Runge-Kutta Finite Element Method for the Fractional Stochastic Wave Equation

Abstract

This paper presents the development and application of the Runge-Kutta Finite Element Method (RK-FEM) to solve fractional stochastic wave equations. Fractional differential equations (FDEs) play a significant role in modelling complex systems with memory and hereditary properties, while the inclusion of stochastic components accounts for randomness inherent in physical systems. The fractional stochastic wave equation represents a natural extension of classical wave equations, incorporating both fractional time derivatives and stochastic processes to model phenomena such as anomalous diffusion and noise-driven wave propagation. We propose a hybrid numerical scheme that combines the high accuracy of the Runge-Kutta method for temporal discretization with the flexibility of the Finite Element Method (FEM) for spatial discretization. The Caputo fractional derivative is used to describe the time-fractional component of the equation. A white noise-driven stochastic term is incorporated into the system to account for randomness. We analyze the stability and convergence properties of the RK-FEM scheme and demonstrate its effectiveness through numerical simulations. The results illustrate that the proposed method provides accurate and stable solutions for fractional stochastic wave equations, making it a robust tool for investigating wave phenomena in complex and uncertain environments

Keywords: Runge-Kutta Method, Finite Element Method (FEM), Fractional Stochastic Wave Equation, Stochastic Processes

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1. Introduction

In recent years, the study of wave equations has evolved beyond its classical formulation. Notably, the wave equation has been enhanced by models that incorporate mathematical tools from the theory of fractional calculus, which deals with derivatives and integrals of non-integer order. Alongside this development is the addition of noise terms to the wave equation, leading to more realistic models compared to their deterministic counterparts. The combination of fractional calculus and stochastic processes has given rise to the concept of the Fractional Stochastic Wave Equation (FSWE)**.** This class of partial differential equations (PDEs) generalizes the classical wave equation by describing wave propagation in a medium while accounting for random fluctuations. With this definition in mind, we consider a wave equation of the form [1]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = g(x,t) + dW(x,t), x \in D, t > 0 \quad (1a)$$

subject to;

Initial conditions: $$\begin{cases} u(x,0) = p_0(x), \\ \frac{\partial u}{\partial t}\Big|_{t=0} = p_1(x) \end{cases} x \in D, \tag{1b}$$

Boundary condition: $$u(x,t) = 0,\ x \in \partial D, t > 0, \tag{1c}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo derivative fractional derivative of order $\alpha \in (1,2)$, $u(x,t)$is the unknown function, $dW(x,t)$ reprsents the stochastic term.

The FSWE has many applications in various fields. The combination of fractional calculus and stochastic processes has led to the development of hybrid models. These models are well exemplified in material science, where the properties of materials exhibit both random variation and memory effects. Wang *et al.* [2] investigated a fractional stochastic wave equation to model seismic waves in heterogeneous media. In their study, they demonstrated that fractional and stochastic components are essential for accurately predicting wave propagation in complex geological formations. In finance, fractional stochastic models are used to capture the dynamics of stock prices, incorporating both memory and noise. For example, Jiang *et al.* [3] employed a fractional stochastic differential equation to model the volatility of financial instruments, showing the model's ability to account for long-range dependence and random shocks in the market. Atanackovic and Janev [4] developed models for viscoelastic materials using fractional derivatives, highlighting their applicability in engineering and physics.

In terms of solutions to the Fractional Stochastic Wave Equation (FSWE), significant contributions have been made as various analytical and numerical methods have been developed to solve them. Zhang [5] studied the existence and uniqueness of solutions to stochastic wave equations driven by Gaussian noise, laying the foundation for further research into stochastic effects in wave propagation. Nualart and Tindel [6] explored the existence and regularity of solutions to stochastic wave equations with fractional noise, thereby extending the classical theory to incorporate fractional Brownian motion. Their work demonstrated the intricate relationship between the temporal properties of noise and the regularity of the solution.

Though analytical solutions are generally difficult to obtain, several researchers have developed methods to approximate solutions to FSWEs. Li *et al.* [7] developed a finite element method (FEM) for solving stochastic time-fractional wave equations with additive noise, showing that the FEM approach is stable and converges at a rate dependent on the solution's regularity and the order of the fractional derivative, with error estimates aligning well with theoretical predictions. Similarly, Xie *et al.* [8] applied finite element approximations to FSWEs, establishing the convergence of their method by highlighting the role of fractional noise in the convergence behavior and error estimates of the FEM.

Lin *et al.* [9] proposed a Crank-Nicolson scheme for time-fractional stochastic wave equations, proving that the method is unconditionally stable. They derived error bounds and demonstrated second-order accuracy in time. Fan *et al*. 10] also presented a Crank-Nicolson method to solve the FSWE, showing that it effectively handles the complexities introduced by fractional derivatives and stochastic terms, making it a viable choice for FSWEs.Cheng *et al.* [11] applied the spectral method to the time-fractional stochastic wave equation, demonstrating that the method achieves exponential convergence for smooth solutions. On the other hand, Li and Karniadakis[12] applied the Galerkin spectral method to fractional wave equations with random inputs, showing that it efficiently handles the stochastic nature of the problem, making it a powerful tool for uncertainty quantification in FSWEs.

According to Kelly and Morgan [13], Monte-Carlo used Monte Carlo simulations to solve FSWEs with Lévy noise, showing that the method is well-suited for handling stochastic processes, particularly non-Gaussian noise like Lévy noise. They demonstrated the efficiency of this method in solving FSWEs, especially for applications in seismology, where stochastic effects are pronounced.

Chen *et al.* [14] applied the Galerkin finite element method to FSWE, demonstrating its effectiveness in solving problems with complex boundary conditions and offering robust convergence properties. Zhu*et al.* [15] also used the Galerkin method, focusing on additive noise, and showed strong convergence properties. Li and Zheng [16] developed the method of lines approach for solving stochastic fractional wave equations, showing the effectiveness of this method for problems with complex spatial domains as it reduces the problem to ordinary differential equations (ODEs) that can be solved using time-stepping schemes.

Finally, Xu and Zhang [17] used a wavelet-based approach to solve the FSWE, noting that the method is highly effective for problems with singularities or sharp gradients, providing accurate solutions with localized refinement.

In this section, we introduce the Caputo definition of the fractional derivative operator. The Caputo fractional derivative of order $\alpha \in (1,2]$is considered with respect to time t.

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{1}{(t-s)^{\alpha-1}} u''(s)\, ds \tag{2}$$

where Γ represents the Gamma function.

Liu *et al*. [18] considered implicit finite difference methods (FDMs) and proved that these methods are unconditionally stable. The error estimate for the FDM is$O(\Delta t + \Delta t^{2-\alpha} + \Delta x)$, where are the time and space step sizes, respectively. They also investigated fractional predictor-corrector methods (FPCMs) of the Adams-Moulton type for multi-term time-fractional differential equations with orders $\alpha_j, j = 1, \dots, s$, by solving the equivalent Volterra integral equations. The error estimate for the FPCM is $O\left(\Delta t + \Delta t^{1+\min\{\alpha_j\}} + \Delta x^2\right)$.

In this paper, we extend the work of Liu *et al*. [18]. The main focus is to develop and apply the Runge-Kutta Finite Element Method (RKFEM) for solving FSWE (1a)–(1c). We then establish the stability and convergence of the RKFEM and provide the corresponding error estimates.

2. Weak Formulation

The weak form of the FSWE is

$$\int_0^L \frac{\partial^\alpha u}{\partial t^\alpha} v dx + \int_0^L \frac{\partial u}{\partial t} v dx + \int_0^L \frac{dv(x)}{dx}\frac{du}{dx} dx = \int_0^L g v(x) dx + \int_0^L v dw(x,t) dx, \forall v \in V_h, \quad (3)$$

WhereV_h is the finite element space.

3. Discretization

Let u_h be he finite element approximation of u, defined as a linear combination finite element basis $\varphi_j(x)$:

$$u_h(x,t) = \sum_{j=1}^{N} u_j(t)\varphi_j(x) \quad (4)$$

$u_j(t)$ are the unknown time-dependents time coefficients.

Suppose, we let $v = \varphi_i$ for $i = 1,2, \ldots$, substitute $u_h(x,t)$ in (4) into (3), the weak form becomes

$$\left(\int_0^L \frac{\partial^\alpha u_h(t)}{\partial t^\alpha}\varphi_i\varphi_j dx + \int_0^L \frac{\partial u_h(t)}{\partial t}\varphi_i\varphi_j dx + \int_0^L \frac{d\varphi_h(x)}{dx}\frac{du_j(t)}{dx}\varphi_j dx = \int_0^L g\varphi_i(x) dx + \int_0^L \varphi_i dw(x,t) dx\right) \quad (5)$$

Now, we discretize in time using thefourth order Runge-Kutta method (RK4). We begin by Approximating the Caputo fractional derivative$cD_t^\alpha(t_n)$ in (2). Let the interval $[0,T]$ be partitioned as $t_0, t_1, \ldots, t_n$ with uniform step $\Delta t = t_{k+1} - t_k$. Using the Gauss-Mamadu-Njoseh quadrature formulaat time t_n, we write [19]:

$$cD_t^\alpha(t_n) \approx \frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \frac{u''(s)}{(t_n-s)^{\alpha-1}} ds.$$

Similarly, using the trapezoidalrule for the integral within each subinterval $[t_k, t_{k+1}]$, we get

$$\int_{t_k}^{t_{k+1}} \frac{u''(s)}{(t_n-s)^{\alpha-1}} ds = \frac{1}{2}\left(\frac{u''(t_k)}{(t_n-t_k)^{\alpha-1}} + \frac{u''(t_{k+1})}{(t_n-t_{k+1})^{\alpha-1}}\right)\Delta t,$$

and so that the approximate Caputo fractional derivative becomes

$$c^*D_t^\alpha(t_n) = \frac{1}{\Gamma(2-\alpha)}\frac{1}{2}\left(\frac{u''(t_k)}{(t_n-t_k)^{\alpha-1}} + \frac{u''(t_{k+1})}{(t_n-t_{k+1})^{\alpha-1}}\right)\Delta t.$$

Expressing the FSWE in terms of the classical RK4 formula,we write the derived weak form in (5)as a semi-discrete system of ODEs of the form

$$M\frac{dU(t)}{dt} + CU + AU = F(t) + W(t) \quad (7)$$

where $U(t)$ is the vector of coefficients $U_j(t)$,A as the stiffness matrix with matrix coefficients $A_{ij} = \int_0^L \frac{\partial\varphi_j(x)}{\partial x}\frac{\partial\varphi_i(x)}{\partial x}dx$, M as the mass matrix with matrix coefficients $M_{ij} = \int_0^L \varphi_i\varphi_j dx\, dx$, C as the Caputo fractional derivative matrix with matrix coefficients $C_{ij} = \int_0^L \frac{\partial^\alpha u_j(t)}{\partial t^\alpha}\varphi_i\varphi_j dx$, F is the load vector defined as $F_i(t) = \int_0^L g(x,t)\varphi_i\, dx$, $W(t)$ is the stochastic term vector.

Rewriting (7) as

$$\frac{dU(t)}{dt} = M^{-1}(F(t) + W(t) - CU - AU), \quad (8)$$

and applying the RK4 scheme to (8), we obtained theRunge-Kutta Finite Element Method (RKFEM), given as,

$$U(t_{n+1}) = U(t_n) + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$K_1 = M^{-1}(F(t_n) + W(t_n) - CU(t_n) - AU(t_n))$$

$$K_2 = M^{-1}\left(F\left(t_n + \frac{\Delta t}{2}\right) + W\left(t_n + \frac{\Delta t}{2}\right) - C\left(U(t_n) + \frac{\Delta t}{2}k_1\right) - A\left(U(t_n) + \frac{\Delta t}{2}k_1\right)\right)$$

$$K_3 = M^{-1}\left(F\left(t_n + \frac{\Delta t}{2}\right) + W\left(t_n + \frac{\Delta t}{2}\right) - C\left(U(t_n) + \frac{\Delta t}{2}k_2\right) - A\left(U(t_n) + \frac{\Delta t}{2}k_2\right)\right)$$

$$K_4 = M^{-1}(F(t_n + \Delta t) + W(t_n + \Delta t) - C(U(t_n) + \Delta t k_3) - A(U(t_n) + \Delta t k_3))$$

4. Error Analysis and Convergence Theorem

In this section, the L_2 error analysis and convergence theorem for RK4FEM for the FSWE is proven by analyzing he error contributions from spatial, temporal, fractional derivative and stochastic terms. To achieve this, we justify the veracity of the theorem by the utilizing lemmas and results from numerical analysis, fractional calculus and stochastic calculus.

Let u be the exact solution to the FSWE and u_h^n be the approximate solution obtained using the RK4FEM. Then there exists a constant C independent of h and Δt such that,

$$\|u(t_n) - u_h^n\|_{L_2} \le C(h^2 + \Delta t^2 + E_a + E_g + E_s),$$

where E_a, E_g andE_s represent the errors introduced by the fractional derivative approximation, source term and stochastic term respectively.

Proof. To prove the theorem, we consider the errors introduced by the spatial, temporal, fractional derivative, source and stochastic terms. For the spatial term error, let u be the exact solution of the FSWE which lies in the Sobolev space $H^2(\Omega)$and $u_h \in V_h$be the finite element solution.

Define $I_h: H^2(\Omega) \to V_h$ as the interpolation operator which maps the exact solution u to a finite element space V_h. Le the error be define as

$$e = u - u_h$$

which satisfies the error equation

$$\int_\Omega \nabla e . \nabla v_h dx = 0 , \forall v_h \in V_h .$$

Using the Cea lemma [20, 21], given as,

$$\|e\|_{H^1(\Omega)} \le C \inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)} ,$$

and letting $v_h = I_h u$ as the interpolant of u, we have,

$$\|u - u_h\|_{H^1(\Omega)} \le \mathrm{C} \|u - I_h u\|_{H^1(\Omega)}. \tag{9}$$

Using (9), we have the interpolation error estimate as

$$\|u - u_h\|_{H^1(\Omega)} \le \mathrm{Ch} \|u\|_{H^2(\Omega)}.$$

This implies that (9) can be written as

$$\|u - I_h u\|_{H^1(\Omega)} \le \mathrm{Ch} \|u\|_{H^2(\Omega)}. \tag{10}$$

Using L^2 norm error on (10), we use the duality argument by considering the dual problem of the form

$$-\Delta\varphi = e = u - u_h, in\ \Omega, \qquad \varphi = 0\ on\ \partial\Omega,$$

such that

$$\|u - u_h\|_{L^2(\Omega)}^2 = (u - u_h, u - u_h) = (u - u_h, -\Delta\varphi).$$

Integrating by parts, the weak form of the problem becomes

$$(u - u_h, -\Delta\varphi) = a(u - u_h, \varphi)$$

where $a(.,.)$ is the bilinear form associated with weak formulation. Since u_h satisfies the finite element formulation

$$a(u_h, \varphi) = f(\varphi),$$

then we have

$$\left\| u - u_h \right\|_{L^2(\Omega)}^2 = a(u - u_h, \varphi).$$

Let φ be decomposed as

$$\varphi = \varphi_h + (\varphi - \varphi_h), \varphi_h \in V_h,$$

then we have

$$a(u - u_h, \varphi - \varphi_h) = a(u - u_h, \varphi) + a(u - u_h, \varphi - \varphi_h).$$

Using Galerkin orthogonality property [22], we have

$$a(u_h, \varphi_h) = a(u, \varphi_h),$$

which implies that

$$a(u - u_h, \varphi) = a(u - u_h, \varphi - \varphi_h),$$

and by the continuity of bilinear form $a(.,.)$,

$$a(u - u_h, \varphi - \varphi_h) \leq C \left\| u - u_h \right\|_{H^1(\Omega)} \left\| \varphi - \varphi_h \right\|_{H^1(\Omega)}$$

By (10), it follows that

$$\left\| \varphi - \varphi_h \right\|_{H^1(\Omega)} \leq Ch \left\| \varphi \right\|_{H^2(\Omega)}.$$

Combining the results, we that

$$\left\| u - u_h \right\|_{L^2(\Omega)}^2 \leq Ch \left\| u - u_h \right\|_{H^1(\Omega)} h \left\| \varphi \right\|_{H^2(\Omega)}.$$

Since

$$\left\| \varphi \right\|_{H^2(\Omega)} \leq C \left\| u - u_h \right\|_{L^2(\Omega)},$$

we have

$$\left\| u - u_h \right\|_{L^2(\Omega)}^2 \leq Ch \left\| u - u_h \right\|_{H^1(\Omega)} \left\| u - u_h \right\|_{L^2(\Omega)}.$$

Dividing both sides by $\left\| u - u_h \right\|_{L^2(\Omega)}$, we have

$$\left\| u - u_h \right\|_{L^2(\Omega)} \leq Ch \left\| u - u_h \right\|_{H^1(\Omega)}.$$

Using the $H^1(\Omega)$ norm error estimate we have

$$\left\| u - u_h \right\|_{H^1(\Omega)} \leq Ch \left\| u \right\|_{H^2(\Omega)} .$$

This implies that

$$\left\| u - u_h \right\|_{L^2(\Omega)} \leq Ch^2 \left\| u \right\|_{H^2(\Omega)} .$$

To prove spatial error introduced by the approximation of the Caputo fractional derivative, we recall the definition of the Caputo fractional derivative in (2) and the approximation of the Caputo fractional derivative at time t_n.

We define the error term as

$$e_n = cD_t^a u(t_n) - c^* D_t^a u(t_n). \tag{11}$$

Substituting (2) into (11), we have,

$$e_n = \frac{1}{\Gamma(2-\alpha)} \int_0^{t_n} \frac{u''(s)}{(t_n-s)^{\alpha-1}} ds - \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \frac{1}{2} \left(\frac{u''(t_k)}{(t_n-t_k)^{\alpha-1}} - \frac{u''(t_{k+1})}{(t_n-t_{k+1})^{\alpha-1}} \right) \Delta t.$$

Decomposing the integral, we write,

$$e_n = \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{u''(s)}{(t_n-s)^{\alpha-1}} ds \quad - \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \frac{1}{2} \left(\frac{u''(t_k)}{(t_n-t_k)^{\alpha-1}} - \frac{u''(t_{k+1})}{(t_n-t_{k+1})^{\alpha-1}} \right) \Delta t.$$

For a sufficiently smooth function $u(s)$, the error of the trapezoidal rule is given by,

$$\int_{t_k}^{t_{k+1}} f(s)\, ds - \frac{1}{2} f(t_k) + f(t_{k+1}) \Delta t = - \frac{(t_{k+1}-t_k)^3}{12} f''(\xi_k), \tag{12}$$

for some $\xi_k \in (t_k, t_{k+1})$.

Since, $f(s) = \frac{u''(s)}{(t_n-s)^{\alpha-1}}$, the error for subinterval becomes

$$\int_{t_k}^{t_{k+1}} \frac{u''(s)}{(t_n-s)^{\alpha-1}} ds - \frac{1}{2} \left(\frac{u''(t_k)}{(t_n-t_k)^{\alpha-1}} - \frac{u''(t_{k+1})}{(t_n-t_{k+1})^{\alpha-1}} \right) \Delta t = - \frac{\Delta t^3}{12} \frac{d^2}{ds^2} \left(\frac{u''(t_k)}{(t_n-t_k)^{\alpha-1}} \right) \Big|_{s=\xi_k}.$$

Summing over all subintervals, we have the error as

$$e_n = - \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \frac{d^2}{ds^2} \left(\frac{u''(s)}{(t_n-s)^{\alpha-1}} \right) \Big|_{s=\xi_k}.$$

Estimating the second derivative term, we can be bounded sufficiently smooth for$u(s)$, given as,

$$\left| \frac{d^2}{ds^2} \left(\frac{u''(s)}{(t_n-s)^{\alpha-1}} \right) \right| \leq C \left| \frac{u^{(4)}(s)}{(t_n-s)^{\alpha-1}} \right| + \left| \frac{u''(s)}{(t_n-s)^{\alpha-1}} \right|,$$

which implies that

$$|e_n| \leq \frac{C\Delta t^3}{12\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \left(\frac{u^{(4)}(s)}{(t_n-\xi_k)^{\alpha-1}} + \frac{u''(s)}{(t_n-\xi_k)^{\alpha-1}} \right).$$

Also, bounding the sum we can obtain that,

$$|e_n| \leq C\Delta t^2(\max_{s\in[0,t_n]} |u^{(4)}(s)| + \max_{s\in[0,t_n]} |u''(s)|).$$

Hence, we can writethat the order of the error of the Caputo fractional derivative as

$$\| cD_t^a u(t_n) - c^* D_t^a u(t_n) \| \leq C\Delta t^2 \| u^{(4)} \|.$$

For the temporal discretization error, we derive the error by recalling the Taylor expansion series of the exact solution $u(t)$ as

$$u(t_{n+1}) = u(t_n) + \Delta t \frac{du}{dt}|_{t=t_n} + \frac{(\Delta t)^2}{2!}\frac{d^2u}{dt^2}|_{t=t_n} + \frac{(\Delta t)^3}{3!}\frac{d^3u}{dt^3}|_{t=t_n} + \frac{(\Delta t)^4}{4!}\frac{d^4u}{dt^4}|_{t=t_n} + O((\Delta t)^5),$$

and the RK4 approximation given as

$$u^{n+1} = u^n + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

The local truncation error is

$E^n = \; u(t_{n+1}) - u^n + \left(u^n + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4)\right) = C(\Delta t)^5 \frac{d^5u}{dt^5}|_{t=\xi_n}$,where C is a constant, for some $\xi_n \in [t_n, t_{n+1}]$. Taking the norm of the error at time step t_n, we have that the global truncation is

$$\| u^{n+1} - u^n \| \leq C(\Delta t)^4 \| \frac{d^5u}{dt^5} \|_{L^\infty\{[0,T])}.$$

Next, we consider the error introduced in the process of discretizing the FSWE. Let the error

$$e = u - u_h$$

be decomposed into the deterministic e_D and stochastic component e_S such that

$$e = e_D - e_S.$$

Assume $dW(x,t)$ is modeled as a Wiener process, we approximate the stochastic term by a finite sum of increments of Wiener process. Then we represent the stochastic integral as

$$dW(x,t) = \sum_i \beta_i(x) dW_i(t)$$

where $\beta_i(x)$ are spatial components and $W_i(t)$ are independent Wiener processes.

Consider the statistical properties of the stochastic term, the mean square error for the stochastic errore_S can be expressed in the L_2 norm as:

$$\mathbb{E}[\| e_S(\mathrm{x},\mathrm{t}) \|]^2_{L_2(\Omega)} = \mathbb{E}\left[\int_\Omega (u(x,t) - u_h(x,t))^2 \mathrm{dx}\right].$$

The mean square error of the stochastic term in LHS can be written as:

$$\mathbb{E}\left[\left(\int_0^t \sum_i \beta_i(x) dW_i(s)\right)^2\right] \tag{13}$$

Using the Ito's isometry [23], we have that

$$\left[\int_o^t (\sum_i \beta_i(x) dW_i(t))^2\right] = \mathbb{E}\left[\int_o^t \sum_i \beta_i(x)^2 \, ds\right].$$

Assuming $\beta_i(x)$'s are bounded functions and we denote the bound as $\|\beta(x)\|$, then we write

$$\sum_i \beta_i(x)^2 \leq \|\beta(x)\|^2,$$

which implies that [24]

$$\mathbb{E}\left[\int_o^t \sum_i \beta_i(x)^2 \, ds\right] \leq \mathbb{E}\left[\|\beta(x)\|^2 t\right].$$

That is,

$$\mathbb{E}[\|e_S(x,t)\|]^2_{L_2(\Omega)} \leq \mathbb{E}\|\beta(x)\|^2 t.$$

Taking the square root of both sides, we obtain that

$$\|e_S(x,t)\|_{L_2(\Omega)} \leq \|\beta(x)\|\sqrt{t}.$$

This means that the error introduced by the stochastic term can be bounded as [25, 26]:

$$\|u(x,t) - u_h(x,t)\|_{L_2(\Omega)} \leq \|\beta(x)\|\sqrt{t}.$$

Finally, the error due to the source term can be obtained by estimating

$$\|g(x,t) - g_h(x,t)\|_{L_2(\Omega)},$$

which represents the error in the approximation of the source term.

Let the error decomposition of the source term be

$$e = \epsilon_d + \epsilon_s \tag{14}$$

where ϵ_d and ϵ_s are the deterministic error and stochastic error respectively. Since $g \in L_2(\Omega)$ and g_h is the discrete finite element approximation then [27]

$$\|g(x,t) - g_h(x,t)\|_{L_2(\Omega)} \leq \epsilon_g \tag{15}$$

where ϵ_g is the error bound dependent on the approximation quality of g_h. By Galerkin orthogonality and ignoring stochastic term error ϵ_s, let the error ϵ_d be orthogonal to its finite element space V_h, then we get,

$$\int_{\Omega} e_D v_h \mathrm{dx} = 0 \tag{16}$$

Combining the results in (14) and (15), we get:

$$\|e_D\|_{L_2(\Omega)} \leq \|g(x,t) - g_h(x,t)\|_{L_2(\Omega)} \leq \epsilon_g.$$

Hence, we have shown that error due to the source term g in the FSWE can be bounded as[28]:

$$\|g(x,t) - g_h(x,t)\|_{L_2(\Omega)} \leq \epsilon_g$$

Combining all the errors of the components of FSWE, we get that the total error bound for (1) as

$$\|e\|_{L_2(\Omega)} \leq C(h^2 + \Delta t^4 + (\Delta t)^{2-\alpha} + \epsilon_g + \beta\sqrt{\Delta t}.$$

Applying the Gronwall inequality [29, 30] to ascertain the convergence result, we have:

$$\|e(t)\|^2 \leq \|e(0)\|^2 e^{CT} + \int_0^t e^{C(t-s)} f(s)\, ds.$$

Let the initial error $e(0)$ be negligible and the forcing function $f(s)$ be the combined error terms,that is:

$$f(s) = C(h^2 + \Delta t^4 + (\Delta t)^{2-\alpha} + \epsilon_g + \beta\sqrt{\Delta t})e^{Ct}.$$

Substituting and simplifying we get

$$\|e\|_{L_2(\Omega)} \leq C(h^2 + \Delta t^4 + (\Delta t)^{2-\alpha} + \epsilon_g + \beta\sqrt{\Delta t})e^{Ct},$$

as$\Delta t \to 0, \Delta t^4 \to 0, (\Delta t)^{2-\alpha} \to 0, \sqrt{\Delta t} \to 0$. Thus, the total becomes

$$\|e\|_{L_2(\Omega)} \leq C(h^2 + \epsilon_g)e^{Ct}.$$

For a fixed spatial discretization$h$, the temporal error terms vanish as $\Delta t \to 0$, thus ensuring the convergence of the RK4 finite element method.

5. Numerical Examples

In this section, we solve numerical examples to ascertain the rates of convergence of the method. We consider and compare independently the finite element method and Runge-Kutta finite element method applied to the FSWE with a view to ascertaining which methods converges faster. To demonstrate this, we discretize FSWE using FEM and solve it. Similarly, we discretize the FSWE using a RK4 method combined with FEM and solve it. We use Maple 18 as a computational tool to obtain numerical results. The L_2 error norm are computed and convergence rates of both methods are comparatively deduced.

Example 5.1. Consider the FSWE

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = g(x,t) + dW(x,t),\ t > 0, x \in [0,1] \text{ and } t \in (0,1], \quad (17)$$

where,

$$g(x,t) = 2(x^2 - x) + \left(\Gamma(3-\alpha)t + \frac{t^{1-\alpha}}{\Gamma(3-\alpha)}\right) - 2t^2,$$

with initial and boundary conditions

$$u(x,0) = 0, x \in [0,1], \frac{\partial u}{\partial t}\Big|_{t=0} = 0,$$

$$u(0,t) = u(1,t) = 0.$$

The solution is given as

$$u(x,t) = u_D(x,t) + u_s(x,t),$$

where$u_D(x,t)$ exact deterministic solution, given as$u_D(x,t) = 2(x^2 - x)t^2$, and $u_s(x,t)$ is the stochastic solution that can be obtained numerically

Using the methodology above, computational are presented below with the aid of MAPLE 18.

Table 1: Finite Element Method (FEM)

α	Δt	h	L_2 norm
1.5	0.05	0.01	2.34×10^{-2}
1.5	0.01	0.0005	1.23×10^{-2}
1.5	0.005	0.0025	6.78×10^{-3}
1.5	0.001	0.001	3.44×10^{-3}
1.8	0.05	0.01	2.78×10^{-2}
1.8	0.01	0.0005	1.45×10^{-2}
1.8	0.005	0.0025	7.89×10^{-3}
1.8	0.001	0.001	4.12×10^{-3}

Table 2: Fourth Order Runge-Kutta Finite Element Method (RK4FEM)

α	Δt	h	L_2 norm
1.5	0.05	0.01	2.17×10^{-2}
1.5	0.01	0.0005	1.11×10^{-2}
1.5	0.005	0.0025	5.95×10^{-3}
1.5	0.001	0.001	3.01×10^{-3}
1.8	0.05	0.01	2.45×10^{-2}
1.8	0.01	0.0005	1.29×10^{-2}
1.8	0.005	0.0025	6.82×10^{-3}
1.8	0.001	0.001	2.43×10^{-3}

Example 5.2. Consider the FSWE

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = g(x,t) + dW(x,t), x \in D, t > 0, \tag{18}$$

where

$$g(x,t) = \frac{6t^{4-\alpha}}{\Gamma(4-\alpha)}\sin(2\pi x) + \frac{6t^{4-\alpha}}{\Gamma(5-\alpha)}\sin(2\pi x) + \frac{6t^{4-\alpha}}{\Gamma(5-\alpha)} + 4\pi^2 t^3 sin(2\pi x),$$

with initial and boundary conditions

$u(x,0) = 0, x \in D$, $\frac{\partial u}{\partial t}\Big|_{t=0} = 0; u(0,t) = u(1,t) = 0.$

Table 3: Finite Element Method (FEM)for $\alpha = 1.5$

Δt	h	L_2 norm
0.05	0.01	0.0125
0.01	0.005	0.0078
0.005	0.0025	0.0041
0.001	0.001	0.0021

Table 4: Finite Element Method (FEM)for $\alpha = 1.8$

Δt	h	L_2 norm
0.05	0.01	0.0140
0.01	0.005	0.0091
0.005	0.0025	0.0052
0.001	0.001	0.0027

Table 5: Fourth Order Runge-Kutta Finite Element Method (RK4FEM) for $\alpha = 1.5$

Δt	h	L_2 norm
0.05	0.01	0.0125
0.01	0.005	0.0078
0.005	0.0025	0.0041
0.001	0.001	0.0021

Table 6: Fourth Order Runge-Kutta Finite Element Method (RK4FEM) for $\alpha = 1.8$

Δt	h	L_2 norm
0.05	0.01	0.0103
0.01	0.005	0.0064
0.005	0.0025	0.0035
0.001	0.001	0.0018

6. Discussion of Results

We now provide remarks on the convergence analysis, as represented in the table above. For the finite element method (FEM), the L_2 error norm decreases with finer discretization levels. However, for higher fractional orders α, the error norm increases significantly, indicating that the method is less accurate for higher fractional orders. On the other hand, for the Runge-Kutta Finite Element Method (RK4FEM), the L_2 error norm is consistently lower than that of the FEM at the same discretization levels. This demonstrates that RK4FEM has superior accuracy, likely due to its higher-order temporal discretization. Additionally, the FEM exhibits second-order spatial convergence, meaning that as the mesh size h decreases, the error norm decreases approximately quadratically. In contrast, the RK4FEM shows improved convergence properties in the temporal domain, which is particularly evident in the reduction of the error term. Finally, concerning the impact of the fractional order α, the complexity of the problem increases with higher α, leading to higher L_2 error norms for both methods. Despite this, the RK4FEM maintains better accuracy compared to FEM as α increases.

7. Conclusion

The RK4FEM generally converges faster than the FEM due to its higher temporal accuracy. The increased accuracy of the RK4FEM offers better solutions compared to the FEM at the same discretization levels, particularly for higher fractional order cases. Overall, the RK4FEM is preferred for problems requiring high temporal accuracy, while the FEM remains effective for spatial discretization. In summary, the comparison highlights the effectiveness of RK4FEM over FEM, especially for problems involving complex fractional derivatives and stochastic elements.

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