

New Explicit Exact Solutions for the (1+1) Dimensional Generalized **shallow** Water Wave Equation

Abstract

The shallow water wave equation describes the propagation of waves in shallow water (where the water depth is small relative to the wavelength). It is widely used in various fields such as fluid dynamics, flood propagation, plasma physics, quantum field theory, and so on. This paper investigates the wave solution of the generalized **shallow** water wave equations using three effective methods: Kudryashov-expansion approach, the modified rational sine-cosine approach and the Hirota bilinear approach. We succeeded in obtaining kink solutions, singular kink solutions, periodic solutions, breathing solutions, bright solitons, and complex value solutions. Furthermore, the different wave solutions are depicted by constructing 2D and 3D diagrams to enhance the understanding and verification of our results. This paper is helpful to study the propagation law of fluctuations and the dynamic characteristics of wave changes, and has an important guiding role in port engineering, water resources dispatching, and the prevention of various natural disasters.

Keywords Generalized **shallow** water wave equation, Kudryashov-expansion approach, Rational sine-cosine approach, Hirota bilinear approach, Exact solutions

1 Introduction

The development of the theory and method of accurate solution of nonlinear partial differential equations (NLPDEs) is very important for the in-depth and improvement of soliton theory and integrable systems and their applications in physics.

NLPDEs have solutions known as propagation wave solutions, and it includes a number of types, including soliton (Solitons are special kinds of solitary waves), kink

(Kink waves are travelling waves which rise or descend from one asymptotic state to another), peakon (Peakons are peaked solitary wave solutions), cuspon (Cuspons are other forms of solitons where solution exhibits cusps at their crests), periodic (Periodic solutions are travelling wave solutions that are periodic), compacton (Compacton is a new class of solitons with compact spatial support such that each compacton is a soliton confined to a finite core) and more. Each of these types has its own features. There is no single way to obtain all of these types of solutions at the same time, and each method has its own specific constructions to generate several types. For partial differential equations, obtaining exact solutions with different physical structures can better understand the underlying mechanisms and processes of the dynamical system corresponding to the equations, which is helpful for the development and maintenance of the corresponding models.

As the exact solutions of nonlinear differential equations are very important and crucial to understand the complex phenomena and behaviors, numerous methods have been created to obtain the analytical solutions of partial differential equations in the literature, such as the Kudryashov-expansion approach[1,2,3], Jacobi elliptic function approach[4], polynomial function approach[5,6], the homogeneous balance approach[7], the Hirota's bilinear approach[8,9,10], the extended tanh function approach[11], the positive quadratic function approach[12], the G/G -expansion approach[13,14], the rational sine-cosine functions approach[15], truncated painlevé technique[16], the improve F-expansion method[17,18], the modified extended auxiliary equation mapping method[19], the adomian decomposition method[20], the improved Bernoulli sub-equation function method[21], the generalized auxiliary equation method[22], the modified Khater (MK) method[23] and many others[24,25,26,27,28].

The shallow water wave equation describes the propagation of waves in shallow water. It is widely used in various fields such as fluid dynamics, flood propagation, plasma physics, quantum field theory, reaction-diffusion processes, and so on. In this study, we consider the generalized shallow water wave (GSWW) equation

$$u_t - u_{xxt} - \alpha u u_t - \beta u_x \int^x u dx + u_x = 0, \quad (1.1)$$

where $u = u(x, t)$ is a function of the spatial variable x and the temporal variable t in the wave propagation path, and α, β are arbitrary non-zero constants. In a previous literature, Ablowitz *et. al.*[29] studied the Eq. (1.1) for the specific case $\alpha = 4$ and $\beta = 2$. Hirota *et. al.*[30] investigated the model equation for the case of $\alpha = \beta = 3$. In this study, we consider the more general case that α and β may be arbitrary constants.

Three different methods are employed to obtain a number of traveling wave solutions, i.e., Kudryashov expansion approach, the modified rational sine-cosine functions approach and the Hirota bilinear approach. We succeeded in obtaining kink solutions, singular kink solutions, periodic solutions, breathing solutions, bright solitons and complex value solutions. In addition, we could also obtain further insight of the wave solutions by using 2D and 3D plots.

The subsequent sections are arranged as follows: In Section 2, we have solved the GSWW equation using three methods and obtained five different solutions. The section 3 discusses and analyzes the obtained results, and presents three-dimensional and two-dimensional images. Finally, in Sections 4, we outline the main conclusions.

2 Generalized shallow water wave (GSWW) equation

We remove the integral term in the GSWW Eq. (1.1) by introducing the potential function $u(x, t) = v_x(x, t)$, and then the GSWW Eq. (1.1) is converted into the following form

$$v_{xt} - v_{xxx} - \alpha v_x v_{xt} - \beta v_{xx} v_t + v_{xx} = 0. \quad (2.2)$$

By using the wave variable $z = x - ct$ and Eq. (2.2) can be transformed into

$$(1-c)U' + c(\alpha + \beta)U'U + cU''' = 0. \quad (2.3)$$

Where c is the wave speed and $U = U(z) = v(x, t)$.

2.1 Kudryashov expansion approach

We consider the general partial differential equation

$$F(u, u_x, u_t, u_{xx}, \dots) = 0 \quad (2.4)$$

by virtue of the wave variable $z = x - ct$ can be transformed into the following ordinary differential equation

$$Q(U, U, U', U'', \dots) = 0 \quad (2.5)$$

where has the form of a finite series

$$U(z) = \sum_{i=0}^M A_i Y^i, \quad (2.6)$$

The number M is a positive integer and is obtained by the balance between the linear terms of highest order and the highest order nonlinear terms. where

$Y = Y(z) = \frac{1}{1 + \lambda e^{\mu z}}$ is the solution of the auxiliary differential equation $Y' = \mu Y(Y-1)$, λ denotes the singularity index and determines the singularity of the solution.

Substitute Eq. (2.6) into Eq. (2.5), setting the each coefficient of Y^i to be zero, we can obtain a set of algebraic systems for $\mu, c, \lambda, A_0, A_1, A_2, \dots, A_M$. from the above relations the values of the parameters can be determined.

By balancing the nonlinear term $U'U$ with the highest derivative term U''' , we can obtain $M = 1$. For the Eq. (2.6), the Kudryashov expansion approach admits a solution in the form of

$$U(z) = A_0 + A_1 Y, \quad (2.7)$$

In order to obtain the derivative term in Eq. (2.3), we need to compute the higher-order derivative of Eq. (2.7) using the implicit differential equation of Eq.(2.7).

Thus, we get the following three derivative relations:

$$\begin{aligned} U &= A_1 \mu Y(Y-1), \\ U' &= A_1 \mu^2 Y(Y-1)(2Y-1), \\ U''' &= A_1 \mu^4 Y(Y-1)(24Y^3 - 36Y^2 + 14Y - 1). \end{aligned} \quad (2.8)$$

We substitute Eq. (2.8) into Eq. (2.3) and then collect the coefficients of the same power of $Y^i (i=1 \cdot 5)$. Let each of the coefficients of Y^i be zero and then give the following nonlinear algebraic systems:

$$A\mu^3(A\alpha+A\beta+1\lambda)=0$$

$$A\mu^2(2A\mu(\alpha+\beta)+2\lambda\mu^2-c+1)=0$$

$$A\mu^2(A\mu(\alpha+\beta)+1\lambda\mu^2-3c+3)=0 \quad (2.9)$$

$$A\mu^2(c\mu^2-c+1)=0$$

By solving equations (2.9), we get

$$A = -\frac{12\mu}{\alpha+\beta}, c = -\frac{1}{\mu^2-1}. \quad (2.10)$$

Accordingly, we get the solution of equation Eq. (1.1), which we define as $u_1(x,t)$, and the expression is as follows:

$$u_1(x,t) = A - \frac{12\mu}{(\alpha+\beta)(1+\lambda e^{\mu(x+\frac{1}{\mu^2-1}t)})}. \quad (2.11)$$

2.2. Rational sine-cosine approach

It is well known that rational sine-cosine function approach admits the following form:

$$U(z) = \frac{A+B\sin(\mu z)}{1+D\cos(\mu z)}, \quad (2.12)$$

where A, B, D, μ and C are determined later. We substitute Eq. (2.12) into Eq. (2.3) and then collect the coefficients of the same power of $\sin(\mu z)\cos(\mu z)$. Let each of the coefficients of $\sin(\mu z)\cos(\mu z)$ be zero and then give the following nonlinear algebraic systems:

$$AD(c\mu^2+1)-1=0$$

$$BD(c\mu^2+1)-1=0$$

$$D(8c\mu^2+2c-2)BD+A^2c\mu(\alpha+\beta)D^2+(c-1-1\lambda\mu^2)BD-B^2c\mu(\alpha+\beta)=0$$

$$AB(2B\mu(\alpha+\beta)-(1\lambda\mu^2-c+1)D)=0$$

$$AD(1\lambda\mu^2+3B\mu(\alpha+\beta)D^2+(c-1-1\lambda\mu^2)D-2B\mu(\alpha+\beta))=0$$

$$(B\mu^2(-1D^3+1D)-(\alpha+\beta)(A^2+3B^2)D^2-B^2)\mu+5BD^2-BD-5BD^2+BD=0,$$

$$(1+2(c-8\mu^2-1)D^4+3(3\mu^2+c-1)D^2-c\mu^2-c)B-A^2c\mu(\alpha+\beta)D^3-2B^2c\mu(\alpha+\beta)D^2=0,$$

$$AD(1+3(3\mu^2-c+1)D^2+2B\mu(\alpha+\beta)D-c\mu^2-c)=0,$$

$$AD((5D^2+3D)\mu^2+B(D^2+1)(\alpha+\beta)\mu-D^3-3D)c+(D^3+3D)=0. \quad (2.13)$$

By solving equation systems (2.13), we get

$$\begin{aligned} A &= -\frac{6\mu\sqrt{D^2-1}}{\alpha+\beta}, B = -\frac{6D\mu}{\alpha+\beta}, c = \frac{1}{\mu^2+1}, \\ \text{or} \\ &= \frac{6\mu\sqrt{D^2-1}}{\alpha+\beta}, B = -\frac{6D\mu}{\alpha+\beta}, c = \frac{1}{\mu^2+1}. \end{aligned} \quad (2.14)$$

Accordingly, the second solution of Eq. (2.2) is

$$\begin{aligned} u_2(x,t) &= -\frac{6\mu\left(\sqrt{D^2-1}+D\sin\left(\mu\left(x-\frac{1}{\mu^2+1}t\right)\right)\right)}{(\alpha+\beta)\left(1+D\cos\left(\mu\left(x-\frac{1}{\mu^2+1}t\right)\right)\right)}, \\ \text{or} \\ &= \frac{6\mu\left(\sqrt{D^2-1}-D\sin\left(\mu\left(x-\frac{1}{\mu^2+1}t\right)\right)\right)}{(\alpha+\beta)\left(1+D\cos\left(\mu\left(x-\frac{1}{\mu^2+1}t\right)\right)\right)}. \end{aligned} \quad (2.15)$$

Similarly, if we substitute

$$U(z) = \frac{A+B\cos(\mu z)}{1+D\sin(\mu z)}, \quad (2.16)$$

into Eq. (2.3), we can obtain the third solution of Eq. (2.2), defined as

$$u_3(x,t) = -\frac{6\mu \left(\sqrt{D^2-1} + D \cos \left(\mu \left(x - \frac{1}{\mu^2+1} t \right) \right) \right)}{(\alpha+\beta) \left(1 + D \sin \left(\mu \left(x - \frac{1}{\mu^2+1} t \right) \right) \right)},$$

or

$$= \frac{6\mu \left(\sqrt{D^2-1} - D \cos \left(\mu \left(x - \frac{1}{\mu^2+1} t \right) \right) \right)}{(\alpha+\beta) \left(1 + D \sin \left(\mu \left(x - \frac{1}{\mu^2+1} t \right) \right) \right)}. \quad (2.17)$$

2.3. Hirota bilinear approach

To get solutions to Eq. (2.2), we apply the simplified Hirota's bilinear approach. We first assume

$$v(x,t) = e^{sx-ct}. \quad (2.18)$$

Then, substituting Eq. (2.18) into the linear terms of Eq. (2.2) and we get the dispersion relationship between c and s

$$c = \frac{s}{1-s^2}, s \neq \pm 1. \quad (2.19)$$

Secondly, we use the auxiliary function

$$k(x,t) = 1 + e^{\frac{sx - \frac{s}{1-s^2}t}{1-s^2}}, \quad (2.20)$$

and the Cole-Hopf transformations

$$v(x,t) = R(\ln k(x,t)). \quad (2.21)$$

Plugging Eq. (2.21) in Eq. (2.2) leads to

$$R = \frac{12}{\alpha+\beta}, \alpha+\beta \neq 0. \quad (2.22)$$

Now, we assume $\alpha = \beta$ and by value of Eq. (2.2), we can obtain

$$v_{xt} - v_{xxx} - \beta v_x v_{xt} - \beta v_{xx} v_t + v_{xx} = 0. \quad (2.23)$$

Then, we use the updated hypothetical function

$$v(x,t) = \psi_x(x,t). \quad (2.24)$$

Substituting Eq. (2.24) into Eq. (2.23), and integrating once we obtain

$$\psi_{xt} - \psi_{xxx} - \beta \psi_{xx} \psi_{xt} + \psi_{xx} = 0 \quad (2.25)$$

Combining Eq. (2.21), Eq. (2.22) and Eq. (2.24), we obtain

$$\psi(x,t) = \frac{6}{\beta} \ln(f(x,t)), \beta \neq 0. \quad (2.26)$$

Finally, substituting Eq. (2.26) into Eq. (2.25) and performing a simplification computation to get the following formula

$$f(-f_{xxx} + f_{xt} + f_{xx}) - f_x(f_t + f_x) + 3f_{xx}f_x - 3f_{xx}f_{xt} + f_{xxx}f_t = 0 \quad (2.27)$$

The bilinear form for the Eq.(2.23) is

$$D_x(D_x D_t - D_x - D_t)f \cdot f = 0. \quad (2.28)$$

Where D_x and D_t are the bilinear differential operators:

$$D_x D_t a \cdot b = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^k \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^l a(x,t) \cdot b(x',t') \Big|_{x=x', t=t'} \quad (2.29)$$

Next, the following polynomial test function is chosen, written as

$$f(x,t) = (ax + bt + c)^2 + (dx + \lambda t + \varepsilon)^2 + \gamma, \quad (2.30)$$

where $a, b, c, d, \lambda, \varepsilon$ and γ are determined later. Substituting Eq. (2.30) into Eq. (2.28)

and then collecting each coefficients of the same power of $t^i x^j$ to zero, we can obtain the following algebraic systems:

$$a^4 + a^3 b + (2d^2 + d\lambda)a^2 + abd + d(d^2 + d^2 \lambda) = 0,$$

$$a^3 b + (b^2 + d\lambda + \lambda)a^2 + abd + d(d^2 \lambda + (b^2 + \lambda)d) = 0,$$

$$a^2 c + (bc + \lambda d + \varepsilon \lambda a^2 + acd + d(ad^2 + (bc + \varepsilon \lambda d))) = 0, \quad (2.31)$$

$$(-bc + \varepsilon \lambda a^2 + (-2(b\varepsilon + c\lambda)d - d(b^2 + \lambda))a + d(bc - \varepsilon \lambda d - d(b^2 + \lambda))) = 0,$$

$$-6a^2 b + (-c^2 - 6d\lambda + \varepsilon^2 + \gamma)a^2 + (-6bd - 4\lambda d - 2c\lambda \varepsilon - b(c^2 - \varepsilon^2 - \gamma))a + d(-6d^2 \lambda + (c^2 - \varepsilon^2 + \gamma)(d + \lambda) - 2b\varepsilon) = 0.$$

By solving equations (2.31), we get

Case I:

$$a = \pm id, \quad b = \mp id, \quad \lambda = -d. \quad (2.32)$$

Accordingly,

$$f(x,t) = (\pm id\bar{x}idt+c)^2 + (dx-dt+\varepsilon)^2 + \gamma. \quad (2.33)$$

Recall Eq. (2.24), we can obtain the fourth solution of Eq. (2.2), written as

$$u_4(x,t) = \frac{12(\varepsilon \pm ic)}{\beta((\pm id\bar{x}idt+c)^2 + (dx-dt+\varepsilon)^2 + \gamma)}. \quad (2.34)$$

Case II:

$$a = -\frac{(-\lambda \pm ib)\lambda + b^2 + \mathcal{L}}{2b}, \quad d = -\frac{\lambda \pm ib}{2}, \quad \gamma = 0. \quad (2.35)$$

Thus,

$$f(x,t) = \left(-\frac{(-\lambda \pm ib)\lambda + b^2 + \mathcal{L}}{2b} x + bt + c \right)^2 + \left(\left(-\frac{\lambda \pm ib}{2} \right) x + \mathcal{L} + \varepsilon \right)^2. \quad (2.36)$$

For this case, we get the fifth solution, expressed as $u_5(x,t)$, written in the form of:

$$u_5(x,t) = \frac{6 \left(\left(-\frac{\lambda \pm ib}{2} \right) x + \mathcal{L} + \varepsilon \right) (-\lambda \pm ib) \left(-\frac{(-\lambda \pm ib)\lambda + b^2 + \mathcal{L}}{2b} x + bt + c \right) \frac{(-\lambda \pm ib)\lambda + b^2 + \mathcal{L}}{b}}{\beta \left(\left(-\frac{(-\lambda \pm ib)\lambda + b^2 + \mathcal{L}}{2b} x + bt + c \right)^2 + \left(\left(-\frac{\lambda \pm ib}{2} \right) x + \mathcal{L} + \varepsilon \right)^2 \right)}. \quad (2.34)$$

3 Graphical analysis and discussion

Based on the above analysis, we have obtained five new solutions to the generalized shallow water wave equation, and they are labeled as u_1, u_2, u_3, u_4 and u_5 .

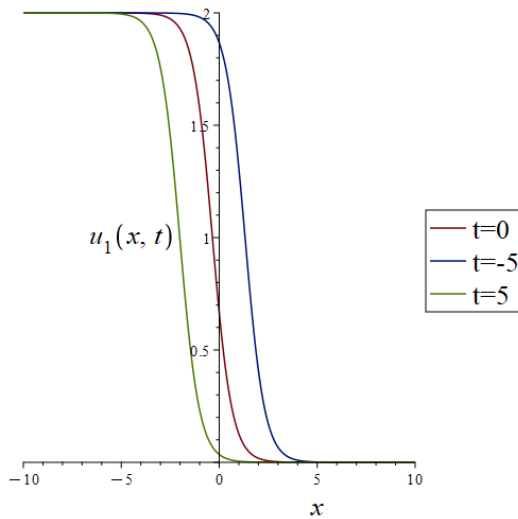
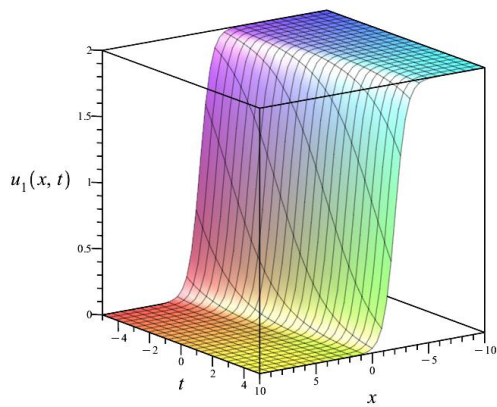
The images of the above results are analyzed below.

Firstly, for the different values of the nonlinear parameters α and β , we have studied the physical structures of $u_1(x,t)$, the singularity of the $u_1(x,t)$ is determined by the positive or negative of the singularity index λ . Now, for $\lambda > 0$, the propagation of u_1 is moving single-kink, see Fig. 1. We find that μ can control the height, the intensity of the change, and the propagation direction of the wave. The larger the value of μ is, the higher the wave is, and the more drastic the waveform

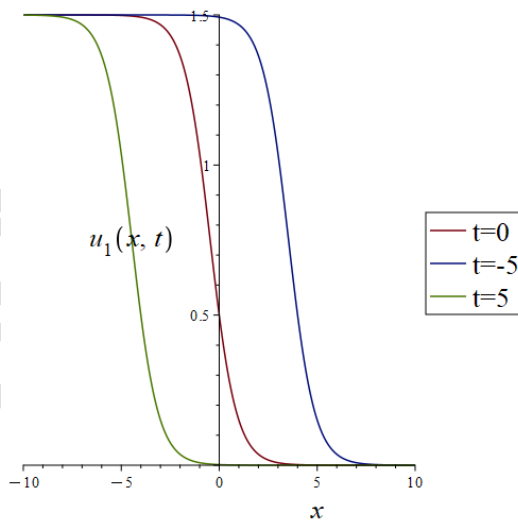
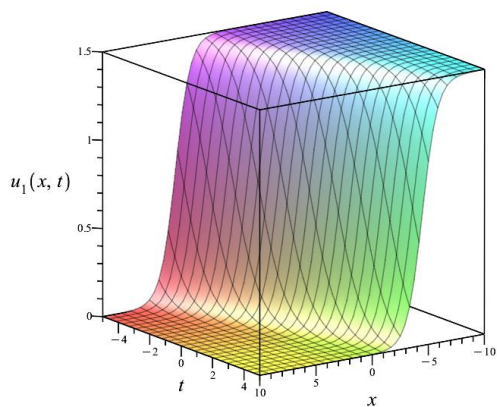
change is. When $\mu \in (1, +\infty)$, the propagation direction of the wave is to the left; When $\mu \in (0, 1)$, the propagation direction is to the right. Moreover, for $\lambda < 0$, different values of μ will result in different images. When $\mu = 2$, u_1 propagates in the form of a breathing wave, which is a periodic wave, see Fig. 2. When $\mu = 1.5$, u_1 propagates in the form of singular kink waves, see Fig. 3.

On the other side, we find that u_2 and u_3 have the same physical structures by using the rational sine-cosine approach. In particular, we use u_2 as an example to make an image of it. For the sake of simplicity, let's choose the parameter $D=1$, in which case the two results of u_2 will be exactly the same. When other variables are fixed and only μ is changed, u_2 is obtained as an even function with respect to μ . We find that the propagation of u_2 is a movement of periodic singular kink-type waves, see Fig. 4.

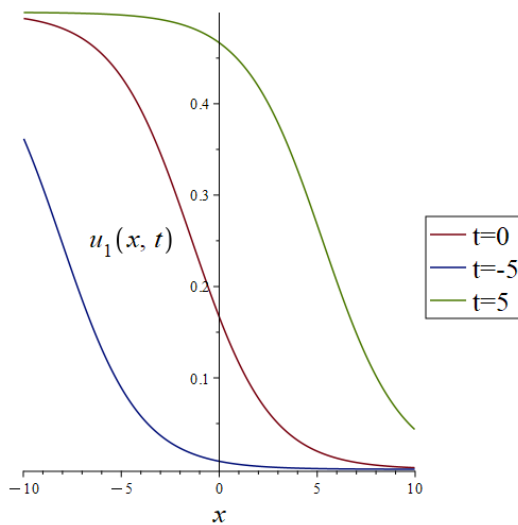
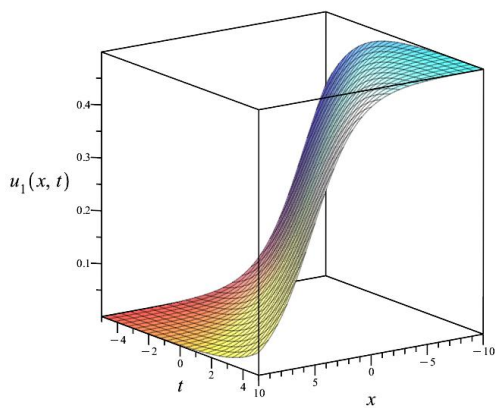
Since some solutions are complex-valued functions, the propagation of their envelopes will be studied. In physics, the envelop function of $u(x, t)$ is defined as $|u(x, t)|^2$ and measures the travelling wave-height, where $|\cdot|$ denotes the norm. In the case of $u_4(x, t)$, the value of C is important. When $C=0$, $u_4(x, t)$ is a real-valued function and it denotes a singular wave, see Fig. 5. But when $C \neq 0$, $u_4(x, t)$ is a complex-valued function and the corresponding envelop is shown in Fig. 6. Obviously, it is a bright soliton.



(a) $\mu=2$



(b) $\mu=1.5$



(c) $\mu=0.5$

Fig.1 3D and 2D plots of the function $u_1(x,t)$ for the Kudryashov-index $\lambda > 0$. Where $A=0, \alpha+\beta=-12, \lambda=2$.

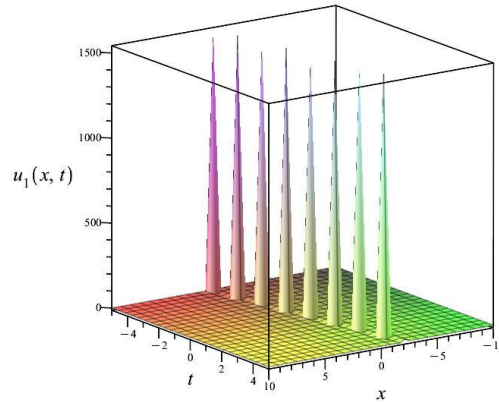


Fig.2 3D plot of the function $u_1(x,t)$ for the Kudryashov-index $\lambda < 0$. Where $\mu=2, A=0, \alpha+\beta=-12, \lambda=-2$.

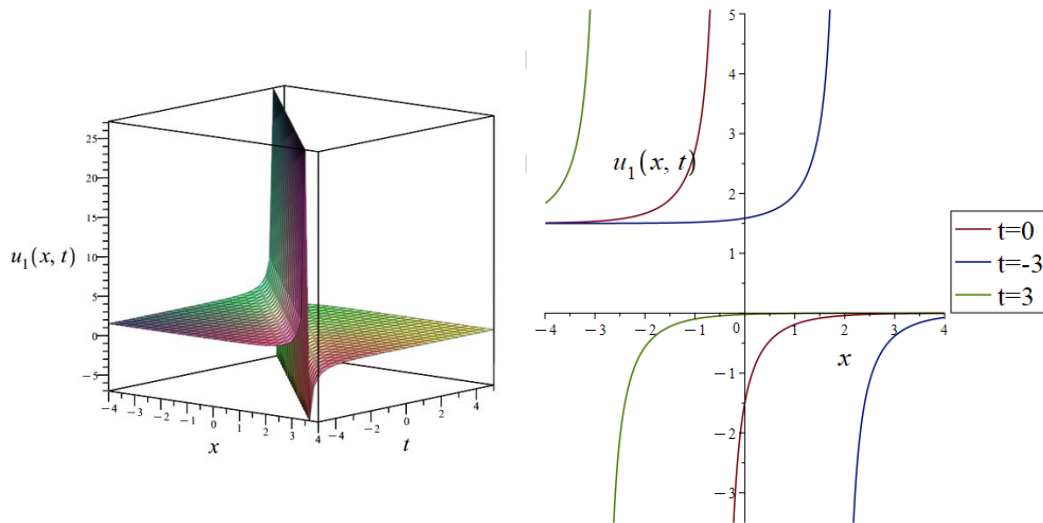


Fig.3 3D and 2D plots of the function $u_1(x,t)$ for the Kudryashov-index $\lambda < 0$. Where $A=0, \alpha+\beta=-12, \lambda=-2, \mu=1.5$.

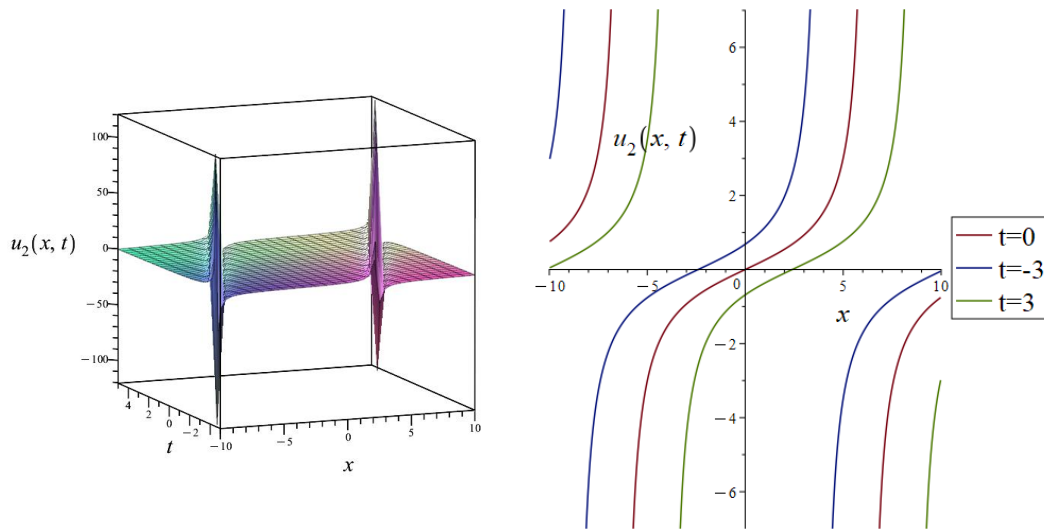


Fig.4 3D and 2D plots of singular kink-type solution of the function $u_2(x, t)$. Where $A=0, \alpha+\beta=-3, \lambda=-2, \mu=0.5$.

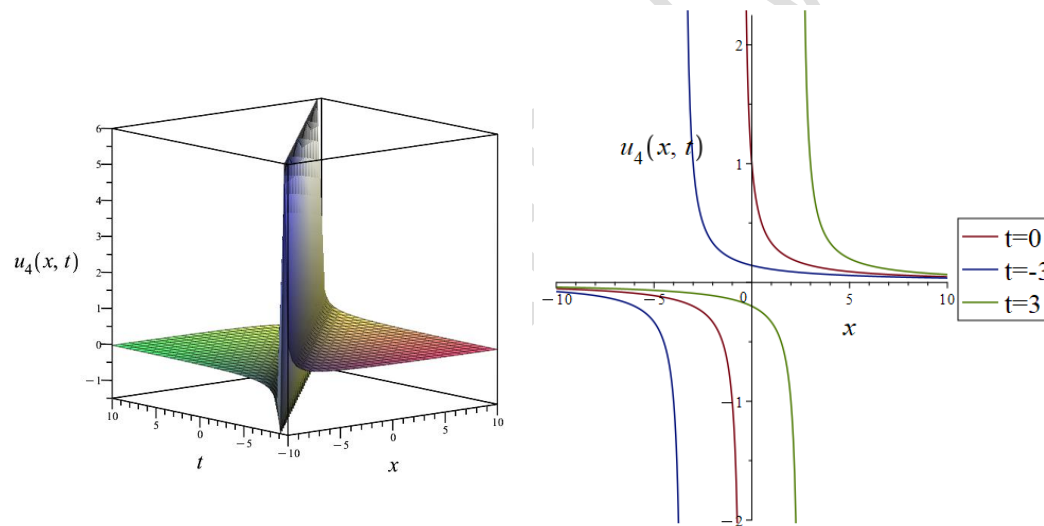


Fig.5 3D and 2D plots of the function $u_4(x, t)$. Where $\beta=1, 2d=\varepsilon=1, c=\gamma=0$.

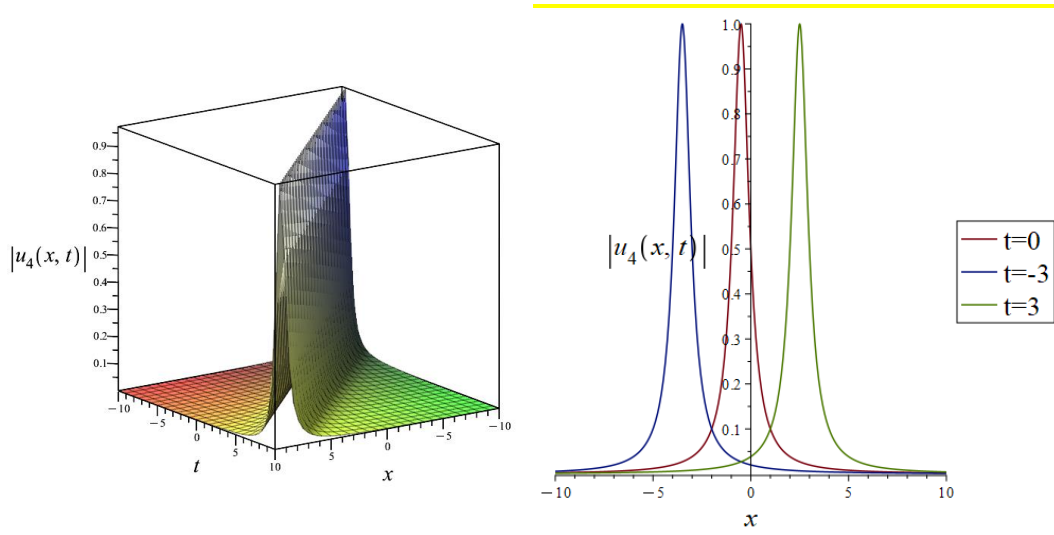


Fig.6 3D and 2D plots of the function $|u_4(x, t)|^2$. Where $\beta=1, d=c=\varepsilon=1, \gamma=0$.

4 Discussion and concluding remarks

In this paper, we have applied wave variable to reduce the generalized shallow wave equation to ordinary differential equation. We have investigated the ordinary differential equation by using the above mentioned three methods, and obtained new exact solutions of the equation with multiple arbitrary parameters, which are explicit traveling wave solutions in the form of trigonometric functions, exponential functions, complex functions and rational functions.

The propagation direction of $u(x, t)$ proves the propagation direction of the wave. The singularity index λ determines the singularity of the solution, which makes the Kudryashov-expansion approach produce singular kink waves and kink waves. For $u_1(x, t)$, we fix the other parameters unchanged and the difference in index λ will result in a very different image. When $u_1(x, t)$ satisfies $\lambda > 0$, it is found that the larger the value of μ is, the higher the image is. The results of rational sine-cosine method and rational cosine-sine method can be obtained by analogy. Both u_2 and u_3 can be obtained by using the rational sine-cosine method. For complex-valued functions u_4 and u_5 , we can discuss its envelope, and also discuss its real and imaginary parts separately.

In our future research, we aim to explore the case of $\alpha \neq \beta$ for the GSWW equation using Hirota bilinear approach, as well as other more complex nonlinear partial differential equations.

Disclaimer (Artificial intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

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