable. The results suggest that for symmetric normal distribution, dt-CI works reasonably well, better than cdt-CI, particularly for small samples. The pb-CI performs equally well like t-CI and bt-CI when sample size is very large. This simulation is in contradiction with Table 2, column 6, of Mokhtar, Yusof & Sapiri, (2024) where dt-CI appears to be very inconsistent and unexpectable even for normal distribution. It is expected that for symmetric distribution or normal distribution dt-CI of mean should have a better coverage probability than what was observed in Mokhtar, Yusof & Sapiri (2024).

Table 5. Simulated coverage probability and length of CI for normal distribution with varying sample size

	Est. coverage probability of CI methods					Est. average length of CI methods				
n	t-CI	pb-CI	bt-CI	dt-CI	cdt-CI	t-CI	pb-CI	bt-CI	dt-CI	cdt-CI
10	0.94	0.89	0.95	0.94	0.89	1.40	1.15	1.51	1.40	1.17
15	0.94	0.92	0.95	0.94	0.89	1.09	0.96	1.12	1.09	0.93
20	0.96	0.94	0.95	0.96	0.91	0.92	0.84	0.94	0.92	0.79
25	0.95	0.93	0.95	0.95	0.91	0.82	0.76	0.83	0.82	0.71
30	0.95	0.94	0.95	0.95	0.91	0.74	0.70	0.75	0.74	0.64
50	0.96	0.95	0.96	0.96	0.92	0.57	0.55	0.57	0.57	0.50
100	0.95	0.95	0.95	0.95	0.92	0.40	0.39	0.40	0.40	0.35
200	0.95	0.94	0.95	0.95	0.91	0.28	0.28	0.28	0.28	0.25
400	0.95	0.95	0.95	0.91	0.95	0.20	0.20	0.20	0.17	0.20

In all simulation while we considered Monte Carlo size M to be 5000, while the bootstrap replications were kept to be 1000. Some other bootstrap replications such as B=250 or 500 were tried in this study while no significant difference in results were noted. Therefore, for the unnecessary redundancies, those results are not accommodated in this paper. It is to be noted that 100 bootstrap iterations are all that is necessary for standard error estimation, while 1000 iterations appear to be good enough for confidence intervals (Efron, 1979, 1987; Efron & Tibshirani,1994)

Examples and applications with real-life data

In this section, we apply various confidence interval methods to construct a 95% confidence interval of the mean μ using data from real-life situations, one following a normal distribution, one with negatively skewed distribution and the other with positively skewed distribution. We wish to evaluate the performance of various confidence interval methods in capturing the unknown population mean and the length of associated confidence interval methods.

Example 1

In this example, we utilize relative poverty (%) in Malaysia from 1970 to 2022 for selected years reported in data.gov.my [27] as follows

19.5, 19.7, 19.0, 20.0, 19.9, 17.4, 19.3, 19.2, 15.6, 15.9, 16.9, 16.2, 16.6

The data skewness value of -0.27, mean 18.1 and median 19.0 (mean<median) along with the histogram and boxplot in Figure 1, all suggest that the distribution of relative poverty is negatively skewed. The Shapiro-Wilk test of normality (W= 0.86128, p-value = 0.04005) suggests that at significance level α = 5%, the relative poverty population distribution fails to be normally distributed. Now, let us have a look at 95% CIs for various underlying methods reported in Table 6.

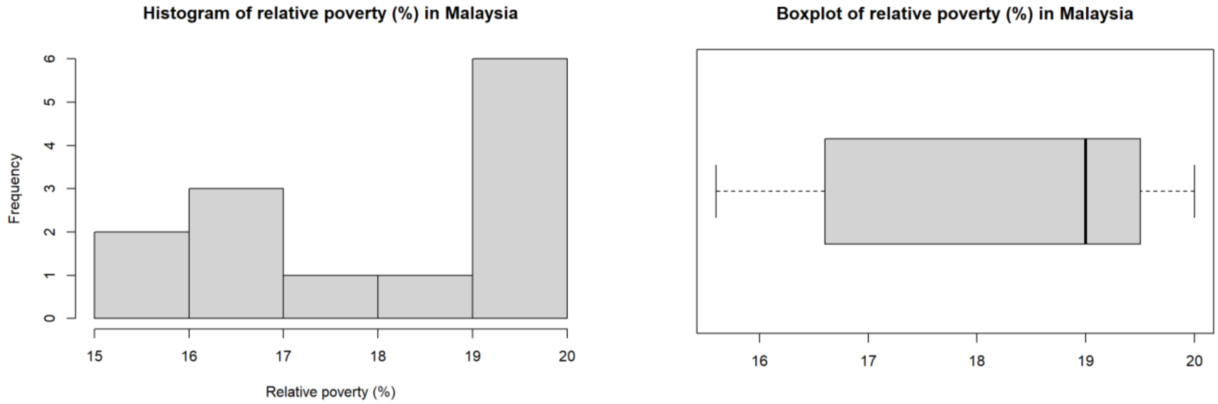

Figure 1. Histogram and boxplot of relative poverty in Malaysia

Table 6. 95% CIs and corresponding length for relative poverty data in Example 1

Methods	CI estimate	Length
t-CI	[17.08, 19.11]	2.03
pb-CI	[17.26, 18.88]	1.62
bt-CI	[17.08, 18.98]	1.90
dt-CI	[17.22, 19.10]	1.88
cdt-CI	[17.08, 19.11]	2.03

The results of Table 6 reveal that the pb-CI has the smallest length (length of 1.62), followed by dt-CI (length of 1.88), bt-CI (length of 1.90) and jointly t-CI and cdt-CI (both having length 2.03). Therefore, with length consideration, dt-CI preferable. But serious attention must be paid given the fact that dt-CI performs badly as has been revealed by the simulation in coverage probability concern, particularly for large sample while computing SD_{DM} may This sample is very small (n=13) and as such the margin of error involving SD_{DM} in the computation of CI may not be that affected to make coverage probability to be higher up to certain extent. In all consideration, cdt-CI prevails to be recommended as we deal with data having skewed distribution.

The results in Table 6 reveal that the pb-CI has the smallest length (1.62), followed by the dt-CI (1.88), bt-CI (1.90), and both the t-CI and cdt-CI (each with a length of 2.03). Therefore, in terms of length, the dt-CI is preferable. However, serious attention must be paid to the fact that the dt-CI performs poorly in terms of coverage probability, as revealed by the simulation, particularly for large sample sizes.

Example 2

In this example, we consider quarterly unemployment rate in Kedah state of Malaysia between 2017 and 2Q 2024 reported in data.gov.my [28] and is represented as follows

2.5	3.1	3.1	2.8	3.7	4.0	3.8	3.3	2.8	2.1
3.0	3.0	3.0	3.2	4.1	3.6	3.8	3.0	2.6	1.9
2.7	2.5	3.0	3.1	4.4	3.9	3.4	3.3	2.3	1.7

The data skewness value of -0.0997, mean 3.09 and median 3.05 (mean appears to be close to the median) along with the histogram and boxplot in Figure 2, might suggest the population to be less skewed. The Shapiro-Wilk test of normality (W= 0.98394, p-value = 0.9178) suggests that at significance level α = 5%, the quarterly unemployment rate in Kedah state might have been normally distributed. As such, let focus our attention to 95% CIs for various underlying methods reported in Table 5.

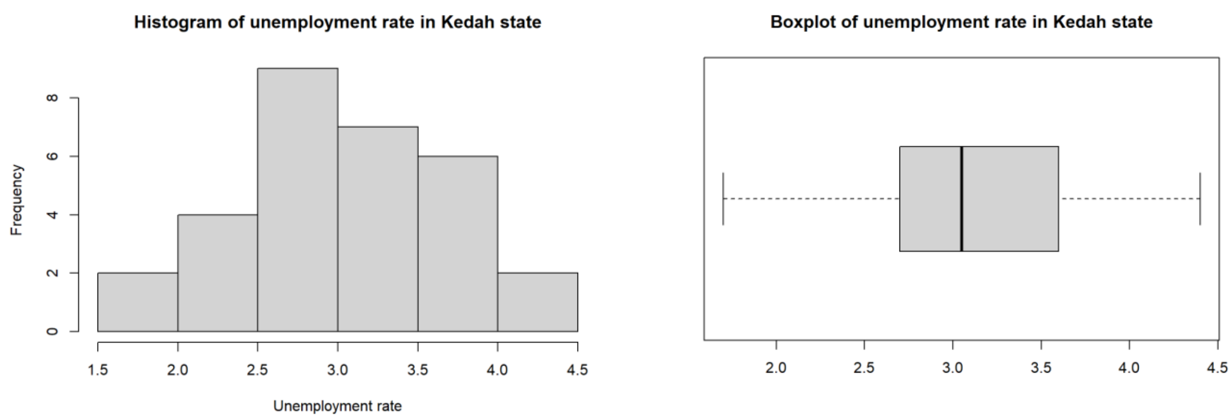

Figure 2. Histogram and boxplot of unemployment rate in Kedah state of Malaysia

Table 7. 95% CIs and length for various CIs for Kedah state unemployment rate data of Example 2

Methods	CI estimate	Length
t-CI	[2.85, 3.33]	0.48
pb-CI	[2.88, 3.33]	0.45
bt-CI	[2.84, 3.33]	0.49
dt-CI	[2.88, 3.30]	0.42
cdt-CI	[2.85, 3.33]	0.48

The results in Table 7 reveal that the dt-CI has the smallest length (0.42), followed by the pb-CI (0.45), with the t-CI and cdt-CI both having a length of 0.48, and the bt-CI having the largest length. Unlike Example 1, where the sample size is small, Example 2 uses a larger sample size (n=30), which leads to a switch in the performance of dt-CI and pb-CI.

Example 3

In this example, we consider the quarterly unemployment rates for all 16 states of Malaysia between 2017 and Q2 2024, with each state having 30 observations, resulting in a total sample size of n=16×30=480, as reported in data.gov.my [28].

The data has a skewness of 1.23, with the mean (3.6) being greater than the median (3.3). The histogram and boxplot in Figure 3 clearly suggest that the data distribution is positively skewed, with the boxplot indicating possible outliers on the right tail. The Shapiro-Wilk test of normality (W = 0.90738, p-value < 0.00001) provides strong evidence that the distribution of the quarterly unemployment rate is not normally distributed.

Given the large sample size (n=480), a t-test can be used to assess whether the population mean is indeed 3.6. The results of the t-test strongly suggest that the population distribution of unemployment could have a mean of 3.6, with a p-value of 0.9783. Estimates of the 95% confidence intervals for various methods are reported in Table 8.

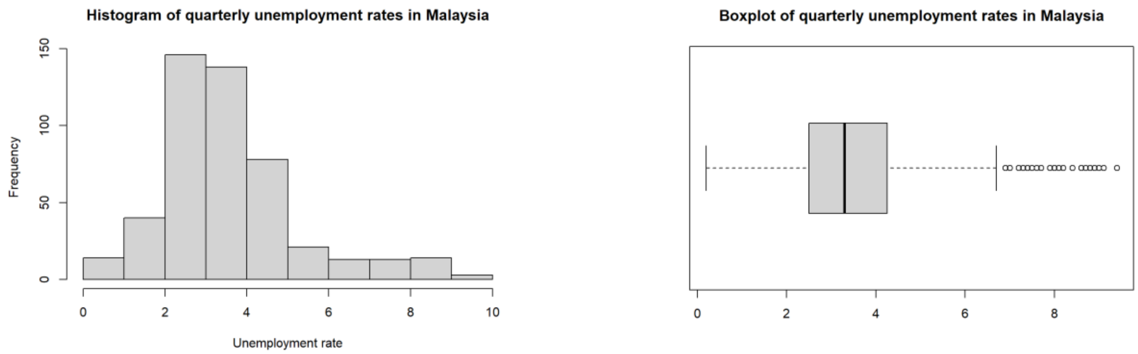

Figure 3. Histogram and boxplot of quarterly unemployment in Malaysia

Table 8. 95% CIs and length for quarterly unemployment rates in Malaysia for Example 3

Methods	CI estimate	Length
t-CI	[3.45, 3.75]	0.30
pb-CI	[3.46, 3.73]	0.27
bt-CI	[3.47, 3.74]	0.27
dt-CI	[3.31, 3.54]	0.23
cdt-CI	[3.45, 3.75]	0.30

As seen in other examples, the dt-CI has the smallest length (0.23), followed by the pb-CI and bt-CI, both with a length of 0.27. However, it is important to note that the dt-CI, [3.31,3.54], does not capture the hypothetical mean of 3.6, which aligns with the simulation results showing the poor performance of the dt-CI under conditions of large sample size and higher skewness. In contrast, the t-CI and the proposed cdt-CI both capture the hypothetical mean of 3.6, as do the other CI methods, except for the dt-CI. The consistent performance of the cdt-CI in terms of coverage probability is well supported, even though it comes at the cost of a relatively larger length.

Example 4

This example has been revisited from Example 2 (Mokhtar, Yusof & Sapiri, 2024), to reinvestigate why pb-CI reported in Table 8 (Mokhtar, Yusof & Sapiri, 2024) differs to a greater extent form other underlying CI.

43.4, 24, 1.8, 0, 0.1, 170.1, 0.4, 150, 31.5, 5.2, 35.7, 27.3, 5, 64.3, 70, 94, 61.9, 9.1, 38.8 and 14.8.

The data has a skewness of 1.45, and the fact that the mean (42.4) is greater than the median (29.4), along with the histogram and boxplot in Figure 2, all suggest that the data is positively skewed. The pb-CI reported in Mokhtar, Yusof & Sapiri (2024) as [0.05,160.55] with a length of 160.5, which differs significantly from the other CIs reported in Table 8 (Mokhtar, Yusof & Sapiri, 2024), possibly indicating an error. Upon revisiting this data, the 95% CIs for the various methods are presented in Table 9.

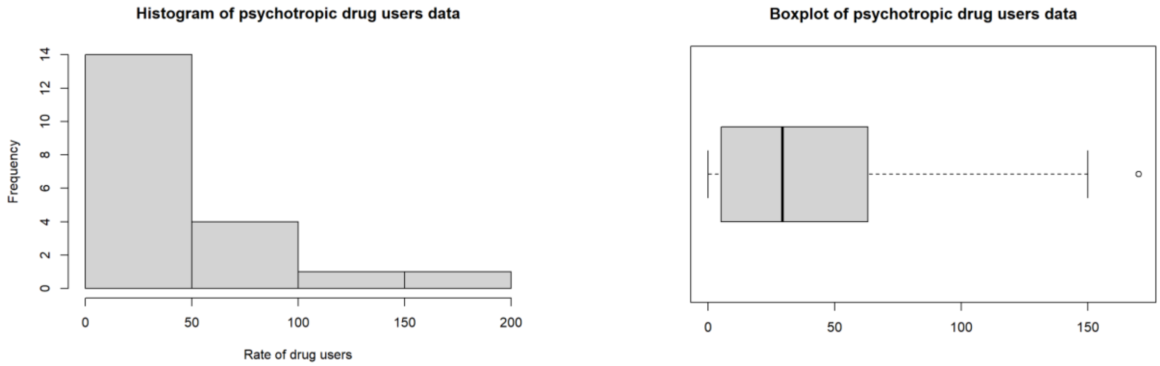

Figure 4. Boxplot and histogram of psychotropic drug user's data of Example 4

Table 9. 95% CIs and corresponding length for data in Example 4

Methods	CI estimate	Length
t-CI	[19.70, 65.04]	45.34
pb-CI	[22.59, 63.88]	41.29
bt-CI	[24.31, 81.09]	56.78
dt-CI	[19.34, 50.28]	30.94
cdt-CI	[19.41, 65.33]	45.92

From the reported results of Table 9, it is noted that length of dt-CI is the smallest (30.94), followed by pb-CI (41.29), t-CI (45.34), cdt-CI (45.92) and bt-CI (56.78). All these results appear to be close to what appears in Table 8 (Mokhtar, Yusof & Sapiri, 2024), except for pb-CI which differ significantly.

Conclusions

Several confidence interval (CI) methods were explored and evaluated through simulations and examples in a recent article by Mokhtar, Yusof & Sapiri (2024). While the authors proposed a new decile t-confidence interval (dt-CI), they noted a significant estimation issue with the method. However, the article provides little or no information on why the dt-CI performed poorly.

In this article, we provide an explanation for the estimation problem identified by Mokhtar, Yusof, and Sapiri (2024) concerning the decile t-confidence interval (dt-CI). Additionally, we propose and evaluate a new and corrective decile t confidence interval, termed the cdt-CI, which significantly mitigates the issues encountered by the dt-CI. We present several examples to demonstrate the usefulness of the cdt-CI method for both small and large samples exhibiting skewness.

Simulation results clearly demonstrate that, for normally distributed data or when skewness is negligible, the bt-CI, cdt-CI, and t-CI methods perform very well. However, when the underlying data exhibits skewness and potential outliers, the bt-CI performs best in most simulation cases in terms of estimated coverage probability. Often, the cdt-CI or t-CI is very close to the bt-CI. The newly proposed cdt-CI consistently outperforms the previously proposed dt-CI (Mokhtar, Yusof & Sapiri, 2024). In all simulations, except for the case of normal distribution, the performance of the bt-CI appears to be the best, followed by the cdt-CI and t-CI. The dt-CI performs very poorly, breaks down with increasing sample sizes and higher skewness values. Therefore, the dt-CI should not be used in cases involving large sample sizes or when skewness is expected.

It is important to note that the t-CI, dt-CI, and cdt-CI do not actually involve bootstrapping. Therefore, unlike Mokhtar, Yusof, and Sapiri (2024), who refer to the dt-CI as the Bootstrap-t DM CI, we have dropped the bootstrap terminology for the dt-CI. Of course, the t-CI, dt-CI, and cdt-CI are computationally less cumbersome and faster compared to the bootstrap-supported percentile CI (pb-CI) or bootstrap-t CI (bt-CI). Therefore, unless bootstrapping is the preferred choice, one can rely on the cdt-CI, as it is very comparable to the bt-CI in performance and better than the pb-CI or dt-CI.

Overall, for normal or nearly normal data with low skewness, the t-CI as well as cdt-CI should be preferred due to simplicity of applications and coverage probability considerations. In the presence of increasing skewness, the cdt-CI should be preferred over the pb-CI and bt-CI because the cdt-CI does not require bootstrapping and is therefore faster. However, if bootstrapping is an option, the bt-CI should be considered, as it is the most robust against increasing skewness and sample sizes. Researchers must also consider whether coverage probability or length is more important when selecting a CI, as it is challenging to identify a CI that achieves a coverage probability close to 0.95 while maintaining a short length. This study will be valuable for researchers with practical experience in deciding on the appropriate method for constructing a CI for the mean.

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