New Explicit Exact Solutions for the (1+1) Dimensional Generalized Short Water Wave Equation

Abstract

Obtaining accurate solutions for a number of nonlinear partial differential equations is very important in a variety of practical applications. This paper investigates the wave solution of the generalized short water wave equations using three effective methods: Kudryashov-expansion approach, the modified rational sine-cosine approach and the Hirota bilinear approach. We succeeded in obtaining kink solutions, singular kink solutions, periodic solutions, breathing solutions, bright solitons, and complex value solutions. Furthermore, the different wave solutions are depicted by constructing 2D and 3D diagrams to enhance the understanding and verification of our results.

Keywords Generalized short water wave equation, Kudryashov-expansion approach, Rational sine-cosine approach, Hirota bilinear approach, Exact solutions

1 Introduction

The development of the theory and method of accurate solution of nonlinear partial differential equations (NLPDEs) is very important for the in-depth and improvement of soliton theory and integrable systems and their applications in physics.

NLPDEs have solutions known as propagation wave solutions, and it includes various types such as soliton, kink, peakon, cusp, periodic, lump, rogue, breather and more. Each of these types has its own features. There is no single way to obtain all of these types of solutions at the same time, and each method has its own specific constructions to generate several types. For partial differential equations, obtaining solutions with different physical structures can better understand the underlying mechanisms and processes of the dynamical system corresponding to the equations, which is helpful for the development and maintenance of the corresponding models.

Since the search for different forms of wave solutions to nonlinear differential equations is one of the goals of this study, many methods for finding wave solutions are constructive and algebraic, for example, the Kudryashov expansion approach (Kudryashov 2012; Alquran 2023), Jacobi elliptic function approach (Fan and Zhang 2002), polynomial function approach (Alquran et al. 2021a; Alquran et al. 2021b), the homogeneous balance approach (Jafari et al. 2014), the Hirota's bilinear approach (Sulaiman et al. 2021; Alquran and Alhami 2022; Ismael et al. 2022), the extended tanh function approach (Raslan et al. 2017), the positive quadratic function approach (Zhu and Liu 2021), the G/G-expansion approach (Ahmet and Özkan 2013), the rational sine-cosine functions approach (Alquran 2022) and many others (Inc et al. 2017; Liu and Ye 2020; Al-Qurashi et al. 2017).

In this study, we consider the generalized short water wave (GSWW) equation

$$u_t - u_{xxt} - \alpha u u_t - \beta u_x \int^x u_t dx + u_x = 0, \tag{1.1}$$

where α and β are non-zero constants. This equation is generally used to describe the motion of shallow water waves in (1+1) dimensional space. In a previous literature, Ablowitz et al. (Ablowitz and Clarkson 1991) studied the equation (1.1) for the specific case $\alpha = 4$ and $\beta = 2$. For the case of $\alpha = \beta = 3$, Hirota et al. (Hirota 2004) investigated the model equation. In this study, we consider the more general case that α and β are arbitrary constants. Three different methods are employed to obtain a number of wave solutions, i.e., Kudryashov expansion approach, the modified rational sine-cosine functions approach and the Hirota bilinear approach. We succeeded in obtaining kink solutions, singular kink solutions, periodic solutions, breathing solutions, bright solitons, and complex value solutions. In addition, we could also obtain further insight of the wave solutions by using 2D and 3D plots.

The subsequent sections are arranged as follows: In Section 2, we have solved the GSWW equation using three methods and obtained five different solutions. The section 3 discusses and analyzes the obtained results, and presents three-dimensional and two-dimensional images. Finally, in Sections 4, we outline the main conclusions.

2 Generalized short water wave (GSWW) equation

We remove the integral term in the GSWW equation (1.1) by introducing the potential function $u(x,t)=v_x(x,t)$, and then the GSWW equation (1.1) is converted into the following form

$$v_{xt}-v_{xxxt}-\alpha v_x v_{xt}-\beta v_{xx} v_t+v_{xx}=0. \tag{2.2}$$

By using the wave variable $z=x-ct$ and equation (2.2) can be transformed into

$$(1-c)U'+c(\alpha+\beta)U'U+cU'''=0. \tag{2.3}$$

Where c is the wave speed and $U=U(z)=v(x,t)$.

2.1 Kudryashov expansion approach

By balancing $U'U$ with U''', we can obtain $M = 1$. For the equation (2.3), the Kudryashov expansion approach admits a solution in the form of

$$U(z)=A+BY, \tag{2.4}$$

where $Y=Y(z)=\dfrac{1}{1+\lambda e^{\mu z}}$ is the solution of the auxiliary differential equation $Y'=\mu Y(Y-1)$, λ denotes the singularity index and determines the singularity of the solution. In order to obtain the derivative term in equation (2.3), we need to compute the higher-order derivative of equation (2.4) using the implicit differential equation of (2.4). Thus, we get the following three derivative relations:

$$\begin{aligned} U'&=B\mu Y(Y-1), \\ U''&=B\mu^2 Y(Y-1)(2Y-1), \\ U'''&=B\mu^4 Y(Y-1)(24Y^3-36Y^2+14Y-1). \end{aligned} \tag{2.5}$$

We substitute (2.5) into (2.3) and then collect the coefficients of the same power of $Y^i(i=1\cdots5)$. Let each of the coefficients of Y^i be zero and then give the following nonlinear algebraic systems:

$$
\begin{aligned}
0&=Bc\mu^3(B\alpha+B\beta+12\mu),\\
0&=B\mu^2(2Bc\mu(\alpha+\beta)+25c\mu^2-c+1),\\
0&=B\mu^2(Bc\mu(\alpha+\beta)+15c\mu^2-3c+3),\\
0&=B\mu^2(c\mu^2-c+1).
\end{aligned}
\tag{2.6}
$$

By solving equations (2.6), we get

$$B=-\frac{12\mu}{\alpha+\beta},\ c=-\frac{1}{\mu^2-1}. \tag{2.7}$$

Accordingly, we get the solution of equation (1.1), which we define as $u_1(x,t)$, and the expression is as follows:

$$u_1(x,t)=A-\frac{12\mu}{(\alpha+\beta)\left(1+\lambda e^{\mu\left(x+\frac{1}{\mu^2-1}t\right)}\right)}. \tag{2.8}$$

2.2. Rational sine-cosine approach

It is well known that rational sine-cosine function approach admits the following form:

$$U(z)=\frac{A+B\sin(\mu z)}{1+D\cos(\mu z)}, \tag{2.9}$$

where A, B, D, μ and c are determined later. We substitute (2.9) into (2.3) and then collect the coefficients of the same power of $\sin^i(\mu z)\cos^j(\mu z)$. Let each of the coefficients of $\sin^i(\mu z)\cos^j(\mu z)$ be zero and then give the following nonlinear algebraic systems:

$$
\begin{aligned}
0&=AD^4(c(\mu^2+1)-1),\\
0&=BD^3(c(\mu^2+1)-1),\\
0&=D((8c\mu^2+2c-2)BD^3+A^2c\mu(\alpha+\beta)D^2+(c-1-11c\mu^2)BD\\
&\quad-B^2c\mu(\alpha+\beta)),\\
0&=AD^3(2Bc\mu(\alpha+\beta)-(11c\mu^2-c+1)D),\\
0&=AD(18D^3c\mu^2+3Bc\mu(\alpha+\beta)D^2+(c-1-11c\mu^2)D-2Bc\mu(\alpha+\beta)),\\
0&=(B\mu^2(-19D^3+11D)-(\alpha+\beta)((A^2+3B^2)D^2-B^2)\mu+5BD^3-BD)c\\
&\quad-5BD^3+BD,\\
0&=(1+2(c-8c\mu^2-1)D^4+3(3c\mu^2+c-1)D^2-c\mu^2-c)B\\
&\quad-A^2c\mu(\alpha+\beta)D^3-2B^2c\mu(\alpha+\beta)D^3,\\
0&=AD(1+3(3c\mu^2-c+1)D^2+2Bc\mu(\alpha+\beta)D-c\mu^2-c),\\
0&=AD((5D^3+3D)\mu^2+B(D^2+1)(\alpha+\beta)\mu-D^3-3D)c+D^3+3D).
\end{aligned}
\tag{2.10}
$$

By solving equations (2.10), we get

$$
\begin{aligned}
&A=-\frac{6\mu\sqrt{D^2-1}}{\alpha+\beta},B=-\frac{6D\mu}{\alpha+\beta},c=\frac{1}{\mu^2+1},\\
&or\\
&=\frac{6\mu\sqrt{D^2-1}}{\alpha+\beta},B=-\frac{6D\mu}{\alpha+\beta},c=\frac{1}{\mu^2+1}.
\end{aligned}
\tag{2.11}
$$

Accordingly, the second solution of equation (2.2) is

$$
\begin{aligned}
&u_2(x,t)=-\frac{6\mu\left(\sqrt{D^2-1}+D\sin\left(\mu\left(x-\frac{1}{\mu^2+1}t\right)\right)\right)}{(\alpha+\beta)\left(1+D\cos\left(\mu\left(x-\frac{1}{\mu^2+1}t\right)\right)\right)},\\
&or\\
&=\frac{6\mu\left(\sqrt{D^2-1}-D\sin\left(\mu\left(x-\frac{1}{\mu^2+1}t\right)\right)\right)}{(\alpha+\beta)\left(1+D\cos\left(\mu\left(x-\frac{1}{\mu^2+1}t\right)\right)\right)}.
\end{aligned}
\tag{2.12}
$$

Similarly, if we substitute

$$
U(z)=\frac{A+B\cos(\mu z)}{1+D\sin(\mu z)},
\tag{2.13}
$$

into equation (2.3), we can obtain the third solution of (2.2), defined as

$$u_3(x,t) = -\frac{6\mu\left(\sqrt{D^2-1}+D\cos\left(\mu\left(x-\frac{1}{\mu^2+1}t\right)\right)\right)}{(\alpha+\beta)\left(1+D\sin\left(\mu\left(x-\frac{1}{\mu^2+1}t\right)\right)\right)},$$

$$or \tag{2.14}$$

$$= \frac{6\mu\left(\sqrt{D^2-1}-D\cos\left(\mu\left(x-\frac{1}{\mu^2+1}t\right)\right)\right)}{(\alpha+\beta)\left(1+D\sin\left(\mu\left(x-\frac{1}{\mu^2+1}t\right)\right)\right)}.$$

2.3. Hirota bilinear approach

To get solutions to equation (2.2) , we apply the simplified Hirota's bilinear approach. We first assume

$$v(x,t) = e^{sx-ct}. \tag{2.15}$$

Then, substituting (2.15) into the linear terms of (2.2) and we get the dispersion relationship between c and s

$$c = \frac{s}{1-s^2}, s \neq \pm 1. \tag{2.16}$$

Secondly, we use the auxiliary function

$$k(x,t) = 1 + e^{sx - \frac{s}{1-s^2}t}, \tag{2.17}$$

and the Cole-Hopf transformations

$$v(x,t) = R(\ln k(x,t))_x. \tag{2.18}$$

Plugging (2.18) in (2.2) leads to

$$R = \frac{12}{\alpha+\beta}, \alpha+\beta \neq 0. \tag{2.19}$$

Now, we assume $\alpha=\beta$ and by value of (2.2) , we can obtain

$$v_{xt} - v_{xxxt} - \beta v_x v_{xt} - \beta v_{xx} v_t + v_{xx} = 0. \tag{2.20}$$

Then, we use the updated hypothetical function

$$v(x,t) = \psi_x(x,t). \tag{2.21}$$

Substituting (2.21) in (2.20), and integrating once we obtain

$$\psi_{xt}-\psi_{xxxt}-\beta\psi_{xx}\psi_{xt}+\psi_{xx}=0. \tag{2.22}$$

Combining (2.18), (2.19), and (2.21), we obtain

$$\psi(x,t)=\frac{6}{\beta}\ln(f(x,t)),\beta\neq 0. \tag{2.23}$$

Finally, substituting (2.23) into (2.22) and performing a simplification to get the following formula

$$f(-f_{xxxt}+f_{xt}+f_{xx})-f_x(f_t+f_x)+3f_{xxt}f_x-3f_{xx}f_{xt}+f_{xxx}f_t=0. \tag{2.24}$$

The bilinear form for the Eq.(2.20) is

$$D_x(D_x^2D_t-D_x-D_t)f\cdot f=0. \tag{2.25}$$

Where D_x and D_t are the bilinear differentials operators:

$$D_t^kD_x^la\cdot b=\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^k\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^l a(x,t)\cdot b(x',t')|_{x=x',t=t'} \tag{2.26}$$

Next, the following polynomial test function is chosen, written as

$$f(x,t)=(ax+bt+c)^2+(dx+\lambda t+\varepsilon)^2+\gamma, \tag{2.27}$$

where $a,b,c,d,\lambda,\varepsilon$ and γ are determined later. Substituting (2.27) into (2.25) and then collecting each coefficients of the same power of t^ix^j to zero, we can obtain the following algebraic systems:

$$\begin{aligned}
&0=a^4+a^3b+(2d^2+d\lambda)a^2+abd^2+d(d^3+d^2\lambda),\\
&0=a^3b+(b^2+d\lambda+\lambda^2)a^2+abd^2+d(d^2\lambda+(b^2+\lambda^2)d),\\
&0=a^3c+(bc+\varepsilon d+\varepsilon\lambda)a^2+acd^2+d(\varepsilon d^2+(bc+\varepsilon\lambda)d),\\
&0=(-b^2+\lambda^2)a^2+(-b^3-4\lambda bd-b\lambda^2)a+d((b^2-\lambda^2)d-b^2\lambda-\lambda^3),\\
&0=(-bc+\varepsilon\lambda)a^2+(-2(b\varepsilon+c\lambda)d-c(b^2+\lambda^2))a+d((bc-\varepsilon\lambda)d-\varepsilon(b^2+\lambda^2)),\\
&0=-6a^3b+(-c^2-6d\lambda+\varepsilon^2+\gamma)a^2+(-6bd^2-4\varepsilon cd-2c\lambda\varepsilon-b(c^2-\varepsilon^2-\gamma))a\\
&\quad+d(-6d^2\lambda+(c^2-\varepsilon^2+\gamma)(d+\lambda)-2bc\varepsilon).
\end{aligned} \tag{2.28}$$

By solving equations (2.28), we get

Case I:

$$a=\pm id,\ b=\mp id,\ \lambda=-d. \tag{2.29}$$

Accordingly,

$$f(x,t)=(\pm idx\mp idt+c)^2+(dx-dt+\varepsilon)^2+\gamma. \tag{2.30}$$

Recall (2.21), we can obtain the fourth solution of (2.2), defined as

$$u_4(x,t)=\frac{12d(\varepsilon\pm ic)}{\beta\left((\pm idx\mp idt+c)^2+(dx-dt+\varepsilon)^2+\gamma\right)}. \tag{2.31}$$

Case II:

$$a=-\frac{(-\lambda\pm ib)\lambda+b^2+\lambda^2}{2b},\ \ d=-\frac{\lambda}{2}\pm\frac{ib}{2},\ \gamma=0. \tag{2.32}$$

Thus,

$$f(x,t)=\left(-\frac{(-\lambda\pm ib)\lambda+b^2+\lambda^2}{2b}x+bt+c\right)^2+\left(\left(-\frac{\lambda}{2}\pm\frac{ib}{2}\right)x+\lambda t+\varepsilon\right)^2. \tag{2.33}$$

For this case, we get the fifth solution, defined as $u_5(x,t)$, written in the form of:

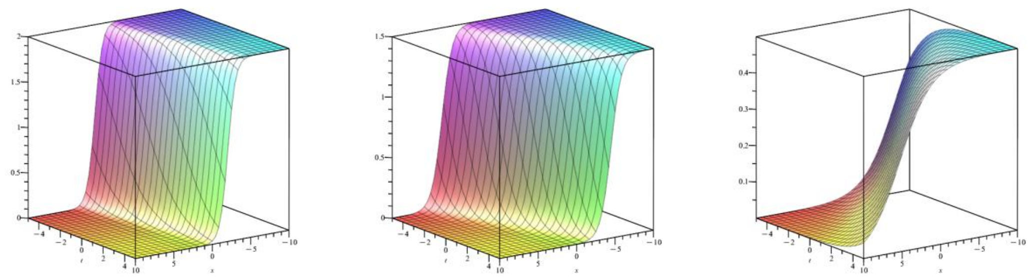
$$u_5(x,t)=\frac{6\left(\left(\left(-\frac{\lambda}{2}\pm\frac{ib}{2}\right)x+\lambda t+\varepsilon\right)(-\lambda\pm ib)-\left(-\frac{(-\lambda\pm ib)\lambda+b^2+\lambda^2}{2b}x+bt+c\right)\frac{(-\lambda\pm ib)\lambda+b^2+\lambda^2}{b}\right)}{\beta\left(\left(-\frac{(-\lambda\pm ib)\lambda+b^2+\lambda^2}{2b}x+bt+c\right)^2+\left(\left(-\frac{\lambda}{2}\pm\frac{ib}{2}\right)x+\lambda t+\varepsilon\right)^2\right)}. \tag{2.34}$$

3 Graphical analysis and discussion

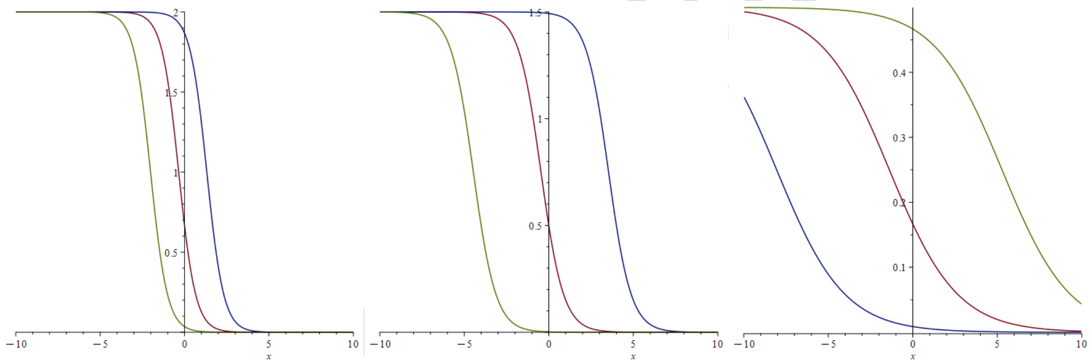
Based on the above analysis, we obtained five new solutions to the generalized short water wave equation, and they are labeled as u_1, u_2, u_3, u_4 and u_5. The images of the above results are analyzed below.

Firstly, for the different values of the nonlinear parameters α and β, we have studied the physical structures of $u_1(x,t)$, the singularity of the $u_1(x,t)$ is determined by the positive or negative of the singularity index λ. Now, for $\lambda>0$, the propagation of u_1 is moving single-kink, see Fig. 1. We found that μ can control the height, the intensity of the change, and the direction of propagation of the wave. The larger the value of μ is, the higher the wave is, and the more drastic the waveform change is. When $\mu\in(1,+\infty)$, the direction of propagation of the wave is to the left; When

$\mu \in (0,1)$, the direction of propagation is to the right. Moreover, for $\lambda < 0$, different values of μ will result in different images. When $\mu = 2$, u_1 propagates in the form of a breathing wave, which is a periodic wave, see Fig. 2. When $\mu = 1.5$, u_1 propagates in the form of singular kink waves, see Fig. 3.



(a) $\mu = 2$ **(b)** $\mu = 1.5$ **(c)** $\mu = 0.5$



(d) $t = 5, 0, -5$ **(e)** $t = 5, 0, -5$ **(f)** $t = -5, 0, 5$

Fig.1 3D and 2D plots of the function $u_1(x,t)$ for the Kudryashov-index $\lambda > 0$. Where $A = 0, \alpha + \beta = -12, \lambda = 2$, other parameters are shown in the figure. (*t* from left to right).

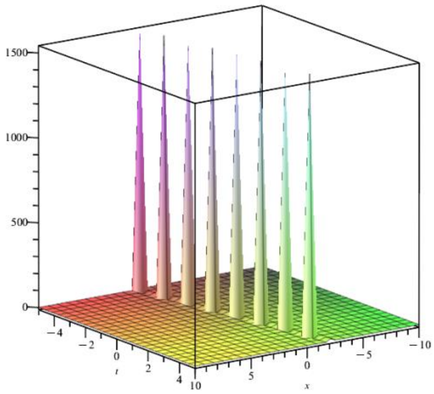
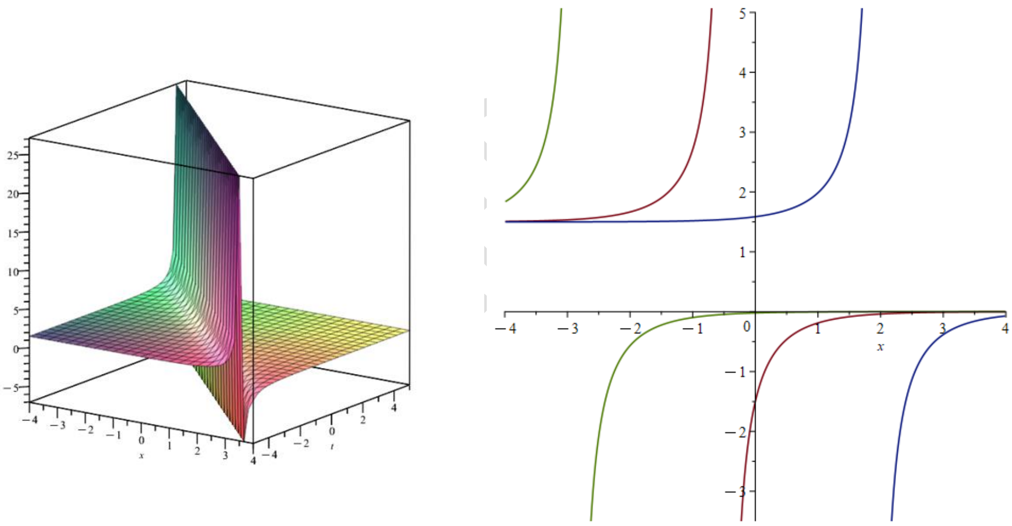


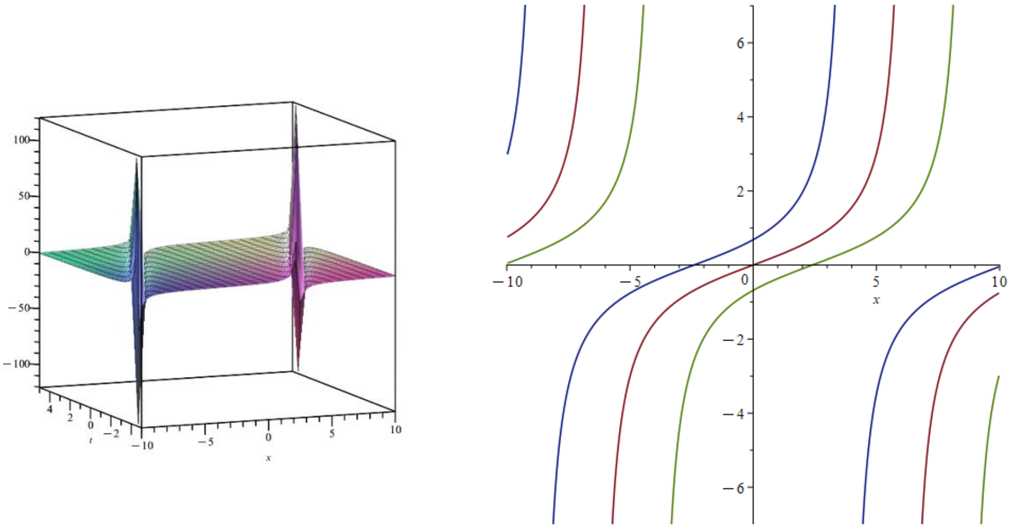
Fig.2 3D plot of the function $u_1(x,t)$ for the Kudryashov-index $\lambda<0$. Where $\mu=2, A=0, \alpha+\beta=-12, \lambda=-2$.



(a) $\mu=1.5$ **(b)** $t=3,0,-3$

Fig.3 3D and 2D plots of the function $u_1(x,t)$ for the Kudryashov-index $\lambda<0$. Where $A=0, \alpha+\beta=-12, \lambda=-2$, other parameters are shown in the figure (t from left to right).

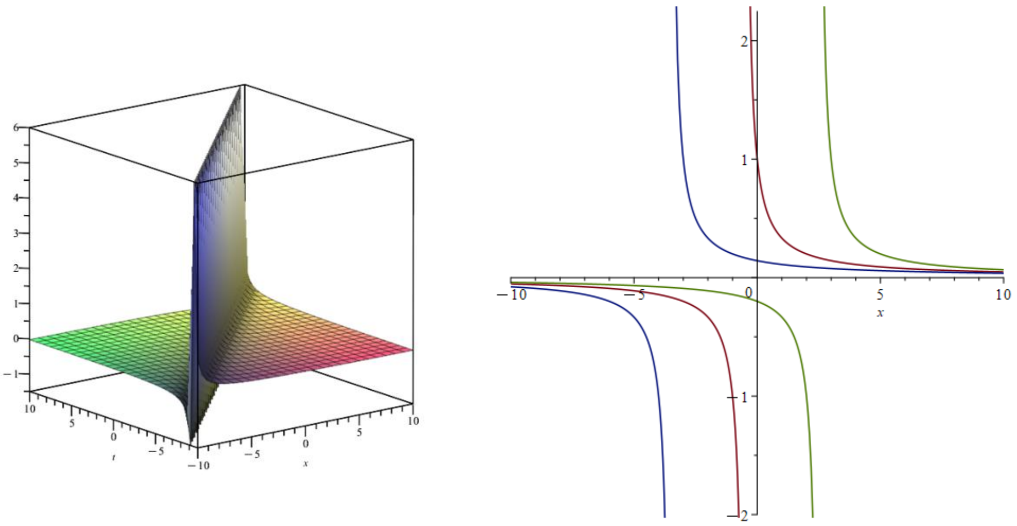
On the other side, we find that u_2 and u_3 have the same physical structures by using the rational sine-cosine approach. In particular, we use u_2 as an example to make an image of it. For the sake of simplicity, let's choose the parameter $D=1$, in which case the two results of u_2 will be exactly the same. When other variables are fixed and only μ is changed, u_2 is obtained as an even function with respect to μ. We find that the propagation of u_2 is a movement of periodic singular kink-type waves, see Fig. 4.



(a) $\mu=0.5$ **(b)** $t=-3,0,3$

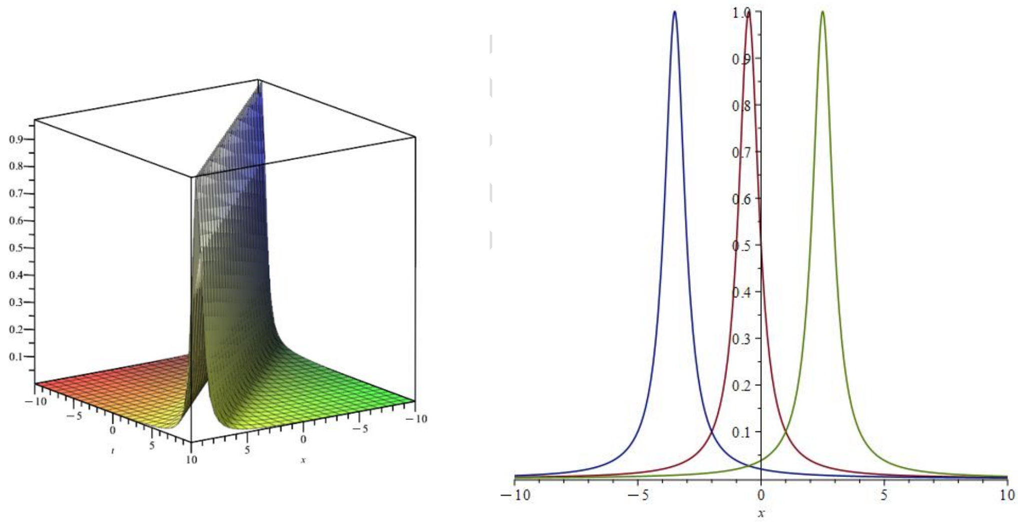
Fig.4 3D and 2D plots of the function $u_2(x,t)$. Where $A=0, \alpha+\beta=-3, \lambda=-2$, other parameters are shown in the figure. (*t* from left to right).

Since some solutions are complex-valued functions, the propagation of the their envelopes will be studied. In physics, the envelop function of $u(x,t)$ is defined as $|u(x,t)|^2$ and measures the travelling wave-height, where $|.|$ denotes the norm. In the case of $u_4(x,t)$, the value of c is important. When $c=0$, $u_4(x,t)$ is a real-valued function and it denotes a singular wave, see Fig. 5. But when $c\neq 0$, $u_4(x,t)$ is a complex-valued function and the corresponding envelop is shown in Fig. 6. Obviously, it is a bright soliton.



(a) $u_4(x,t)$ **(b)** $t=-3,0,3$

Fig.5 3D and 2D plots of the function $u_4(x,t)$. Where $\beta=12,d=\varepsilon=1,c=\gamma=0$, other parameters are shown in the figure. (*t* from left to right).



(a) $|u_4(x,t)|^2$ **(b)** $t=-3,0,3$

Fig.6 3D and 2D plots of the function $|u_4(x,t)|^2$. Where $\beta=12,d=c=\varepsilon=1,\gamma=0$, other parameters are shown in the figure. (*t* from left to right).

4 Discussion and concluding remarks

In this paper, we have applied wave variable to reduce the generalized short wave equation to ordinary differential equation. We study the ordinary differential equation by using the above mentioned three methods, and obtain new exact solutions of the equation with multiple arbitrary parameters, which are explicit traveling wave solutions in the form of trigonometric functions, exponential functions, complex functions and rational functions.

The direction of propagation of $u(x,t)$ proves the direction of propagation of the wave. The singularity index λ determines the singularity of the solution, which makes the Kudryashov-expansion approach produce singular kink waves and kink waves. For $u_1(x,t)$, we fix the other parameters unchanged and the difference in index λ will result in a very different image. When $u_1(x,t)$ satisfies $\lambda > 0$, it is found that the larger the value of μ is, the higher the image is. The results of rational sine-cosine method and rational cosine-sine method can be obtained by analogy. Both u_2 and u_3 can be obtained by using the rational sine-cosine method. For complex-valued functions u_4 and u_5, we can discuss its envelope, or discuss its real and imaginary parts separately.

In our future research, we aim to explore the case of $\alpha \neq \beta$ for the GSWW equation using Hirota bilinear approach, as well as other more complex nonlinear partial differential equations.

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