ITERATIVE METHODS FOR THE SOLUTION OF HESTON EQUATION WITH MAMADU-NJOSEH POLYNOMIALS

Abstract

In this paper, a modified form of the Homotopy Perturbation Method (HPM) and the Variational Iteration Method (VIM), both developed by J.H. He, using the Mamadu-Njoseh polynomials (MNPs) which is new orthogonal polynomial as developed by E.j. Mamadu and I.N. Njoseh as modifier and basis function is presented and comparison made to determine which approximates the Heston Stochastic Partial Differential Equation (HSPDE) faster to its exact solution. The Heston SPDE is a volatility model for determining the European bond and currency options as determined by stock pricing. Other methods exist in literature in determining the numerical solution to the HSPDE whose analytic solution is difficult to arrive at. Modifying the HPM and the VIM by the MNPs has created a new technique in such regards of which comparing with those in existing literature has proven that our method has a faster approximation to the exact solution. However in this work, it was observed that the Modified VIM approximates faster than the modified HPM.

Keyword: Homotopy Perturbation Method (HPM), Variational Iteration Method (VIM), orthogonal polynomials, Mamadu-Njoseh Polynomials (MNPs), Heston Stochastic Partial Differential Equation (HSPDE).

1. Introduction

Based on the limitations of most perturbation methods which served as an- alytical techniques for solving nonlinear problems, Ji-Huan He in 1998 developed the Homotopy perturbation method (HPM), a special technique which combines principles from the field of topology and the usual perturbation techniques. Gupta [1] used the HPM to derive an exact solution for the coupled one-dimensional time fractional nonlinear shallow water system which is a system of PDEs involved with the flow of fluids in riverine areas. Syed [2] employed the HPM to find the solution of some linear and non- linear PDEs, while [3] employed the HE's HPM in their article to derive a solution for a nonlinear system of two-dimensional Volterra-Fredholm integral equations. A year later, [4] developed an iterative method for solving differential and integral equations. This method, the Variational Iteration Method (VIM) was devoid of the limitations of the Grid point techniques, the Spline solution, the Perturbation method and the Adomian method [5]. The method has the ability to treat linear and non-linear equations alike without any unrealistic assumptions. Safari[6] applied the VIM to derive the analytic solution space fractional diffusion equation, while [7] employed the method on the non-linear free vibration of conservative oscillator. Elsheikh [8] ap- plied a modified form of the VIM to solve a fourth order parabolic partial differential equation with variable coefficients and [9] in same vain applied a modified VIM to obtain the solution of some non-linear, nonhomogeneous differential equations. Boazan [10] compared the VIM against the ADM and HPM on the numerical solutions of the Heston partial differentia equation where they concluded that the method is much easier, more convenient, more stable and efficient than the other two iterative methods. Mamadu-Njoseh Polynomials (MNPs) was developed and employed on the numerical solu- tions of fifth order boundary value problems by [11]. The MNPs, a $C^{[a,b]}$ orthogonal polynomial was also used to derive the numerical solutions of the Voltera equation using Galerkin method [11]. This paper is geared towards using the MNPs as a modifier and basis function to modify the HPM and the VIM in arriving at the approximate solution of Heston stochastic partial differential equation via the orthogonal collocation method and comparisons made to determine which approximates the HSPDE faster to its exact solution. The swift convergence of the MVIM is represented graphically as against the MHPM.

2. Heston Stochastic Partial Differential Equation (HSPDE)

According to [12], the Heston model is a typical stochastic volatility model of the form
$\alpha(t, S(t), V(t)) = (a - bV(t))$ and $(t, S(t), V(t)) = \sigma\sqrt{(V(t))}$, while [10] gives the Heston model as

$$\begin{aligned} \frac{(dS(t))}{(S(t))} &= rdt + \sigma\sqrt{V}d\widehat{W}_1 t \\ \frac{dV(t)}{d(t)} &= (a - bV(t)) + \sigma\sqrt{V}\widehat{W}_2 t \end{aligned} \tag{1}$$

where α is the option price, β is the price of the volatility risk, r is the interest rate, $S(t)$ is the asset price at time t, $V(t)$ is the volatility of the asset price at time t with $\sqrt{V}$ as the variance of the volatility, a is the long-run mean, b is the speed of the mean reversion, σ is the volatility of the of the variance process, while $d\widehat{W_1}(t)$ and $d\widehat{W_2}(t)$ are correlated Brownian motions under the risk-neutral measure with the correlation coefficient $\rho \in (1,1)$ such that

$$d\widehat{W_1}(t)d\widehat{W_2}(t) \tag{2}$$

The risk-neutral price of a call expiring at time $t \leqslant T$ in the Heston stochastic volatility model is given as

$$c(t, S(t), V(t)) = \widehat{E}\left[e^{-r(T-t)}(S(T) - K^{+}\right], 0 \leqslant t \leqslant T \tag{3}$$

The equation

$$\frac{\partial c}{\partial t} + rs\frac{\partial c}{\partial s} + (a - bv)\frac{\partial c}{\partial v} + \frac{1}{2}s^2 v\frac{\partial^2 c}{\partial s^2} + \rho\sigma sv\frac{\partial^2 c}{\partial s\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 c}{\partial v^2} - rc = 0 \tag{4}$$

is the Heston partial differential equation (PDE) for the fair values of European style options forming a time dependant convection diffusion reaction equation with mixed spatial derivative terms. The Heston PDE (2.4) has the initial and boundary conditions given as

$$\begin{aligned} C(S, v, t) &= max(0, s - K) \\ C(0, v, t) &= 0 \end{aligned} \tag{5}$$

where k $>$ 0 is the given strike price.

3. He's Homotopy Perturbation Method (HPM)

According to He [13], the applications of the homotopy perturbation method mainly cover in nonlinear differential equations, nonlinear integral equations, nonlinear differential-integral equations, difference differential equations, and fractional differential equations. He [14] noted that the earlier perturbation methods were limited based on what was called the "small parameter as- sumption" where it was assumed that a small parameter whose appropriate choice that leads to ideal results must exist in an equation. Should these so called "small parameters" be chosen without care and suitability, the given result will become inappropriate. With the existence of this assumption, it became difficult to explore greatly many nonlinear problems as most of them have big parameters. The work of the HPM was to eliminate this limitation existing in the traditional perturbation methods.
By the homotopy technique, a homotopy $V(r, \rho) : \infty \to [0, 1] \to \Re$ is constructed satisfying

$$x(V, \rho) = (1 - \rho)[L(V) - L(U_0)] + \rho[(V) - f(r)] = 0, \quad \rho \in [0, 1], \quad r \in \infty \tag{6}$$

or

$$x(V, \rho) = L(V) - L(U0) + \rho L(U0) + \rho[N(V) - f(r)] = 0 \tag{7}$$

4. He's Homotopy Perturbation Method (HPM)

Ji Huan He in 1999 developed a Variational Iteration Method (VIM) in solving linear and nonlinear equations alike without any unrealistic assumptions[15] concluded that the method converges faster to the exact solution by successive approximation. Unlike the Adomian Decomposition Method (ADM) developed by George Adomian in 1982, the VIM does not require any form of polynomials to obtain its approximate solution. According to [16], the basic concept of the VIM is to construct a correction functional for nonlinear systems. This is given as

$$Un + 1(x, t) = Un(x, t) + \int_0^t \lambda(LUn(\tau) + N\widehat{U}n(\tau)g(\tau))d\tau \tag{8}$$

where λ is a general Langrange multiplier, which can be identified optimally via the variational theory. $NU\widehat{n}(\tau)$ is considered as the restricted variations. If we set the Langrange multiplier $\lambda = 1$, then (8) will be given as

$$Un + 1(x, t) = Un(x, t)(LUn(\tau) + \int_0^t NUn(\tau) - g(\tau))d\tau \tag{9}$$

$NUn(\tau)$ is called the correction term and (9) can be solved iteratively using $U0(x)$ as the initial approximation with possible unknowns.

5. Mamadu-Njoseh Polynomials (MNPs)

The MNPs are a set of orthogonal polynomials having an interval of $[1, 1]$ and a weight function of $w(x) = (1+x^2)$. It is given as

$$\int_{-1}^{1} \varphi_m(x)\varphi_n(x)(1+x^2)dx = 0 \tag{10}$$

with the first four MNPs given as

$$\begin{aligned} \varphi_0(x) &= 1 \\ \varphi_1(x) &= 0 \\ \varphi_2(x) &= \frac{1}{3}(5x^2 - 2) \\ \varphi_3(x) &= \frac{1}{5}(14x^3 - 9x) \end{aligned} \tag{11}$$

6. Modified HPM for Heston SPDE

The scheme for generating the initial approximation through the OCM with the Mamadu-Njoseh polynomials as basis function is described as follows:

Let the initial approximation be given as

$$C_O = \sum_{i=1}^{N} a_i \varphi_i(x) \tag{12}$$

where a_i are unknown constants to be determined and $\varphi_i(x)$ are the Mamadu-Njoseh Polynomials with interval of orthogonality $[-1, 1]$. According to [10], the Heston SPDE has a generalized initial condition

$$C_(O) = 2s^2t^2 \tag{13}$$

Incorporating (12) and (13), we have

$$C_{(O)} = \sum_{i=0}^{N} a_i \varphi_i(x) = 2s^2t^2 \tag{14}$$

Solving (14) at $N = 3$ (chosen arbitrarily) and substituting the $\varphi_i(x), i = 0, 1, 2, 3$, we have

$$a_0 + a_1 s + a_2 \left(\frac{5}{3}s^2 - \frac{2}{3}\right) + a_3 \left(\frac{14}{5}s^3 - \frac{9}{5}s\right) = 2s^2t^2 \tag{15}$$

In [10], the value of t is defined within the range $0 \le t \le T$. Thus, collocating (6.4) at the zeroes of $\varphi_4(x)$, that is,

$$s = 0.3676425560, -0.3676425560, 0.8756710201, -0.8756710201$$

and writing the resulting linear algebraic equations in the form

$$A\underline{X} = \underline{b} \tag{16}$$

where,

$$A = \begin{bmatrix} 1 & 0.3676425560 & -0.4413982517 & -0.5226219309 \\ 1 & -0.3676425560 & -0.4413982517 & 0.5226219309 \\ 1 & 0.8756710201 & 0.6113328923 & 0.303892222 \\ 1 & -0.8756710201 & 0.6113328923 & -0.303892222 \end{bmatrix}$$

$$\underline{X} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} 52.98313121 \\ 52.98313121 \\ 300.5854963 \\ 300.5854963 \end{bmatrix}$$

Solving (16) using Gaussian elimination method, we obtain the following values for the constant $a_i's$

$$a_0 = 156.800000$$
$$a_1 = 0.000000$$
$$a_2 = 235.200000$$
$$a_3 = 0.000000$$

Thus, substituting the above in (14), we obtain

$$c_0 = 392.00000s^2 \tag{17}$$

We use the initial approximation $c_0(s, v, 0) = 392.00000s^2$ as given in (17) satisfying the initial condition for the Heston SPDE. We introduce the structure of the HPM as relating to the HSPDE given as

$$H(s, v, t) = (1-p)(L(c) - L(v_0)) + p(\frac{\partial c}{\partial t} - rs\frac{\partial c}{\partial s} + (a - bv)\frac{\partial c}{\partial v} + \frac{1}{2}s^2v\frac{\partial^2 c}{\partial s^2} + \rho\sigma sv\frac{\partial^2 c}{\partial s\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 c}{\partial v^2} - rc) = 0 \tag{18}$$

with given algorithm

$$\begin{aligned}
P^0 &= \frac{\partial c_0}{\partial t} - \frac{\partial v_0}{\partial t} = 0 \\
P^1 &= \frac{\partial c_1}{\partial t} - rs\frac{\partial c_0}{\partial s} - (a - bv)\frac{\partial c_0}{\partial v} - \frac{1}{2}s^2v\frac{\partial^2 c_0}{\partial s^2} - \rho\sigma sv\frac{\partial^2 c_0}{\partial s\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 c_0}{\partial v^2} + rc_0 + \frac{\partial v_0}{\partial t} = 0 \\
P^2 &= \frac{\partial c_2}{\partial t} - rs\frac{\partial c_1}{\partial s} - (a - bv)\frac{\partial c_1}{\partial v} - \frac{1}{2}s^2v\frac{\partial^2 c_1}{\partial s^2} - \rho\sigma sv\frac{\partial^2 c_1}{\partial s\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 c_1}{\partial v^2} + rc_1 = 0 \\
P^3 &= \frac{\partial c_3}{\partial t} - rs\frac{\partial c_2}{\partial s} - (a - bv)\frac{\partial c_2}{\partial v} - \frac{1}{2}s^2v\frac{\partial^2 c_2}{\partial s^2} - \rho\sigma sv\frac{\partial^2 c_2}{\partial s\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 c_2}{\partial v^2} + rc_2 = 0 \\
P^4 &= \frac{\partial c_4}{\partial t} - rs\frac{\partial c_3}{\partial s} - (a - bv)\frac{\partial c_3}{\partial v} - \frac{1}{2}s^2v\frac{\partial^2 c_3}{\partial s^2} - \rho\sigma sv\frac{\partial^2 c_3}{\partial s\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 c_3}{\partial v^2} + rc_3 = 0 \\
&\vdots
\end{aligned} \tag{19}$$

and the required approximate solution obtained is

$$c(s, v, t) = \sum_{n=0}^{\infty} P^n c_n \tag{20}$$

Executing the HPM methodology as described above, MAPLE 18 software is brought into play. Using the following parameters $a = 0.16, b = 0.055, \delta = 0.9, \rho = -0.5, T = 15, K = 100$, we have;

Table 1: Numerical results for MHPM

$C(t, s, v)$	HPM(Biazar et al , 2015)	MHPM	Error MHPM-HPM
C(1, 10, 0.1)	413.2583333	3.796325819 E5 10^1	409.462007
C(2, 50, 0.2)	81406.34666	5.476248101 E6 10^6	81400.8704
C(4, 70, 0.3)	1.734519680 E6	4.40834821 E6 10^6	2.6738285
C(6, 90, 0.4)	1.335877440 E7	-8.9567091 E5 10^6	10.2925865
C(8, 120, 0.5)	7.485848702 E7	-1.24852579 E7 10^6	8.73437449
C(10, 150, 0.6)	2.937374856 E8	-3.139821922 E7 10^6	6.07719678
C(14, 200, 0.8)	2.198733197 E9	-7.467534155 E7 10^7	9.66626735

7. Modified VIM for Heston SPDE

We now proceed to modify the He's VIM using the MNPs as modifier and basis function via the orthogonal collocation method (OCM). Given the general formulation of the VIM in (8), the correction function as related to the HSPDE is thus given as

$$c_{(n+1)}(s, v, t) = c_n(s, v, t) + \int_0^t \lambda(\xi)\left[\frac{\partial c}{\partial \xi} - rs\frac{\partial c}{\partial s} + (a - bv)\frac{\partial c}{\partial v} + \frac{1}{2}s^2v\frac{\partial^2 c}{\partial s^2} + \rho\sigma sv\frac{\partial^2 c}{\partial s\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 c}{\partial v^2} + rc\right]d\xi \tag{21}$$

Hence, the initial approximation for the modified VIM is given by (17). From (8), we have that $\lambda(\zeta)$ is the general Langrange multiplier which can be obtained optimally via the variational theory. If we set

$$\lambda(\zeta) = -1 \tag{22}$$

and substituting (22) into (8) gives,

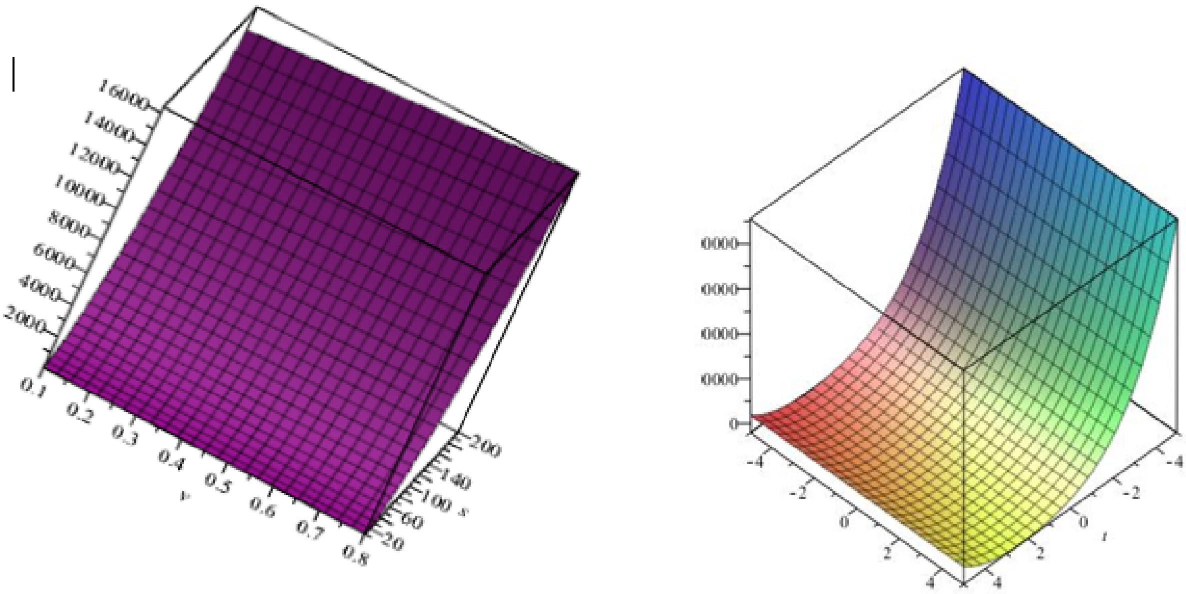
$$c_{(n+1)}(s,v,t) = c_n(s,v,t) - \int_0^t \left[\frac{\partial c}{\partial \xi} - rs\frac{\partial c}{\partial s} + (a-bv)\frac{\partial c}{\partial v} + \frac{1}{2}s^2 v\frac{\partial^2 c}{\partial s^2} + \rho\sigma s v\frac{\partial^2 c}{\partial s \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 c}{\partial v^2} + rc\right] d\xi \qquad (23)$$

which is the MVIM with initial approximation $c_0(s,v,t) = 392.00000s^2$. Evaluating (23) with the aid of MAPLE 18 application software for $n \geq 0$ and with the following parametric values $a = 0.16, b = 0.055, \delta = 0.9, \rho = -0.5, T = 15, K = 100$ yields the following approximation

Table 2: Numerical results for MVIM

$C(t,s,v)$	VIM(Biazar et al , 2015)	MVIM	Error MVIM-VIM
C(1, 10, 0.1)	209.5038332	38456.89887	38247.395
C(2, 50, 0.2)	82474.33729	9.740327147 10^6	82464.597
C(4, 70, 0.3)	1.147814970 E6	1.873262944 10^6	0.725448
C(6, 90, 0.4)	1.950175983 E7	3.017060928 10^6	1.0668849
C(8, 120, 0.5)	7.79650567 E7	5.276105707 10^6	2.52039996
C(10, 150, 0.6)	2.317375113 E8	8.112385120 10^6	5.79501
C(14, 200, 0.8)	2.576414314 E9	1.380834321 10^7	1.19557999

Hence the approximate solution of the HSPDE given by the MVIM is thus



MHPM for HSPDE

MVIM for HSPDE

Figure 1: HSPDE given by the MVIM.

8. Conclusion

The MNPs being a new orthogonal polynomial was used as a modifier and basis function for the He's HPM and VIM via the OCM after which both modified methods were applied on the HSPDE. The compatibility of the MNPs and the numerical method created a new iterative method which was able to approximate the Heston Stochastic Partial Differential Equation to its exact solution with the MVIM standing out as a faster approximant for the HSPDE than the MHPM.

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