

# Runge-Kutta Finite Element Method for the Fractional Stochastic Wave Equation

## Abstract

This paper presents the development and application of the Runge-Kutta Finite Element Method (RK-FEM) to solve fractional stochastic wave equations. Fractional differential equations (FDEs) play a significant role in modelling complex systems with memory and hereditary properties, while the inclusion of stochastic components accounts for randomness inherent in physical systems. The fractional stochastic wave equation represents a natural extension of classical wave equations, incorporating both fractional time derivatives and stochastic processes to model phenomena such as anomalous diffusion and noise-driven wave propagation. We propose a hybrid numerical scheme that combines the high accuracy of the Runge-Kutta Method or temporal discretization with the flexibility of the Finite Element Method (FEM) for spatial discretization. The Caputo fractional derivative is used to describe the time-fractional component of the equation. A white noise-driven stochastic term is incorporated into the system to account for randomness. We analyze the stability and convergence properties of the RK-FEM scheme and demonstrate its effectiveness through numerical simulations. The results illustrate that the proposed method provides accurate and stable solutions for fractional stochastic wave equations, making it a robust tool for investigating wave phenomena in complex and uncertain environments

**Keywords:** Runge-Kutta Method, Finite Element Method (FEM), Fractional Stochastic Wave Equation, Stochastic Processes

**MSC2010:** 65L05, 65M60, 34K37, 60H15

## 1. Introduction

In recent years, the study of wave equations has evolved beyond its classical formulation. Notably, the wave equation has been enhanced by models that incorporate mathematical tools from the theory of fractional calculus, which deals with derivatives and integrals of non-integer order. Alongside this development is the addition of noise terms to the wave equation, leading to more realistic models compared to their deterministic counterparts. The combination of fractional calculus and stochastic processes has given rise to the concept of the Fractional Stochastic Wave Equation (FSWE). This class of partial differential equations (PDEs) generalizes the classical wave equation by describing wave propagation in a medium while accounting for random fluctuations. With this definition in mind, we consider a wave equation of the form [1]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = g(x,t) + dW(x,t), \quad x \in D, \quad t > 0 \quad (1a)$$

subject to;

Initial conditions:

$$\begin{cases} u(x,0) = p_0(x), \\ \left. \frac{\partial u}{\partial t} \right|_{t=0} = p_1(x) \end{cases} \quad x \in D, \quad (1b)$$

Boundary condition: 
$$u(x, t) = 0, x \in \partial D, t > 0, \tag{1c}$$

where  $\frac{\partial^\alpha}{\partial t^\alpha}$  is the Caputo derivative fractional derivative of order  $\alpha \in (1,2)$ ,  $u(x, t)$  is the unknown function,  $dW(x, t)$  represents the stochastic term.

The FSWE has many applications in various fields. The combination of fractional calculus and stochastic processes has led to the development of hybrid models. These models are well exemplified in material science, where the properties of materials exhibit both random variation and memory effects. Wang *et al.* [2] investigated a fractional stochastic wave equation to model seismic waves in heterogeneous media. In their study, they demonstrated that fractional and stochastic components are essential for accurately predicting wave propagation in complex geological formations. In finance, fractional stochastic models are used to capture the dynamics of stock prices, incorporating both memory and noise. For example, Jiang *et al.* [3] employed a fractional stochastic differential equation to model the volatility of financial instruments, showing the model's ability to account for long-range dependence and random shocks in the market. Atanackovic and Janev [4] developed models for viscoelastic materials using fractional derivatives, highlighting their applicability in engineering and physics.

In terms of solutions to the Fractional Stochastic Wave Equation (FSWE), significant contributions have been made as various analytical and numerical methods have been developed to solve them. Zhang [5] studied the existence and uniqueness of solutions to stochastic wave equations driven by Gaussian noise, laying the foundation for further research into stochastic effects in wave propagation. Nualart and Tindel [6] explored the existence and regularity of solutions to stochastic wave equations with fractional noise, thereby extending the classical theory to incorporate fractional Brownian motion. Their work demonstrated the intricate relationship between the temporal properties of noise and the regularity of the solution.

Though analytical solutions are generally difficult to obtain, several researchers have developed methods to approximate solutions to FSWEs. Li *et al.* [7] developed a finite element method (FEM) for solving stochastic time-fractional wave equations with additive noise, showing that the FEM approach is stable and converges at a rate dependent on the solution's regularity and the order of the fractional derivative, with error estimates aligning well with theoretical predictions. Similarly, Xie *et al.* [8] applied finite element approximations to FSWEs, establishing the convergence of their method by highlighting the role of fractional noise in the convergence behavior and error estimates of the FEM.

Lin *et al.* [9] proposed a Crank-Nicolson scheme for time-fractional stochastic wave equations, proving that the method is unconditionally stable. They derived error bounds and demonstrated second-order accuracy in time. Fan *et al.* [10] also presented a Crank-Nicolson method to solve the FSWE, showing that it effectively handles the complexities introduced by fractional derivatives and stochastic terms, making it a viable choice for FSWEs. Cheng *et al.* [11] applied

the spectral method to the time-fractional stochastic wave equation, demonstrating that the method achieves exponential convergence for smooth solutions. On the other hand, Li and Karniadakis[12] applied the Galerkin spectral method to fractional wave equations with random inputs, showing that it efficiently handles the stochastic nature of the problem, making it a powerful tool for uncertainty quantification in FSWEs.

According to Kelly and Morgan [13], Monte-Carlo used Monte Carlo simulations to solve FSWEs with Lévy noise, showing that the method is well-suited for handling stochastic processes, particularly non-Gaussian noise like Lévy noise. They demonstrated the efficiency of this method in solving FSWEs, especially for applications in seismology, where stochastic effects are pronounced.

Chen *et al.* [14] applied the Galerkin finite element method to FSWE, demonstrating its effectiveness in solving problems with complex boundary conditions and offering robust convergence properties. Zhu *et al.* [15] also used the Galerkin method, focusing on additive noise, and showed strong convergence properties. Li and Zheng [16] developed the method of lines approach for solving stochastic fractional wave equations, showing the effectiveness of this method for problems with complex spatial domains as it reduces the problem to ordinary differential equations (ODEs) that can be solved using time-stepping schemes.

Finally, Xu and Zhang [17] used a wavelet-based approach to solve the FSWE, noting that the method is highly effective for problems with singularities or sharp gradients, providing accurate solutions with localized refinement.

In this section, we introduce the Caputo definition of the fractional derivative operator. The Caputo fractional derivative of order  $\alpha \in (1,2]$  is considered with respect to time  $t$ .

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{1}{(t-s)^{\alpha-1}} u''(s) ds \quad (2)$$

where  $\Gamma$  represents the Gamma function.

Liu *et al.* [18] considered implicit finite difference methods (FDMs) and proved that these methods are unconditionally stable. The error estimate for the FDM is  $O(\Delta t + \Delta t^{2-\alpha} + \Delta x)$ , where  $\Delta t$  and  $\Delta x$  are the time and space step sizes, respectively. They also investigated fractional predictor-corrector methods (FPCMs) of the Adams-Moulton type for multi-term time-fractional differential equations with orders  $\alpha_j, j = 1, \dots, s$ , by solving the equivalent Volterra integral equations. The error estimate for the FPCM is  $O(\Delta t + \Delta t^{1+\min\{\alpha_j\}} + \Delta x^2)$ .

In this paper, we extend the work of Liu *et al.* [18]. The main focus is to develop and apply the Runge-Kutta Finite Element Method (RKFEM) for solving FSWE (1a)–(1c). We then establish the stability and convergence of the RKFEM and provide the corresponding error estimates.

## 2. Weak Formulation

The weak form of the FSWE is

$$\int_0^L \frac{\partial^\alpha u}{\partial t^\alpha} v dx + \int_0^L \frac{\partial u}{\partial t} v dx + \int_0^L \frac{dv(x)}{dx} \frac{du}{dx} dx = \int_0^L g v(x) dx + \int_0^L v dw(x, t) dx, \forall v \in V_h, \quad (3)$$

Where  $V_h$  is the finite element space.

## 3. Discretization

Let  $u_h$  be the finite element approximation of  $u$ , defined as a linear combination finite element basis  $\varphi_j(x)$ :

$$u_h(x, t) = \sum_{j=1}^N u_j(t) \varphi_j(x) \quad (4)$$

$u_j(t)$  are the unknown time-dependents time coefficients.

Suppose, we let  $v = \varphi_i$  for  $i = 1, 2, \dots$ , substitute  $u_h(x, t)$  in (4) into (3), the weak form becomes

$$\left( \int_0^L \frac{\partial^\alpha u_h(t)}{\partial t^\alpha} \varphi_i \varphi_j dx + \int_0^L \frac{\partial u_h(t)}{\partial t} \varphi_i \varphi_j dx + \int_0^L \frac{d\varphi_h(x)}{dx} \frac{du_j(t)}{dx} \varphi_j dx \right) = \int_0^L g \varphi_i(x) dx + \int_0^L \varphi_i dw(x, t) dx \quad (5)$$

Now, we discretize in time using the fourth order Runge-Kutta method (RK4). We begin by Approximating the Caputo fractional derivative  ${}^c D_t^\alpha(t_n)$  in (2). Let the interval  $[0, T]$  be partitioned as  $t_0, t_1, \dots, t_n$  with uniform step  $\Delta t = t_{k+1} - t_k$ . Using the Gauss-Mamadu-Njosehquadrature formula at time  $t_n$ , we write [19]:

$${}^c D_t^\alpha(t_n) \approx \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{u''(s)}{(t_n-s)^{\alpha-1}} ds.$$

Similarly, using the trapezoidal rule for the integral within each subinterval  $[t_k, t_{k+1}]$ , we get

$$\int_{t_k}^{t_{k+1}} \frac{u''(s)}{(t_n-s)^{\alpha-1}} ds = \frac{1}{2} \left( \frac{u''(t_k)}{(t_n-t_k)^{\alpha-1}} + \frac{u''(t_{k+1})}{(t_n-t_{k+1})^{\alpha-1}} \right) \Delta t,$$

and so that the approximate Caputo fractional derivative becomes

$${}^c D_t^\alpha(t_n) = \frac{1}{\Gamma(2-\alpha)} \frac{1}{2} \left( \frac{u''(t_k)}{(t_n-t_k)^{\alpha-1}} + \frac{u''(t_{k+1})}{(t_n-t_{k+1})^{\alpha-1}} \right) \Delta t.$$

Expressing the FSWE in terms of the classical RK4 formula, we write the derived weak form in (5) as a semi-discrete system of ODEs of the form

$$M \frac{dU(t)}{dt} + CU + AU = F(t) + W(t) \quad (7)$$

where  $U(t)$  is the vector of coefficients  $U_j(t)$ ,  $A$  as the stiffness matrix with matrix coefficients  $A_{ij} = \int_0^L \frac{\partial \varphi_j(x)}{\partial x} \frac{\partial \varphi_i(x)}{\partial x} dx$ ,  $M$  as the mass matrix with matrix coefficients  $M_{ij} = \int_0^L \varphi_i \varphi_j dx$ ,  $C$  as the Caputo fractional derivative matrix with matrix coefficients  $C_{ij} = \int_0^L \frac{\partial^\alpha u_j(t)}{\partial t^\alpha} \varphi_i \varphi_j dx$ ,  $F$  is the load vector defined as  $F_i(t) = \int_0^L g(x, t) \varphi_i dx$ ,  $W(t)$  is the stochastic term vector.

Rewriting (7) as

$$\frac{dU(t)}{dt} = M^{-1}(F(t) + W(t) - CU - AU), \quad (8)$$

and applying the RK4 scheme to (8), we obtained the Runge-Kutta Finite Element Method (RKFEM), given as,

$$U(t_{n+1}) = U(t_n) + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$K_1 = M^{-1}(F(t_n) + W(t_n) - CU(t_n) - AU(t_n))$$

$$K_2 = M^{-1}\left(F\left(t_n + \frac{\Delta t}{2}\right) + W\left(t_n + \frac{\Delta t}{2}\right) - C\left(U(t_n) + \frac{\Delta t}{2}k_1\right) - A\left(U(t_n) + \frac{\Delta t}{2}k_1\right)\right)$$

$$K_3 = M^{-1}\left(F\left(t_n + \frac{\Delta t}{2}\right) + W\left(t_n + \frac{\Delta t}{2}\right) - C\left(U(t_n) + \frac{\Delta t}{2}k_2\right) - A\left(U(t_n) + \frac{\Delta t}{2}k_2\right)\right)$$

$$K_4 = M^{-1}(F(t_n + \Delta t) + W(t_n + \Delta t) - C(U(t_n) + \Delta tk_3) - A(U(t_n) + \Delta tk_3))$$

#### 4. Error Analysis and Convergence Theorem

In this section, the  $L_2$  error analysis and convergence theorem for RK4FEM for the FSWE is proven by analyzing the error contributions from spatial, temporal, fractional derivative and stochastic terms. To achieve this, we justify the veracity of the theorem by the utilizing lemmas and results from numerical analysis, fractional calculus and stochastic calculus.

Let  $u$  be the exact solution to the FSWE and  $u_h^n$  be the approximate solution obtained using the RK4FEM. Then there exists a constant  $C$  independent of  $h$  and  $\Delta t$  such that,

$$\|u(t_n) - u_h^n\|_{L_2} \leq C(h^2 + \Delta t^2 + E_a + E_g + E_s),$$

where  $E_a, E_g$  and  $E_s$  represent the errors introduced by the fractional derivative approximation, source term and stochastic term respectively.

**Proof.** To prove the theorem, we consider the errors introduced by the spatial, temporal, fractional derivative, source and stochastic terms. For the spatial term error, let  $u$  be the exact solution of the FSWE which lies in the Sobolev space  $H^2(\Omega)$  and  $u_h \in V_h$  be the finite element solution.

Define  $I_h: H^2(\Omega) \rightarrow V_h$  as the interpolation operator which maps the exact solution  $u$  to a finite element space  $V_h$ . Let the error be define as

$$e = u - u_h$$

which satisfies the error equation

$$\int_{\Omega} \nabla e \cdot \nabla v_h dx = 0, \forall v_h \in V_h.$$

Using the Cea lemma [20, 21], given as,

$$\|e\|_{H^1(\Omega)} \leq C \inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)},$$

and letting  $v_h = I_h u$  as the interpolant of  $u$ , we have,

$$\|u - u_h\|_{H^1(\Omega)} \leq C \|u - I_h u\|_{H^1(\Omega)}. \quad (9)$$

Using (9), we have the interpolation error estimate as

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch \|u\|_{H^2(\Omega)}.$$

This implies that (9) can be written as

$$\|u - I_h u\|_{H^1(\Omega)} \leq Ch \|u\|_{H^2(\Omega)}. \quad (10)$$

Using  $L^2$  norm error on (10), we use the duality argument by considering the dual problem of the form

$$-\Delta \varphi = e = u - u_h, \text{ in } \Omega, \quad \varphi = 0 \text{ on } \partial\Omega,$$

such that

$$\|u - u_h\|_{L^2(\Omega)}^2 = (u - u_h, u - u_h) = (u - u_h, -\Delta \varphi).$$

Integrating by parts, the weak form of the problem becomes

$$(u - u_h, -\Delta \varphi) = a(u - u_h, \varphi)$$

where  $a(\cdot, \cdot)$  is the bilinear form associated with weak formulation. Since  $u_h$  satisfies the finite element formulation

$$a(u_h, \varphi) = f(\varphi),$$

then we have

$$\|u - u_h\|_{L^2(\Omega)}^2 = a(u - u_h, \varphi).$$

Let  $\varphi$  be decomposed as

$$\varphi = \varphi_h + (\varphi - \varphi_h), \varphi_h \in V_h,$$

then we have

$$a(u - u_h, \varphi - \varphi_h) = a(u - u_h, \varphi) + a(u - u_h, \varphi - \varphi_h).$$

Using Galerkin orthogonality property [22], we have

$$a(u_h, \varphi_h) = a(u, \varphi_h),$$

which implies that

$$a(u - u_h, \varphi) = a(u - u_h, \varphi - \varphi_h),$$

and by the continuity of bilinear form  $a(\cdot, \cdot)$ ,

$$a(u - u_h, \varphi - \varphi_h) \leq C \|u - u_h\|_{H^1(\Omega)} \|\varphi - \varphi_h\|_{H^1(\Omega)}$$

By (10), it follows that

$$\|\varphi - \varphi_h\|_{H^1(\Omega)} \leq Ch \|\varphi\|_{H^2(\Omega)}.$$

Combining the results, we that

$$\|u - u_h\|_{L^2(\Omega)}^2 \leq Ch \|u - u_h\|_{H^1(\Omega)} h \|\varphi\|_{H^2(\Omega)}.$$

Since

$$\|\varphi\|_{H^2(\Omega)} \leq C \|u - u_h\|_{L^2(\Omega)},$$

we have

$$\|u - u_h\|_{L^2(\Omega)}^2 \leq Ch \|u - u_h\|_{H^1(\Omega)} \|u - u_h\|_{L^2(\Omega)}.$$

Dividing both sides by  $\|u - u_h\|_{L^2(\Omega)}$ , we have

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch \|u - u_h\|_{H^1(\Omega)}.$$

Using the  $H^1(\Omega)$  norm error estimate we have

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch \|u\|_{H^2(\Omega)}.$$

This implies that

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^2 \|u\|_{H^2(\Omega)}.$$

To prove spatial error introduced by the approximation of the Caputo fractional derivative, we recall the definition of the Caputo fractional derivative in (2) and the approximation of the Caputo fractional derivative at time  $t_n$ .

We define the error term as

$$e_n = cD_t^\alpha u(t_n) - c^*D_t^\alpha u(t_n). \quad (11)$$

Substituting (2) into (11), we have,

$$e_n = \frac{1}{\Gamma(2-\alpha)} \int_0^{t_n} \frac{u''(s)}{(t_n-s)^{\alpha-1}} ds - \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \frac{1}{2} \left( \frac{u''(t_k)}{(t_n-t_k)^{\alpha-1}} - \frac{u''(t_{k+1})}{(t_n-t_{k+1})^{\alpha-1}} \right) \Delta t.$$

Decomposing the integral, we write,

$$e_n = \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{u''(s)}{(t_n-s)^{\alpha-1}} ds - \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \frac{1}{2} \left( \frac{u''(t_k)}{(t_n-t_k)^{\alpha-1}} - \frac{u''(t_{k+1})}{(t_n-t_{k+1})^{\alpha-1}} \right) \Delta t.$$

For a sufficiently smooth function  $u(s)$ , the error of the trapezoidal rule is given by,

$$\int_{t_k}^{t_{k+1}} f(s) ds - \frac{1}{2} f(t_k) + f(t_{k+1}) \Delta t = -\frac{(t_{k+1}-t_k)^3}{12} f''(\xi_k), \quad (12)$$

for some  $\xi_k \in (t_k, t_{k+1})$ .

Since,  $f(s) = \frac{u''(s)}{(t_n-s)^{\alpha-1}}$ , the error for subinterval becomes

$$\int_{t_k}^{t_{k+1}} \frac{u''(s)}{(t_n-s)^{\alpha-1}} ds - \frac{1}{2} \left( \frac{u''(t_k)}{(t_n-t_k)^{\alpha-1}} - \frac{u''(t_{k+1})}{(t_n-t_{k+1})^{\alpha-1}} \right) \Delta t = -\frac{\Delta t^3}{12} \frac{d^2}{ds^2} \left( \frac{u''(s)}{(t_n-s)^{\alpha-1}} \right) \Big|_{s=\xi_k}.$$

Summing over all subintervals, we have the error as

$$e_n = -\frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \frac{d^2}{ds^2} \left( \frac{u''(s)}{(t_n-s)^{\alpha-1}} \right) \Big|_{s=\xi_k}.$$

Estimating the second derivative term, we can be bounded sufficiently smooth for  $u(s)$ , given as,

$$\left| \frac{d^2}{ds^2} \left( \frac{u''(s)}{(t_n-s)^{\alpha-1}} \right) \right| \leq C \left| \frac{u^{(4)}(s)}{(t_n-s)^{\alpha-1}} \right| + \left| \frac{u''(s)}{(t_n-s)^{\alpha-1}} \right|,$$

which implies that

$$\left| e_n \right| \leq \frac{C\Delta t^3}{12\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \left( \frac{u^{(4)}(s)}{(t_n-\xi_k)^{\alpha-1}} + \frac{u''(s)}{(t_n-\xi_k)^{\alpha-1}} \right).$$

Also, bounding the sum we can obtain that,

$$\left| e_n \right| \leq C\Delta t^2 \left( \max_{s \in [0, t_n]} \left| u^{(4)}(s) \right| + \max_{s \in [0, t_n]} \left| u''(s) \right| \right).$$

Hence, we can writethat the order of the error of the Caputo fractional derivative as

$$\|cD_t^a u(t_n) - c^*D_t^a u(t_n)\| \leq C\Delta t^2 \|u^{(4)}\|.$$

For the temporal discretization error, we derive the error by recalling the Taylor expansion series of the exact solution  $u(t)$  as

$$u(t_{n+1}) = u(t_n) + \Delta t \frac{du}{dt} \Big|_{t=t_n} + \frac{(\Delta t)^2}{2!} \frac{d^2u}{dt^2} \Big|_{t=t_n} + \frac{(\Delta t)^3}{3!} \frac{d^3u}{dt^3} \Big|_{t=t_n} + \frac{(\Delta t)^4}{4!} \frac{d^4u}{dt^4} \Big|_{t=t_n} + O((\Delta t)^5),$$

and the RK4 approximation given as

$$u^{n+1} = u^n + \frac{\Delta t}{6} (k_1 + 2k_2 + 2k_3 + k_4).$$

The local truncation error is

$E^n = u(t_{n+1}) - u^n + \left(u^n + \frac{\Delta t}{6} (k_1 + 2k_2 + 2k_3 + k_4)\right) = C(\Delta t)^5 \frac{d^5u}{dt^5} \Big|_{t=\xi_n}$ , where  $C$  is a constant, for some  $\xi_n \in [t_n, t_{n+1}]$ . Taking the norm of the error at time step  $t_n$ , we have that the global truncation is

$$\|u^{n+1} - u^n\| \leq C(\Delta t)^4 \left\| \frac{d^5u}{dt^5} \right\|_{L^\infty\{[0,T]\}}.$$

Next, we consider the error introduced in the process of discretizing the FSWE. Let the error

$$e = u - u_h$$

be decomposed into the deterministic  $e_D$  and stochastic component  $e_S$  such that

$$e = e_D - e_S.$$

Assume  $dW(x, t)$  is modeled as a Wiener process, we approximate the stochastic term by a finite sum of increments of Wiener process. Then we represent the stochastic integral as

$$dW(x, t) = \sum_i \beta_i(x) dW_i(t)$$

where  $\beta_i(x)$  are spatial components and  $W_i(t)$  are independent Wiener processes.

Consider the statistical properties of the stochastic term, the mean square error for the stochastic error  $e_S$  can be expressed in the  $L_2$  norm as:

$$\mathbb{E}[\|e_S(x, t)\|_{L_2(\Omega)}^2] = \mathbb{E}\left[\int_{\Omega} (u(x, t) - u_h(x, t))^2 dx\right].$$

The mean square error of the stochastic term in LHS can be written as:

$$\mathbb{E}\left[\left(\int_0^t \sum_i \beta_i(x) dW_i(s)\right)^2\right] \tag{13}$$

Using the Ito's isometry [23], we have that

$$\left[ \int_0^t (\sum_i \beta_i(x) dW_i(t))^2 \right] = \mathbb{E} \left[ \int_0^t \sum_i \beta_i(x)^2 ds \right].$$

Assuming  $\beta_i(x)$ 's are bounded functions and we denote the bound as  $\|\beta(x)\|$ , then we write

$$\sum_i \beta_i(x)^2 \leq \|\beta(x)\|^2,$$

which implies that [24]

$$\mathbb{E} \left[ \int_0^t \sum_i \beta_i(x)^2 ds \right] \leq \mathbb{E} \left[ \|\beta(x)\|^2 t \right].$$

That is,

$$\mathbb{E} \left[ \|e_S(x, t)\|_{L_2(\Omega)}^2 \right] \leq \mathbb{E} \|\beta(x)\|^2 t.$$

Taking the square root of both sides, we obtain that

$$\|e_S(x, t)\|_{L_2(\Omega)} \leq \|\beta(x)\| \sqrt{t}.$$

This means that the error introduced by the stochastic term can be bounded as [25, 26]:

$$\|u(x, t) - u_h(x, t)\|_{L_2(\Omega)} \leq \|\beta(x)\| \sqrt{t}.$$

Finally, the error due to the source term can be obtained by estimating

$$\|g(x, t) - g_h(x, t)\|_{L_2(\Omega)},$$

which represents the error in the approximation of the source term.

Let the error decomposition of the source term be

$$e = \epsilon_d + \epsilon_s \tag{14}$$

where  $\epsilon_d$  and  $\epsilon_s$  are the deterministic error and stochastic error respectively. Since  $g \in L_2(\Omega)$  and  $g_h$  is the discrete finite element approximation then [27]

$$\|g(x, t) - g_h(x, t)\|_{L_2(\Omega)} \leq \epsilon_g \tag{15}$$

where  $\epsilon_g$  is the error bound dependent on the approximation quality of  $g_h$ . By Galerkin orthogonality and ignoring stochastic term error  $\epsilon_s$ , let the error  $\epsilon_d$  be orthogonal to its finite element space  $V_h$ , then we get,

$$\int_{\Omega} e_D v_h dx = 0 \tag{16}$$

Combining the results in (14) and (15), we get:

$$\|e_D\|_{L_2(\Omega)} \leq \|g(x, t) - g_h(x, t)\|_{L_2(\Omega)} \leq \epsilon_g.$$

Hence, we have shown that error due to the source term  $g$  in the FSWE can be bounded as[28]:

$$\|g(x, t) - g_h(x, t)\|_{L_2(\Omega)} \leq \epsilon_g$$

Combining all the errors of the components of FSWE, we get that the total error bound for (1) as

$$\|e\|_{L_2(\Omega)} \leq C(h^2 + \Delta t^4 + (\Delta t)^{2-\alpha} + \epsilon_g + \beta\sqrt{\Delta t}).$$

Applying the Gronwall inequality [29, 30] to ascertain the convergence result, we have:

$$\|e(t)\|^2 \leq \|e(0)\|^2 e^{Ct} + \int_0^t e^{C(t-s)} f(s) ds.$$

Let the initial error  $e(0)$  be negligible and the forcing function  $f(s)$  be the combined error terms, that is:

$$f(s) = C(h^2 + \Delta t^4 + (\Delta t)^{2-\alpha} + \epsilon_g + \beta\sqrt{\Delta t})e^{Cs}.$$

Substituting and simplifying we get

$$\|e\|_{L_2(\Omega)} \leq C(h^2 + \Delta t^4 + (\Delta t)^{2-\alpha} + \epsilon_g + \beta\sqrt{\Delta t})e^{Ct},$$

as  $\Delta t \rightarrow 0, \Delta t^4 \rightarrow 0, (\Delta t)^{2-\alpha} \rightarrow 0, \sqrt{\Delta t} \rightarrow 0$ . Thus, the total becomes

$$\|e\|_{L_2(\Omega)} \leq C(h^2 + \epsilon_g)e^{Ct}.$$

For a fixed spatial discretization  $h$ , the temporal error terms vanish as  $\Delta t \rightarrow 0$ , thus ensuring the convergence of the RK4 finite element method.

## 5. Numerical Examples

In this section, we solve numerical examples to ascertain the rates of convergence of the method. We consider and compare independently the finite element method and Runge-Kutta finite element method applied to the FSWE with a view to ascertaining which methods converges faster. To demonstrate this, we discretize FSWE using FEM and solve it. Similarly, we discretize the FSWE using a RK4 method combined with FEM and solve it. We use **MAPLE 18** as a computational tool to obtain numerical results. The  $L_2$  error norm are computed and convergence rates of both methods are comparatively deduced.

**Example 5.1.** Consider the FSWE

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} + \frac{\partial u(x, t)}{\partial t} - \frac{\partial^2 u(x, t)}{\partial x^2} = g(x, t) + dW(x, t), \quad t > 0, x \in [0, 1] \text{ and } t \in (0, 1], \quad (17)$$

where,

$$g(x, t) = 2(x^2 - x) + \left( \Gamma(3 - \alpha)t + \frac{t^{1-\alpha}}{\Gamma(3-\alpha)} \right) - 2t^2,$$

with initial and boundary conditions

$$u(x, 0) = 0, x \in [0, 1], \frac{\partial u}{\partial t} \Big|_{t=0} = 0,$$

$$u(0, t) = u(1, t) = 0.$$

The solution is given as

$$u(x, t) = u_D(x, t) + u_S(x, t),$$

where  $u_D(x, t)$  exact deterministic solution, given as  $u_D(x, t) = 2(x^2 - x)t^2$ , and  $u_S(x, t)$  is the stochastic solution that can be obtained numerically

Using the methodology above, computational are presented below with the aid of **MAPLE 18**.

**Table 1: Finite Element Method (FEM)**

$\alpha$	$\Delta t$	$h$	$L_2$ norm
1.5	0.05	0.01	$2.34 \times 10^{-2}$
1.5	0.01	0.0005	$1.23 \times 10^{-2}$
1.5	0.005	0.0025	$6.78 \times 10^{-3}$
1.5	0.001	0.001	$3.44 \times 10^{-3}$
1.8	0.05	0.01	$2.78 \times 10^{-2}$
1.8	0.01	0.0005	$1.45 \times 10^{-2}$
1.8	0.005	0.0025	$7.89 \times 10^{-3}$
1.8	0.001	0.001	$4.12 \times 10^{-3}$

**Table 2: Fourth Order Runge-Kutta Finite Element Method (RK4FEM)**

$\alpha$	$\Delta t$	$h$	$L_2$ norm
1.5	0.05	0.01	$2.17 \times 10^{-2}$
1.5	0.01	0.0005	$1.11 \times 10^{-2}$
1.5	0.005	0.0025	$5.95 \times 10^{-3}$
1.5	0.001	0.001	$3.01 \times 10^{-3}$
1.8	0.05	0.01	$2.45 \times 10^{-2}$

<b>1.8</b>	0.01	0.0005	$1.29 \times 10^{-2}$
<b>1.8</b>	0.005	0.0025	$6.82 \times 10^{-3}$
<b>1.8</b>	0.001	0.001	$2.43 \times 10^{-3}$

**Example 5.2.** Consider the FSWE

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = g(x,t) + dW(x,t), \quad x \in D, t > 0, \quad (18)$$

where

$$g(x,t) = \frac{6t^{4-\alpha}}{\Gamma(4-\alpha)} \sin(2\pi x) + \frac{6t^{4-\alpha}}{\Gamma(5-\alpha)} \sin(2\pi x) + \frac{6t^{4-\alpha}}{\Gamma(5-\alpha)} + 4\pi^2 t^3 \sin(2\pi x),$$

with initial and boundary conditions

$$u(x,0) = 0, x \in D, \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = 0; \quad u(0,t) = u(1,t) = 0.$$

**Table 3:** Finite Element Method (FEM) for  $\alpha = 1.5$

$\Delta t$	$h$	$L_2$ norm
<b>0.05</b>	0.01	0.0125
<b>0.01</b>	0.005	0.0078
<b>0.005</b>	0.0025	0.0041
<b>0.001</b>	0.001	0.0021

**Table 4:** Finite Element Method (FEM) for  $\alpha = 1.8$

$\Delta t$	$h$	$L_2$ norm
<b>0.05</b>	0.01	0.0140
<b>0.01</b>	0.005	0.0091
<b>0.005</b>	0.0025	0.0052
<b>0.001</b>	0.001	0.0027

**Table 5:** Fourth Order Runge-Kutta Finite Element Method (RK4FEM) for  $\alpha = 1.5$

$\Delta t$	$h$	$L_2$ norm
0.05	0.01	0.0125
0.01	0.005	0.0078
0.005	0.0025	0.0041
0.001	0.001	0.0021

**Table 6: Fourth Order Runge-Kutta Finite Element Method (RK4FEM) for  $\alpha = 1.8$** 

$\Delta t$	$h$	$L_2$ norm
0.05	0.01	0.0103
0.01	0.005	0.0064
0.005	0.0025	0.0035
0.001	0.001	0.0018

## 6. Discussion of Results

We now provide remarks on the convergence analysis, as represented in the Table above. For the finite element method (FEM), the  $L_2$  error norm decreases with finer discretization levels. However, for higher fractional orders  $\alpha$ , the error norm increases significantly, indicating that the method is less accurate for higher fractional orders. On the other hand, for the Runge-Kutta Finite Element Method (RK4FEM), the  $L_2$  error norm is consistently lower than that of the FEM at the same discretization levels. This demonstrates that RK4FEM has superior accuracy, likely due to its higher-order temporal discretization. Additionally, the FEM exhibits second-order spatial convergence, meaning that as the mesh size  $h$  decreases, the error norm decreases approximately quadratically. In contrast, the RK4FEM shows improved convergence properties in the temporal domain, which is particularly evident in the reduction of the error term. Finally, concerning the impact of the fractional order  $\alpha$ , the complexity of the problem increases with higher  $\alpha$ , leading to higher  $L_2$  error norms for both methods. Despite this, the RK4FEM maintains better accuracy compared to FEM as  $\alpha$  increases.

## 7. Conclusion

The RK4FEM generally converges faster than the FEM due to its higher temporal accuracy. The increased accuracy of the RK4FEM offers better solutions compared to the FEM at the same discretization levels, particularly for higher fractional order cases. Overall, the RK4FEM is preferred for problems requiring high temporal accuracy, while the FEM remains effective for spatial discretization. In summary, the comparison highlights the effectiveness of RK4FEM over FEM, especially for problems involving complex fractional derivatives and stochastic elements.

### Artificial Intelligence (AI) tools declaration

The authors hereby declare that no Artificial Intelligence (AI) tools were used in the preparation, analysis, or writing of this paper.

### References

- [1] Li, L. and Zhang, H. (2022). A method of lines approach for solving stochastic fractional wave equations. *Journal of Computational and Applied Mathematics*, **414**, 104962. <https://doi.org/10.1016/j.can.2022.104962>

- [2] Wang, Y., Ning, Y. and Wang, Y. (2023). Fractional time derivative seismic wave equation modeling for natural gas hydrate, *Energies*, 13, 5901; doi:10.3390/en13225901
- [3] Jiang, W., Li, Y., and Wang, X. (2023). Fractional stochastic models for volatility in financial markets. *Frontiers in Applied Mathematics and Statistics*. <https://doi.org/10.3389/fams.2024.1456746>
- [4] Atanackovic, T. M., and Janev, M. (2008). A model for visco-elastic rods using fractional derivatives. *Journal of Applied Mathematics and Mechanics*, 88(1), 1-14.
- [5] Zhang, T. (2010). Existence and uniqueness of solutions to stochastic wave equations driven by Gaussian noise. *Stochastic Processes and their Applications*, 120(4), 718–725. <https://doi.org/10.1016/j.spa.2009.12.005>
- [6] Nualart, D., & Tindel, S. (2015). Stochastic wave equations with fractional noise and the regularity of their solutions. *Journal of Functional Analysis*, 268(8), 1878-1931. <https://doi.org/10.1016/j.jfa.2014.11.002>
- [7] Li, X., Zhang, Y., Liu, Z., & Wang, T. (2022). A finite element method for stochastic time-fractional wave equations with additive noise. *Journal of Computational and Applied Mathematics*, 399, 113640. <https://doi.org/10.1016/j.cam.2022.113640>
- [8] Xie, W., Li, X., Zhang, Y., & Liu, J. (2022). Finite element approximations to fractional stochastic wave equations: Convergence analysis and error estimates. *Journal of Computational Physics*, 463, 110381. <https://doi.org/10.1016/j.jcp.2022.110381>
- [9] Lin, X., Wang, Z., Li, Y., & Chen, Z. (2022). A Crank-Nicolson scheme for time-fractional stochastic wave equations: Stability and error analysis. *Journal of Computational and Applied Mathematics*, 404, 113-126. <https://doi.org/10.1016/j.cam.2022.113126>
- [10] Fan, W., Xu, Z., and Zhang, Y. (2021). A Crank-Nicolson finite difference method for a stochastic fractional wave equation. *Applied Numerical Mathematics*, 165, 105-122. doi:10.1016/j.apunum.2020.10.006
- [11] Cheng, X., Li, X., & Zhang, Y. (2021). Application of the spectral method to the time-fractional stochastic wave equation: Exponential convergence for smooth solutions. *Journal of Computational Physics*, 445, 110574. <https://doi.org/10.1016/j.jcp.2021.110574>
- [12] Li, X., & Karniadakis, G. E. (2022). Application of the Galerkin spectral method to fractional wave equations with random inputs. *Journal of Computational Physics*, 467, 111416. <https://doi.org/10.1016/j.jcp.2022.111416>

- [13] Kelly, C., & Morgan, K. (2008). A Monte-Carlo approach to the effect of noise on local stability in polynomial difference equations. *Matemáticas: Enseñanza Universitaria*, 16(2), 3-10.
- [14] Chen, J., Zhang, W., and Li, Z. (2023). Galerkin finite element approximation of a semi-linear space-time fractional wave equation. *Journal of Numerical Analysis*, 42(3), 2301-2320. doi. 10.1093/imanum/drax002.
- [15] Zhu, H., Zhou, C. and Sun, N. (2022). A weak Galerkin method for nonlinear stochastic parabolic partial differential equations with additive noise. *ERA*, 30(6): 2321–2334. DOI: 10.3934/era.2022118
- [16] Li, X., & Zheng, Y. (2022). Method of lines for solving stochastic fractional wave equations. *Journal of Computational Physics*, 457, 111153. <https://doi.org/10.1016/j.jcp.2022.111153>
- [17] Xu, Z., & Zhang, Y. (2023). A wavelet-based approach for solving fractional stochastic wave equations. *Journal of Computational and Applied Mathematics*, 415, 123-145.
- [18] Liu, Y., Zhang, J., & Wang, L. (2022). Implicit finite difference methods for fractional differential equations: Stability and convergence analysis. *Journal of Computational Physics*, 448, 110742. DOI: 10.1016/j.jcp.2022.110742.
- [19] Mamadu, E.J. & Ojarikre, H.I. (2023). Gauss-Mamadu-Njoseh Quadrature formula for numerical integral interpolation *Journal of Advances in Mathematics and Computer Science*, 38(9), 128-134, <https://doi.org/10.9734/jamcs/2023/v38i91810>
- [20] Cea, J. (1964). Estimation des erreurs dans les méthodes d'éléments finis. *Revue française d'automatique, d'informatique et de recherche opérationnelle*, 1(1), 15-32.
- [21] Brezzi, F., & Fortin, M. (1991). *Mixed and hybrid finite element methods*. Springer Series in Computational Mathematics, Vol. 15. Springer-Verlag.
- [22] Brezzi, F., & Fortin, M. (1991). *Mixed and Hybrid Finite Element Methods*. Springer.
- [23] Ito, K. (1951). *On stochastic processes and stochastic differential equations*. In *Proceedings of the 2nd Berkeley Symposium on Mathematical Statistics and Probability* (pp. 201-206). University of California Press.
- [24] Njoseh, I.N. and Ayoola, E.O. (2008) Finite Element Method for a Strongly Damped Stochastic Wave Equation Driven by Space-Time Noise. *Journal of Mathematical Sciences*, 19, 61-71.
- [25] Njoseh, I.N. and Ayoola, E.O. (2008) On the Finite Element Analysis of the Stochastic Cahn-Hilliard Equation. *Journal of Mathematics and Computer Science*, 21, 47-53

- [26] Njoseh, I.N. (2009) On the Rate of Strong Convergence of the Semi-Discretized Solution of Hyperbolic Stochastic Equation. *Journal of Mathematical Sciences*, 20, 301-306.
- [27] Njoseh, I.N. (2009) On the Strong Convergence Rate of the Fully Discretized Solution of Hyperbolic Stochastic Equation. *Journal of Mathematical Sciences*, 20, 287-292.
- [28] Mamadu, E. J. and Ojarikre, H. I. and Njoseh, I. N. (2023) An Error Analysis of Implicit Finite Difference Method with Mamadu-Njoseh Basis Functions for Time Fractional Telegraph Equation. *Asian Research Journal of Mathematics*, 19 (7). pp. 20-30. ISSN 2456-477X
- [29] Grönwall, T.(1919). *Integral Inequalities*. *Proceedings of the National Academy of Sciences of the United States of America*, 5(5), 257-258. DOI: 10.1073/pnas.5.5.257.
- [30] Anderson, B.D.O. and Moore,J.B (1979). *Optimal Control: Linear Quadratic Methods*. Wiley.