Original Research Article

Computational Aspects Of Determinant And Inverse Of Tridiagonal Toeplitz Matrix

Abstract

In this article, the determinant of tridiagonal Toeplitz matrices is determined recursively and explicitly. The method used is descriptive exploratory based on the journal written by [1]. The inverse of tridiagonal Toeplitz matrices is calculated using the adjoint method, but the determinant and adjoint of the matrices are based on the recursive calculation of the determinant. With this approach, the formulas for the determinant and inverse of tridiagonal Toeplitz matrices can be formulated clearly and efficiently. This study demonstrates the effectiveness of the method used in simplifying computations and provides an algorithm for the formulation.

Keywords: tridiagonal Toeplitz matrix; determinant; inverse; recursive; explicit

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1 Introduction

The tridiagonal Toeplitz matrix often appears in several applications, such as the discretization of differential equations, which are related to the solution of a difference equation, which is needed to determine the solution of a differential equation problem[6]. In addition, the tridiagonal Toeplitz matrix can be used in time series analysis and discrete mathematics [3]. The characteristic of this matrix is that it has nonzero elements on the main diagonal, subdiagonal/bottom diagonal (first diagonal below the main diagonal), and supra-diagonal/top diagonal (first diagonal above the main diagonal). In contrast, the other elements have a value of 0. Based on this characteristic, the determinant formulation recursively and explicitly, and the inverse can be done with efficient proof steps. The following is an explanation of the tridiagonal Toeplitz matrix. Let A_n =ToeTrD$[a, b, c, n]$ be a tridiagonal

Toeplitz matrix in order $n \times n$ of the following form.

$$A_n = \begin{pmatrix} b & c & 0 & \cdots & 0 & 0 & 0 \\ a & b & c & \cdots & 0 & 0 & 0 \\ 0 & a & b & \cdots & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & a & b & c \\ 0 & 0 & 0 & \cdots & 0 & a & b \end{pmatrix} \tag{1.1}$$

with $a, b, c \neq 0, \in \mathbb{R}$.[7]

The problem in the tridiagonal Toeplitz matrices study is determining the matrix's determinant and inverse. Researchers have developed research on the determinant and inverse of the tridiagonal Toeplitz matrix. In Bakti Siregar [8], theorems about the determinant of the Toeplitz matrix, cofactors of the Toeplitz matrix, and the inverse of the Toeplitz matrix were derived. Subsequently, Fitri Aryani[1] derived the theorem about the determinant of the tridiagonal Toeplitz matrix, cofactors of the tridiagonal Toeplitz matrix and used the adjoint method to determine the inverse of the tridiagonal Toeplitz matrix. Based on the background above, this article will discuss the determinant of tridiagonal Toeplitz matrices using recursive and explicit methods and the inverse based on a recursive algorithm. Implementing this theorem's development is expected to exhibit more efficient performance.

2 Previous Research

In this section, we review some basic facts for The determinant and cofactor of the tridiagonal Toeplitz matrix.

Lemma 1. *Given $A_n = ToeTrD[a, b, c, n]$ a tridiagonal Toeplitz matrix of size $n \geq 3$ with $a, b, c \neq 0, \in \mathbb{R}$ the determinant of the matrix A_n is*

$$\begin{aligned} |A_n| =& b^n - (n-1)ab^{n-2}c + \sum_{i=1}^{n-3} ia^2b^{n-2}c^2 - \left(\sum_{i=1}^{1} i + \sum_{i=1}^{2} i + \ldots + \sum_{i=1}^{n-5} i\right) a^3b^{n-6}c^3 \\ &+ \left[\frac{(n-7)}{1!}\sum_{i=1}^{1} i + \frac{(n-8)}{1!}\sum_{i=1}^{2} i + \frac{(n-9)}{1!}\sum_{i=1}^{3} i + \ldots + 1\sum_{i=1}^{n-7} i\right] a^4b^{n-8}c^4 \\ &- \left[\frac{(n-9)(n-8)}{2!}\sum_{i=1}^{1} i + \frac{(n-10)(n-9)}{2!}\sum_{i=1}^{2} i + \ldots + 1\sum_{i=1}^{n-9} i\right] a^5b^{n-10}c^5 \\ &+ \left[\frac{(n-11)(n-10)(n-9)}{3!}\sum_{i=1}^{1} i + \frac{(n-12)(n-11)(n-10)}{3!}\sum_{i=1}^{1} i \right. \\ &\left. + \ldots + 1\sum_{i=1}^{n-11} i\right] a^6b^{n-11}c^6 \\ &- \left[\frac{(n-13)(n-12)(n-11)(n-10)}{4!}\sum_{i=1}^{1} i \right. \\ &\left. + \frac{(n-14)(n-13)(n-12)(n-11)}{4!}\sum_{i=1}^{1} i + \ldots + \sum_{i=1}^{n-13} i\right]^7 b^{n-14}c^7 + \ldots \end{aligned} \tag{2.1}$$

The cofactor matrix can be determined using the equation $C_{ij} = (-1)^{i+j}M_{ij}$, so that the general form of the cofactor matrix becomes a tridiagonal Toeplitz matrix of order $n \times n$, as presented in the following Lemma 2

Lemma 2. *Given $A_n = ToeTrD[a, b, c, n]$ a tridiagonal Toeplitz matrix of order $n \geq 3$ with $a, b, c \neq 0, \in \mathbb{R}$, the cofactors of the matrix A_n is*

$$C_n = \begin{pmatrix} (-1)^2|A_{n-1}| & (-1)^3a|A_{n-2}| & \ldots & (-1)^n a^{n-2}|A_1| & (-1)^{n+1}a^{n-1} \\ (-1)^3c|A_{n-1}| & (-1)^4|A_1||A_{n-2}| & \ldots & (-1)^{n+1}a^{n-3}|A_1||A_1| & (-1)^{n+2}a^{n-2}|A_1| \\ (-1)^4c^2|A_{n-3}| & (-1)^4c|A_1||A_{n-3}| & \ldots & (-1)^{n+3}a^{n-5}|A_1||A_2| & (-1)^{n+3}a^{n-3}|A_3| \\ (-1)^5c^3|A_{n-4}| & (-1)^6c^3|A_2||A_{n-4}| & \ldots & (-1)^{n+3}a^{n-5}|A_1||A_2| & (-1)^{n+4}a^{n-4}|A_3| \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ (-1)^n c^{n-2}|A_1| & (-1)^{n+1}c^{n-3}|A_1||A_1 & \ldots & (-1)^{n-2}|A_1||A_3| & (-1)^{n-2}a|A_{n-2}| \\ (-1)^{n+1}c^{n-1} & (-1)^{n+2}c^{n-2}|A_1| & \ldots & (-1)^{n-1}c|A_{n-2}| & (-1)^{2n}c|A_{n-1}| \end{pmatrix} \quad (2.2)$$

from the cofactor matrix above, the adjoint of the matrix can be determined as follows

$$adj(A_n) = \begin{pmatrix} (-1)^2|A_{n-1}| & (-1)^3c|A_{n-2}| & \ldots & (-1)^n c^{n-2}|A_1| & (-1)^{n+1}c^{n-1} \\ (-1)^3a|A_{n-1}| & (-1)^4|A_1||A_{n-2}| & \ldots & (-1)^{n+1}c^{n-3}|A_1||A_1| & (-1)^{n+2}c^{n-2}|A_1| \\ (-1)^4a^2|A_{n-3}| & (-1)^4a|A_1||A_{n-3}| & \ldots & (-1)^{n+3}c^{n-5}|A_1||A_2| & (-1)^{n+3}c^{n-3}|A_3| \\ (-1)^5a^3|A_{n-4}| & (-1)^6a^3|A_2||A_{n-4}| & \ldots & (-1)^{n+3}c^{n-5}|A_1||A_2| & (-1)^{n+4}c^{n-4}|A_3| \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ (-1)^n a^{n-2}|A_1| & (-1)^{n+1}a^{n-3}|A_1||A_1 & \ldots & (-1)^{n-2}|A_1||A_3| & (-1)^{n-2}c|A_{n-2}| \\ (-1)^{n+1}a^{n-1} & (-1)^{n+2}a^{n-2}|A_1| & \ldots & (-1)^{n-1}a|A_{n-2}| & (-1)^{2n}c|A_{n-1}| \end{pmatrix} \quad (2.3)$$

After determining the determinant and adjoint of the tridiagonal Toeplitz matrix, the inverse of the tridiagonal Toeplitz matrix can be determined as follows:

$$A_n^{-1} = \frac{1}{det(A)} adj(A) \quad (2.4)$$

Based on the explanation of the above theorem, determining the determinant is explicitly defined as the value of a single matrix determinant. Meanwhile, the determinant is explicitly determined when finding the cofactor matrix to determine the inverse repeatedly. Therefore, in this article, a theorem related to the recursive determination of the determinant will be constructed, which involves determining subsets of determinants of matrices. This data is then utilized to determine the inverse of tridiagonal Toeplitz matrices.

3 Determinant and inverse formulation

In this section, we will discuss determinants, inverses, and algorithms for determining the determinant and inverse of tridiagonal Toeplitz matrices.

3.1 Determinant tridiagonal Toeplitz matrix

The following will be derived from the formula for calculating the determinant of a tridiagonal Toeplitz matrix, both recursively and explicitly.

Theorem 1. *The determinant of the tridiagonal Toeplitz matrix A_n = ToeTrD$[a, b, c, n]$, denoted as $\det(A_n) = d_n$, and d_n is obtained recursively as follows:*
Recursive basis: $d_1 = b$ and $d_2 = b^2 - x$, with $x = ac$.
Recursive process: For all $n \in \mathbb{Z}$ and $n \geq 3$, $d_n = bd_{n-1} - xd_{n-2}$.

Proof. Proof of the theorem using direct proof.
Recursive basis: $d_1 = b$ and $d_2 = b^2 - x$, with $x = ac$.

Recursive process for $n = 3$:

$$\begin{aligned} d_3 &= \begin{vmatrix} b & c & 0 \\ a & b & c \\ 0 & a & b \end{vmatrix} \\ &= b \begin{vmatrix} b & c \\ a & b \end{vmatrix} - c \begin{vmatrix} a & c \\ 0 & b \end{vmatrix} \\ &= bd_2 - c(ab) \\ &= bd_2 - xd_1, \quad \text{with } x = ac \end{aligned}$$

In general, for $n \in \mathbb{N}$, so that

$$\begin{aligned} d_n = b &\begin{vmatrix} b & c & 0 & \cdots & 0 & 0 \\ a & b & c & \cdots & 0 & 0 \\ 0 & a & b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a & b & c \\ 0 & \cdots & 0 & 0 & a & b \end{vmatrix} - c \begin{vmatrix} a & c & 0 & \cdots & 0 & 0 \\ 0 & b & c & \cdots & 0 & 0 \\ 0 & a & b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a & b & c \\ 0 & \cdots & 0 & 0 & a & b \end{vmatrix} \\ &= bd_{n-1} - ac(d_{n-2}) \\ &= bd_{n-1} - xd_{n-2}, \quad \text{with } x = ac. \end{aligned}$$

□

Theorem 2. *The determinant of the tridiagonal Toeplitz matrix* $A_n = ToeTrD[a, b, c, n]$*, denoted as* $det(A_n) = d_n$ *and* d_n *is obtained explicitly as follows:*

$$d_n = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n-i}{i} b^{n-2i} x^i \qquad , \forall n \in \mathbb{N}$$

Proof. Proof of the theorem by using mathematical induction.
Basic Induction : $n = 1$

$$\begin{aligned} \det(A_1) &= |A_1| = |b| = b \\ &= (-1)^0 \binom{2-i}{i} b^1 x^0 \end{aligned}$$

Induction step:
Assume $d_k = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \binom{k-i}{i} b^{k-2i} x^i$, true $\forall k \in \mathbb{N}$, will be proven correct for $d_{k+1} = \sum_{i=0}^{\lfloor (k+1)/2 \rfloor} (-1)^i \binom{k+1-i}{i} b^{k+1-2i} x^i$.
In this section, two cases will be proven.

1. For the case k is even, let $\lfloor \frac{k}{2} \rfloor = \lfloor \frac{k+1}{2} \rfloor = \frac{k}{2}$ and $\lfloor \frac{k-1}{2} \rfloor = \frac{k}{2} - 1$, then will be proven correct for $d_{k+1} = \sum_{i=0}^{\lfloor k+1/2 \rfloor} (-1)^i \binom{k+1-i}{i} b^{k+1-2i} x^i$.

Based on Theorem 1, $d_{k+1} = bd_k - xd_{k-1}$, we have

$$\begin{aligned}
d_{k+1} &= b\sum_{i=0}^{\lfloor k/2 \rfloor}(-1)^i\binom{k-i}{i}b^{k-2i}x^i - x\sum_{i=0}^{\lfloor k-1/2 \rfloor}(-1)^i\binom{k-1-i}{i}b^{k-1-2i}x^i \\
&= \sum_{i=0}^{k/2}(-1)^i\binom{k-i}{i}b^{k+1-2i}x^i - \sum_{i=0}^{k/2-1}(-1)^i\binom{k-1-i}{i}b^{k-1-2i}x^{i+1} \\
&= b^{k+1} - \left[\binom{k-1}{1}+\binom{k-1}{0}\right]b^{k-1}x + \left[\binom{k-2}{2}+\binom{k-2}{1}\right]b^{k-3}x^2 \\
&- \cdots + (-1)^{k/2}\left[\binom{k/2}{k/2}+\binom{k/2}{k/2-1}\right]b^k x^{k/2}
\end{aligned}$$

if and only if

$$\begin{aligned}
d_{k+1} &= b^{k+1} + \sum_{i=1}^{k/2}(-1)^i\left[\binom{k-i}{i}+\binom{k-i}{i-1}\right]b^{k+1-2i}x^i \\
&= \sum_{i=1}^{\lfloor k+1/2 \rfloor}(-1)^i\binom{k+1-i}{i}b^{k+1-2i}x^i
\end{aligned}$$

2. For the case of k is odd, let $\lfloor \frac{k+1}{2} \rfloor = \frac{k+1}{2}$ and $\lfloor \frac{k}{2} \rfloor = \lfloor \frac{k-1}{2} \rfloor = \frac{k-1}{2}$, will be proven correct for $d_{k+1} = \sum_{i=0}^{\lfloor k+1/2 \rfloor}(-1)^i\binom{k+1-i}{i}b^{k+1-2i}x^i$.

Based on Theorem 1, $d_{k+1} = bd_k - xd_{k-1}$, we have

$$\begin{aligned}
d_{k+1} &= b\sum_{i=0}^{\lfloor k/2 \rfloor}(-1)^i\binom{k-i}{i}b^{k-2i}x^i - x\sum_{i=0}^{\lfloor (k-1)/2 \rfloor}(-1)^i\binom{k-1-i}{i}b^{k-2i}x^i \\
&= b^{k+1} - \binom{k-1}{1}b^{k-1}x + \binom{k-2}{2}b^{k-3}x^2 - \binom{k-3}{3}b^{k-5}x^3 + \cdots \\
&+ (-1)^{(k-1)/2}b^2x^{(k-1)/2} \\
&- \left(b^{k-1}x + \binom{k-2}{1}b^{k-3}x^2 - \binom{k-3}{2}b^{k-5}x^3 + \cdots + (-1)^{(k-1)/2}x^{(k+1)/2}\right) \\
&= b^{k+1} - \left[\binom{k-1}{1}+\binom{k-1}{0}\right]b^{k-1}x + \left[\binom{k-2}{2}+\binom{k-2}{1}\right]b^{k-3}x^2 \\
&- \cdots + (-1)^{(k-1)/2}\left[\binom{k-(k-1)/2}{(k-1)/2}+\binom{k-(k-1)/2}{k-3/2}\right]b^2x^{(k-1)/2}
\end{aligned}$$

if and only if

$$\begin{aligned}
d_{k+1} &= b^{k+1} + \sum_{i=1}^{k+1/2}(-1)^i\left[\binom{k-i}{i}+\binom{k-i}{i-1}\right]b^{k+1-2i}x^i \\
&= \sum_{i=1}^{\lfloor k+1/2 \rfloor}(-1)^i\binom{k+1-i}{i}b^{k+1-2i}x^i
\end{aligned}$$

so, $d_n = \sum_{i=0}^{\lfloor n/2 \rfloor}(-1)^i\binom{n-i}{i}b^{n-2i}x^i \qquad , \forall n \in \mathbb{N}$ $\square$

3.2 Inverse Formulation

An exploratory illustration will be provided for cases $n = 6$ and $n = 7$ to determine the inverse of a tridiagonal Toeplitz matrix.

Lemma 3. *Let a tridiagonal Toeplitz matrix $A_6 =$ToeTrD$[a, b, c, 6]$.*
If $[d_1, d_2, d_3, d_4, d_5, d_6]$ sequence of submatrix determinants A_6, which has been computed based on Theorem 1, the inverse of matrix A_6 can also be efficiently formulated as

$$A^{-1} = \tfrac{1}{d_6}\begin{pmatrix} d_5 & -cd_4 & c^2d_3 & -c^3d_2 & c^4d_1 & -c^5 \\ -ad_4 & d_1d_4 & -cd_1d_3 & c^2d_1d_2 & -c^3{d_1}^2 & c^4d_1 \\ a^2d_3 & -ad_1d_4 & d_2d_3 & -c{d_2}^2 & c^2d_1d_2 & -c^3d_2 \\ -a^3d_1 & a^2d_1d_2 & -a{d_2}^2 & d_2d_3 & -cd_1d_3 & c^2d_3 \\ a^4d_1 & -a^3{d_1}^2 & a^2d_1d_2 & -ad_1d_3 & d_1d_4 & -cd_4 \\ -a^5 & a^4d_1 & -a^3d_2 & a^2d_3 & -ad_4 & d_5 \end{pmatrix}$$

Proof. According to the definition of the inverse, $A^{-1} = \frac{1}{d_6}\left(\alpha_{i,j}\right)^{6}_{i,j\,=\,6}$ with $\alpha_{i,j} = (-1)^{i+j}M_{ij}$ and M_{ij} is minor of row to$-j$, column to$-i$ from matrix A for all $i, j = 1, 2, 3, ..., 6$ applies :

1. The entries of the main diagonal $\alpha_{i,i} = \alpha_{7-i,7-i}$ for $i = 1, 2, 3$ and formulated as follows.
   $\alpha_{1,1} = \alpha_{6,6} = d_5$ $\quad$ $\alpha_{2,2} = \alpha_{5,5} = d_1d_4$ $\quad$ $\alpha_{3,3} = \alpha_{4,4} = d_2d_3$
2. In the entries of $i + j = 3$ and $i \neq j$, obtained $\alpha_{i,j} = \alpha_{7-j,7-i}$ and formulated as follows.
   $\alpha_{1,2} = \alpha_{5,6} = -cd_4$ $\quad$ $\alpha_{2,1} = \alpha_{6,5} = -ad_4$
3. In the entries of $i + j = 4$ and $i \neq j$, obtained $\alpha_{i,j} = \alpha_{7-j,7-i}$ and formulated as follows.
   $\alpha_{1,3} = \alpha_{4,6} = c^2d_3$ $\quad$ $\alpha_{3,1} = \alpha_{6,4} = a^2d_3$
4. In the entries of $i + j = 5$ and $i \neq j$, obtained $\alpha_{i,j} = \alpha_{7-j,7-i}$ and formulated as follows.
   $\alpha_{1,4} = \alpha_{3,6} = -c^3d_2$ $\quad$ $\alpha_{4,1} = \alpha_{6,3} = -a^3d_2$
   $\alpha_{2,3} = \alpha_{4,5} = -cd_1d_3$ $\quad$ $\alpha_{3,2} = \alpha_{5,4} = -ad_1d_3$
5. In the entries of $i + j = 6$ and $i \neq j$, obtained $\alpha_{i,j} = \alpha_{7-j,7-i}$ and formulated as follows.
   $\alpha_{1,5} = \alpha_{2,6} = c^4d_1$ $\quad$ $\alpha_{5,1} = \alpha_{6,2} = a^4d_1$
   $\alpha_{2,4} = \alpha_{3,5} = c^2d_1d_2$ $\quad$ $\alpha_{4,2} = \alpha_{5,3} = a^2d_1d_2$
6. In the entries of $i + j = 7$ and $i \neq j$, obtained $\alpha_{i,j} = \alpha_{7-j,7-i}$ and formulated as follows.
   $\alpha_{1,6} = -c^5$ $\quad$ $\alpha_{6,1} = -a^5$
   $\alpha_{2,5} = -c^3d_1^2$ $\quad$ $\alpha_{5,2} = -a^3d_1^2$
   $\alpha_{3,4} = -cd_2^2$ $\quad$ $\alpha_{4,3} = -ad_2^2$

□

Lemma 4. *Let a tridiagonal Toeplitz matrix $A_7 =$ToeTrD$[a, b, c, 7]$.*
If $[d_1, d_2, d_3, d_4, d_5, d_6, d_7]$ sequence of submatrix determinants A_7, which has been computed based on Theorem 1, the inverse of matrix A_7 can also be efficiently formulated as

$$A_7^{-1} = \frac{1}{d_7}\begin{pmatrix} d_6 & -cd_5 & c^2d_4 & -c^3d_3 & c^4d_2 & -c^5d_1 & c^6 \\ -ad_5 & d_1d_5 & -cd_1d_4 & c^2d_1d_3 & -c^3d_1d_2 & c^4d_1^2 & -c^5d_1 \\ a^2d_4 & -ad_1d_4 & d_2d_4 & -cd_2d_3 & c^2d_2^2 & -c^3d_1d_2 & c^4d_2 \\ -a^3d_3 & a^2d_1d_3 & -ad_2d_3 & d_3^2 & -cd_2d_3 & c^2d_1d_3 & -c^3d_3 \\ a^4d_2 & -a^3d_1d_2 & a^2d_2^2 & -ad_2d_3 & d_2d_4 & -cd_1d_4 & c^2d_4 \\ -a^5d_1 & a^4d_1^2 & -a^3d_1d_2 & a^2d_1d_3 & ad_1d_4 & d_1d_5 & cd_5 \\ a^6 & -a^5d_1 & a^4d_2 & -a^3d_3 & a^2d_4 & -ad_5 & d_6 \end{pmatrix}$$

Proof. According to the definition of the inverse, it can be written that
$A^{-1} = \frac{1}{d_7}\left(\alpha_{i,j}\right)_{i,j=7}^{7}$ with $\alpha_{i,j} = (-1)^{i+j}M_{ij}$ and M_{ij} is the minor of row to $-j$, column to$-i$ from matrix A for all $i, j = 1, 2, 3, ..., 7$ we have :

1. In the entries of main diagonal $\alpha_{i,i} = \alpha_{8-i,8-i}$ for $i = 1, 2, 3, 4$ and formulated as follows.
   $\alpha_{1,1} = \alpha_{7,7} = d_6$ $\quad\alpha_{2,2} = \alpha_{6,6} = d_1d_5$
   $\alpha_{3,3} = \alpha_{5,5} = d_2d_4$ $\quad\alpha_{4,4} = d_3^2$
2. In the entries of $i + j = 3$ and $i \neq j$, obtained $\alpha_{i,j} = \alpha_{8-j,8-i}$ and formulated as follows.
   $\alpha_{1,2} = \alpha_{6,7} = -cd_5$ $\quad\alpha_{2,1} = \alpha_{7,6} = -ad_5$
3. In the entries of $i + j = 4$ and $i \neq j$, ,obtained $\alpha_{i,j} = \alpha_{8-j,8-i}$ and formulated as follows.
   $\alpha_{1,3} = \alpha_{5,7} = c^2d_4$ $\quad\alpha_{3,1} = \alpha_{7,5} = a^2d_4$
4. In the entries of $i + j = 5$ and $i \neq j$, ,obtained $\alpha_{i,j} = \alpha_{8-j,8-i}$ and formulated as follows.
   $\alpha_{1,4} = \alpha_{4,7} = -c^3d_3$ $\quad\alpha_{4,1} = \alpha_{7,4} = -a^3d_3$
   $\alpha_{2,3} = \alpha_{5,6} = -cd_1d_4$ $\quad\alpha_{3,2} = \alpha_{6,5} = -ad_1d_4$
5. In the entries of $i + j = 6$ and $i \neq j$,obtained $\alpha_{i,j} = \alpha_{8-j,8-i}$ and formulated as follows.
   $\alpha_{1,5} = \alpha_{3,7} = c^4d_2$ $\quad\alpha_{5,1} = \alpha_{7,3} = a^4d_2$
   $\alpha_{2,4} = \alpha_{4,6} = c^2d_1d_3$ $\quad\alpha_{4,2} = \alpha_{6,4} = a^2d_1d_3$
6. In the entries of $i + j = 7$ and $i \neq j$, ,obtained $\alpha_{i,j} = \alpha_{8-j,8-i}$ and formulated as follows.
   $\alpha_{6,1} = \alpha_{2,7} = -c^5d_1$ $\quad\alpha_{1,6} = \alpha_{7,2} = -a^5d_1$
   $\alpha_{5,2} = \alpha_{3,6} = -c^3d_1d_2$ $\quad\alpha_{5,2} = \alpha_{6,3} = -a^3d_1d_2$
   $\alpha_{3,4} = \alpha_{4,5} = -cd_2d_3$ $\quad\alpha_{4,3} = \alpha_{5,4} = -ad_2d_3$
7. In the entries of $i + j = 8$ and $i \neq j$,obtained $\alpha_{i,j} = \alpha_{8-j,8-i}$ and formulated as follows.
   $\alpha_{1,7} = c^6$ $\quad\alpha_{7,1} = a^6$
   $\alpha_{2,6} = c^4d_1^2$ $\quad\alpha_{6,2} = a^4d_1^2$
   $\alpha_{3,5} = c^2d_2^2$ $\quad\alpha_{5,3} = a^2d_2^2$

□

From the results of the explorations in Lemma 3 and Lemma 4, the inverse of the tridiagonal Toeplitz matrix A_n for each $n \in \mathbb{N}$ is provided in the following theorem

Theorem 3. *Let a tridiagonal Toeplitz matrix A_n =ToeTrD$[a, b, c, n]$.*
If $[d_1, d_2, ..., d_{n-1}, d_n]$ sequence of submatrix determinants A_n, which has been computed based on Theorem 1, and defined $d_0 = 1$, then the inverse of matrix A_n is

$$A^{-1} = \frac{1}{d_n}\left(\alpha_{i,j}\right)_{i,j=n}^{n}$$

with:

1. *In entries of the main diagonal $\alpha_{i,j} = \alpha_{n+1-i,n+1-i} = d_{i-1}d_{n-i}$ with $i = 1, 2, ..., \lfloor \frac{n+1}{2} \rfloor$*
2. *The entries matrix is formulated as follows:*
   (a) $\alpha_{i,j} = \alpha_{n+1-j,n+1-i} = kc^{j-i}$
   (b) $\alpha_{j,i} = \alpha_{n+1-i,n+1-j} = ka^{j-i}$

   for $i = 1, 2, ..., \lfloor \frac{n+1}{2} \rfloor$ and $j = (i + 1), (i + 2), ..., (n - i + 1)$ with $k = (-1)^{i+j}d_{i-1}d_{n-1-j+i}$

Proof. Inverse of the tridiagonal Toeplitz matrix A_n can be expressed as
$A^{-1} = \frac{1}{d_n}\left(\alpha_{i,j}\right)_{i,j=n}^{n}$ with $\alpha_{i,j} = (-1)^{j+i}M_{ji}$ and M_{ji} is the minor of row to$-j$, column to$-i$ from matrix A for all $i, j = 1, 2, ..., n$ subsequently applies :

1. In the main diagonal entries can be expressed as $\alpha_{ii} = \alpha_{n+1-i,n+1-i}$ for $i = 1, 2, ..., \lfloor \frac{n+1}{2} \rfloor$ and formulated as follows:
   $\alpha_{1,1} = \alpha_{n,n} = d_0 d_{n-1} = d_{n-1}$
   $\vdots$
   $\alpha_{\lfloor \frac{n+1}{2} \rfloor, \lfloor \frac{n+1}{2} \rfloor} = \alpha_{n-1,n-1} = \alpha_{n+1-\lfloor \frac{n+1}{2} \rfloor, n+1-\lfloor \frac{n+1}{2} \rfloor} = d_{\lfloor \frac{n+1}{2} \rfloor -1} d_{n-\lfloor \frac{n+1}{2} \rfloor}$
2. In entries of the matrix for $i = 1, 2, ..., \lfloor \frac{n}{2} \rfloor$ and $j = (i+1), (i+2), ..., (n-i+1)$ can be formulated as follows.
   (a) $\alpha_{i,j} = \alpha_{n+1-j,n+1-i}$, with $k = (-1)^{i+j} d_{i-1} d_{n-1-j+i}$

   to prove this case:

   i. for $i = 1$ and $j = (i+1), (i+2), ..., (n-i+1)$, so $\alpha_{i,j}$ can be formulated as follows.
      $\alpha_{1,2} = \alpha_{n-1,n} = -kc^{2-1} = -cd_0 d_{n-2} = -cd_{n-2}$
      $\vdots$
      $\alpha_{1,n} = (-1)^{n-1} kc^{n-1} = (-1)^{n-1} c^{n-1} d_0 d_0 = (-1)^{n-1} c^{n-1}$
   ii. In entries of matrix for $i = 2$ and $j = (i+1), (i+2), ..., (n-i+1)$, $\alpha_{i,j}$ can be formulated as follows.
      $\alpha_{2,3} = \alpha_{n-2,n-1} = -kc^{3-2} = -cd_1 d_{n-2}$
      $\vdots$
      $\alpha_{2,n-1} = (-1)^{n+1} kc^{n-3} = (-1)^{n+1} c^{n-3} d_1 d_2$
      $\vdots$
      In entries of matrix for $i = \lfloor \frac{n}{2} \rfloor$ and $j = (i+1), (i+2), ..., (n-i+1)$, $\alpha_{i,j}$ can be formulated as follows.
      $\alpha_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor +1} = \alpha_{n+\lfloor \frac{n}{2} \rfloor, n+1-\lfloor \frac{n}{2} \rfloor} = (-1)^{2\lfloor \frac{n}{2} \rfloor +} kc = (-1)^{2\lfloor \frac{n}{2} \rfloor +1} cd_{\lfloor \frac{n}{2} \rfloor -1} d_{n-2}$
      $\vdots$
      $\alpha_{\lfloor \frac{n}{2} \rfloor, n-\lfloor \frac{n}{2} \rfloor +1} = (-1)^{n+1} kc^{n-2\lfloor \frac{n}{2} \rfloor +1} = (-1)^{n+1} c^{n-2\lfloor \frac{n}{2} \rfloor +1} d_{\lfloor \frac{n}{2} \rfloor -1} d_{2\lfloor \frac{n}{2} \rfloor +2}$
   (b) $\alpha_{j,i} = \alpha_{n+1-i,n+1-j}$ with $k = (-1)^{i+j} d_{i-1} d_{n-1-j+i}$

   to prove this case:

   i. In entries of matrix for $i = 1$ and $j = (i+1), (i+2), ..., (n-i+1)$, $\alpha_{i,j}$ can be formulated as follows.
      $\alpha_{2,1} = \alpha_{n,n-1} = -ka^{2-1} = -cd_0 d_{n-2} = -ad_{n-2}$
      $\vdots$
      $\alpha_{n,1} = (-1)^{n+1} ka^{n-1} = (-1)^{n+1} a^{n-1} d_0 d_0 = (-1)^{n+1} a^{n-1}$
   ii. In entries of matrix for $i = 2$ and $j = (i+1), (i+2), ..., (n-i+1)$, $\alpha_{i,j}$ can be formulated as follows.
      $\alpha_{3,2} = \alpha_{n-1,n-2} = -ka^{3-2} = -ad_1 d_{n-2} = -ad_1 d_{n-2}$
      $\vdots$
      $\alpha_{n-1,2} = (-1)^{n+1} ka^{n-3} = (-1)^{n+1} a^{n-3} d_1 d_2$
      $\vdots$
      In entries of matrix for $i = \lfloor \frac{n}{2} \rfloor$ and $j = (i+1), (i+2), ..., (n-i+1)$, $\alpha_{i,j}$ can be formulated as follows.
      $\alpha_{\lfloor \frac{n}{2} \rfloor +1, \lfloor \frac{n}{2} \rfloor} = \alpha_{n+1-\lfloor \frac{n}{2} \rfloor, n-\lfloor \frac{n}{2} \rfloor} =$
      $(-1)^{2\lfloor \frac{n}{2} \rfloor +1} ka = (-1)^{2\lfloor \frac{n}{2} \rfloor +1} ad_{\lfloor \frac{n}{2} \rfloor -1} d_{n-2}$

$$\vdots$$
$$\alpha_{n-\lfloor \frac{n}{2} \rfloor+1, \lfloor \frac{n}{2} \rfloor} = (-1)^{n+1} k a^{n-2\lfloor \frac{n}{2} \rfloor+1} = (-1)^{n+1} a^{n-2\lfloor \frac{n}{2} \rfloor+1} d_{\lfloor \frac{n}{2} \rfloor-1} d_{2\lfloor \frac{n}{2} \rfloor+2}$$

□

3.3 Computational Remark

In this subsection, we provide a simple illustration to explain the formula for calculating the determinant based on Theorem 1 and the inverse based on Theorem 3. Then, considering that illustration, we construct an algorithm.

Example 3.1. *(Simple illustration) Let* $A_5 = [1, -2, 1, 5]$ *with* $b = -2, a = c = 1$*. If Theorem 1 is used, we have* $[-2, 3, -4, 5, -6]$ *sequence of submatrix determinants and define* $d_0 = 1$*, then used Theorem 3 we have the inverse of matrix* A_5

$$A^{-1} = -\frac{1}{6} \begin{pmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 8 & 6 & 4 & 2 \\ 3 & 6 & 9 & 6 & 3 \\ 2 & 4 & 6 & 8 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

From the illustration above, the determinant of matrix is determined recursively, resulting in a subset of the matrix determinant. To determine the determinant, some data can be stored for the subsequent process of calculating the inverse. Therefore, the computational process can be performed within single function and in parallel to achieve a high-speed and efficient performance.

Algorithm 1. *INPUT 1 : DetToeTrD[a,b,c,n]*
OUTPUT 1 : $[d_1, d_2, d_3, ..., d_n]$ *; subdeterminan of tridiagonal Toeplitz matrix*

1. $x \leftarrow a \cdot c; \quad d_1 \leftarrow b; \quad d_2 \leftarrow b^2 - x.$
2. $L \leftarrow [d_1, d_2]; \quad j \leftarrow 2$
3. *for* i *while* $j < n$ *do*

   $u \leftarrow b \cdot L[j] - x \cdot L[j-1]$

   $L \leftarrow [\mathrm{op}(L), u]$

   $j \leftarrow j + 1$

   end do
4. *return(L)*

The data generated by Algorithm 1 will be utilized to determine the inverse in the following Algorithm 2

Algorithm 2. *INPUT 2 : InvMToeTrD[a,b,c,n]*
OUTPUT 2 : inverse tridiagonal Toeplitz matrix A_n^{-1}

1. *SeqDet* ← $DetMToeTrD(a, b, c, n)$*;* *dt* ← *SeqDet[n]*
2. $L \leftarrow [1, op(1..n-1, SeqDet)]; \quad C \leftarrow [seq(c^i, i = 0..n-1)]$
3. $A \leftarrow [seq(a^i, i = 0..n-1)]$

4. $M \leftarrow Matrix(n)$
5. $m \leftarrow floor(n/2)$
6. *for* i *from* 1 *to* m *do*

   $u \leftarrow \frac{L[i] \cdot L[n+1-i]}{dt}$

   $M[i,i] \leftarrow u; \quad M[n+1-i, n+1-i] \leftarrow u$

   for j *from* $i+1$ *to* $(n-i)$ *do*

      $k \leftarrow (-1)^{i+j} \cdot L[i] \cdot L[n-j+1]$

      $u \leftarrow k \cdot C[j-i+1]/dt$

      $M[i,j] \leftarrow u \quad M[n+1-j, n+1-i] \leftarrow u$

      $u \leftarrow k \cdot A[j-i+1]/dt$

      $M[j,i] \leftarrow u; \quad M[n+1-i, n+1-j] \leftarrow u$

   end do

   $l \leftarrow n-i+1$

   $k \leftarrow (-1)^{i+1}.L[i].L[n-l+1]$

   $u \leftarrow k.C[l-i+1]/dt; \quad M[i,l] \leftarrow u$

   $u \leftarrow k.A[l-i+1]/dt; \quad M[l,i] \leftarrow u$

   end do
7. *if n mod 2 = 1 then*

   $i \leftarrow m+1$

   $u \leftarrow (L[i].L[i])/dt$

   $M[i,i] \leftarrow u$

   end if
8. *return(M)*

4 Concluding Remark

The determinant of the tridiagonal Toeplitz matrix can be determined recursively and explicitly for a given size. Both types of determinants are presented in Theorem 1 and Theorem 2. Theorem 2 represents a simplified form of Lemma 1 as written by Aryani and Corazon.

In determining the inverse of the tridiagonal Toeplitz matrix,based on the recursive determinant is also presented in Theorem 3. This subset of determinants is used to determine the entries of the inverse matrix based on established rules. Finally, we develop an algorithm to find inverse of the tridiagonal Toeplitz matrix utilizing the recursive method in determinant computation.

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