Basics of cyclical sequences theorem

Abstract

This paper introduces two recursive algorithms that create a new subject of numerical sequences which are called **cyclical sequences** and the Collatz sequence is classified in them. They also divide integers into **tree partitions**. This theory studies the structure of cyclical sequences. One of the obstacles to proving Collatz's conjecture has been the lack of sufficient knowledge about the Collatz sequence. This article provides a new and complete understanding of Collatz's conjecture, in which, we have also redefined the collatz sequence with a parameter.

Keywords: arithmetic algorithm; geometric algorithm; cyclical sequences; tree partition; isocyclic partition

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1 Introduction

In the reaserch, we introduce a new subject of numerical sequences called cyclical sequences, in which, the Collatz sequence is classified. This theory studies the structure of cyclical sequences. One of the important obstacles to prove Collatz's conjecture[2] was the complete lack of knowledge of the Collatz sequence.[1] study of this paper is necessary to understand Collatz's conjecture. We also redefine the Collatz sequence by a parameter, based on which, the collatz conjecture is proved. In the section 4, by extending the division algorithm, sequences similar to the Collatz sequence are created, which also reach a cycle. Such sequences and the Collatz sequences are placed in a new mathematical subject which are called cyclical sequences . We also create new partitions of rational number and integers by recursive algorithms which are in the form of tree.

We have used the following sets of numbers in this article:

Naturals N = f1; 2; : : :g, arithmatics W = N □ 1 = f0; 1; 2; : : :g,
integers Z = f: : : ;□2;□1; 0; 1; 2; : : :g, rationals Q = f $\frac{a}{b}$: a; b 2 Z; b 6= 0g.

An unsolved open problem in mathematics which Lothar Collatz introduced in 1937 and during research that lasted 6 years, we completed its proof on May 12, 2024 after 87 years.

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2 Cyclical sequences

You should be familiar with properties of sequences. In a sequence, repetition is allowed and order is important. in view of these evidences, it is better a sequence is denoted by symbols [and g.

Definition 2.1 (Periodic sequence and cycle)**.** A sequence consists of a finite and ordered set of numbers that repeats indefinitely. [2] [3]

Each Periodic sequence is denoted by all members respectively as follows:

$C_p = [c_t g^{+1}_{t=1} = [c_1; c_2; : : : ; c_p; c_{p+1}; c_{p+2}; : : : ; c_{2p}; : : : g$; so that $c_{t+k\_p} = c_t$; k 2 W;

where p 2 N is the **base period**.

Thus by definition, a Periodic sequence can be represented by a **cycle**[3]. Each cycle is also denoted with all element, respectively, between two simbols _ and _.

Let C_3 = [a; b; c; a; b; c; : : :g be a periodic sequence which has period p = 3 then, it is represented by these forms: by simbols of cycle _a; b; c _, and by directed cycle graph, a & b & i c or like this figure:

a
b
c

Definition 2.2. The sequence that eventually reaches (falls into) a cycle are called a **cyclical sequence**, the final cycle is called **stop cycle**.

Cyclical sequences are created by **special functions** or **recursive algorithms**.

Let r_{00} = a be the starter term,[4] then a cyclical sequence is represented as follows:

$[r_m g^{+1}_{m=00} = [r_{00}; r_0; r_1; r_2; : : : ; r_m; r_{m+1}; r_{m+2}; : : : ; r_{m+p□1}; r_{m+p}; : : : g$; Where, $r_m = r_{m+k\_p}$: k 2 W;

therefore, its stop cycle is $C_p = \_r_m; r_{m+1}; r_{m+2}; : : : ; r_{m+p□1}\_$, in which, p is the base period.

Definition 2.3. The cyclical sequence from a point onwards are placed in a cycle, the value of the first stop from which the cycle begins, is called **stop value**. The number of generated terms up to the stop point is called **stop time** which is denoted by M. Thus, the stop time M = m + p □ 1, the **stop value** $r_M = r_{m+p□1}$. See example 4.2, 4.1.

Thus by definition, final cycle with some different starter point is as follows:

$C_p = \_r_{M□p+1}; r_{M□p+2}; r_{M□p+3}; : : : ; r_M\_ = \_r_M; r_{M+1}; r_{M+2}; : : : ; r_{M+p□1}\_$:

3 Regular algorithms

Definition 3.1 (Regular algorithm)**.** A relation that has a specific mathematical formula and its output variables have one or more conditions is called a **regular algorithm**.

Based on breaking an integer into a certain number or dividing it by a certain value and whether we want the remainder to be a positive or negative value, we can propose two arithmetic algorithm theorems which are also proved by the **well-order principle** of natural numbers and integers.

Notation 3.1. The normal state is that negative integer have a negative remainder and positive integer have a positive remainder, but theoretically, the remainder can be positive or negative, that is, the remainder condition optional.

[3] A cycle is a path in which only the first and last vertice is equal.

[4] The starter term a may not be the same as the initial term r_0. we can denote a starter term by r_{00} when it is out of the condition.

Example 3.2. Breaking the number 13 into 3 equal pices: 13 =
3equal pieces z }| {
4 + 4 + 4 +1 : 0 _ 1 < 4, and if we
want its remainder to be negative, we can write: 13 =
4equal pieces z }| {
4 + 4 + 4 + 4□3 : □4 < □3 _ 0.
Breaking the number □13 into 3 equal pices: □13 =
3equal pieces z }| {
□4 □ 4 □ 4□1 : : □4 < □1 _ 0, and if we want
its remainder to be positive, we can write: □13 =
4equal pieces z }| {
□4 □ 4 □ 4 □ 4 +3 : 0 _ 3 < 4.

Definition 3.2 (Critical quotient)**.** Let a; d 2 Z; d 6= 0 and let n 2 W; a_n _ d _ 0 or a_n _ d _ 0,[5] then the maximum value of n that does not violate one of these inequalitys is called **critical quotient** of d respect to a, for that inequality. We denoted by symbols [;) as: [a; d) = n. We can get the value of the critical quocient by the lemma 3.3.

Lemma 3.3. Let [a; d) be the critical quocient in which a; d 2 Z; d 6= 0 and let n 2 W then we have:
a _ n _ d _ 0; a _ (n + 1) _ d > 0) [a; d) = n or a _ n _ d _ 0; a _ (n + 1) _ d < 0) [a; d) = n :

Notation 3.4. Holding the main inequality according to the sign of the integers a; d determines which of the + or □ operators to use.

Intuitive proof. We consider the following successive inequalities:
a_0_d _ 0; a_1_d _ 0; a_2_d _ 0; : : : ; a _ n _ d _ 0; a _ (n + 1) _ d > 0 ; a_(n+2)_d > 0; : : : ;
a_0_d _ 0; a_1_d _ 0; a_2_d _ 0; : : : ; a _ n _ d _ 0; a _ (n + 1) _ d < 0 ; a_(n+2)_d < 0; : : : ;
because n is the largest value that does not violate inequality, thus by definition 3.2, it is equal to the critical quotient, then we can write: [a; d) = n.

Corollary 3.5. Given that the inequality is violated by an infinite well-ordered sequence of natural numbers, then the critical quotient n is unique.

Theorem 3.6 (Arithmetic algorithm theorem with positive remainder[6])**.** For any two integer a; d 6= 0, we can find two unique numbers q 2 W and integer r such that:
a _ q _ d = r : q 2 W; 0 _ r _ jdj: [4]

Theorem 3.7 (Arithmetic algorithm theorem with negative remainder)**.** For any two integer a; d 6= 0, we can find two unique numbers q 2 W and integer r such that:
a _ q _ d = r : q 2 W; □jdj < r _ 0:

Proof. Let a 2 Z and let n 2 W be the value of the critical quotient [a; d), thus by Lemma 3.3 and properties of inequalities, we can get following relations.
[a; d) = n) a _ n _ d _ 0) a _ n _ d = r : r 2 Z; r _ 0; thus, it must be q = n; then; q 2 W:
Since, a _ (n + 1) _ d > 0) a _ n _ d _ d > 0) r _ d > 0) r > _d, on the other hand, r _ 0,
thus, it follows that, □jdj < r _ 0.
Suppos we have another remainder r_0, such that, a=_q _ d + r_0 : □jdj < r_0 _ 0.
thus by corollary 3.5, the critical quotient q is unique, then _q _ d + r_0 = _q _ d + r) r_0 = r,
therefore, the remainder is also unique. In this way, the theorem is proved.

[5] Because we want to use theorems by natural counting,we assume n 2 W, which is the reason for the appearance of + and □ signs, otherwise we can assume that n 2 Z.

[6] This theorem is the same division algorithm theorem which has been proved

Corollary 3.8. Holding the remainder condition according to the sign of the integers a; d determines which of the + or □ operators to use.

Remark 3.1. The **congruence theorem** is actually the same arithmetic algorithm theorem regardless

of the remainder condition, which is as follows: a _ q _ d = r , a
d_
r : q 2 W; r 2 Z

Definition 3.3 (critical power)**.** Let a; d 2 Z; jdj _ 2 and let n 2 W; d_n j a, then the maximum value of n that does not violate this divisibility relation is called **critical power** of d respect to a or **divisibility degree** of a respect to d. We denoted by symbols d;) as: da; d) = n. We can get the critical power value by the lemma 3.9.

Lemma 3.9. Let da; d) be the critical power in which a; d 2 Z; jdj _ 2 and let - be the indivisibility symbol. Let n 2 W, then we have:

d_n j a; d_{n+1} - a , da; d) = n:

Intuitive proof. We consider the following successive divisibility and indivisibility relations:

d_0 j a; d_1 j a; d_2 j a; d_3 j a; : : : ; d_n j a; d_{n+1} - a ; d_{n+2} - a; d_{n+3} - a; : : : ;

because n is the largest value that does not violate divisibility, thus by definition 3.3, it is equal to the divisibility degree, then we can write: da; d) = n.

Corollary 3.10. Given that the divisibility relation is violated by an infinite well-ordered sequence of natural numbers, then the critical power n is unique.

Theorem 3.11 (Geometric algorithm theorem)**.** The geometric algorithm states that given any two integers or fractionals a and integer d so that jdj _ 2,we can find two uniche numbers q 2 W and integer or fractional number r such that:

a
d_q = r or a _ d_q = r : q 2 W; d - r:

Proof. Let a 2 Z and let n 2 W be the value of the critical power da; d), thus by Lemma 3.9 and the definition of the divisibility, we can get following relations.

da; d) = n) d_n j a) a = d_n _ r : r 2 Z; thus, it must be q = n; then; q 2 W: (3.2)

we assume that d j r, then, d_n _ d j d_n _ r) d_{n+1} j a, but this result is a contradiction, since, according to the lemma 3.9 d_{n+1} - a, therefore, it follows that: d - r.

Suppos we have another remainder r_0, such that, a=b_q _ r_0 : d - r_0.

Thus by corollary 3.10, the critical power q is unique, then $b_q r_0$ = b_qr) r_0 = r,

therefore, the remainder is also unique.

Notation 3.12. The condition of undivisibility d - r is the same to inequvalancy relation r
d
6_ 0.

4 Recursive forms of regular algorithms

To expand a regular algorithm, we must rewrite its relation recursively, then we change the previous remainder with a suitable function, so that the value of the function is outside the condition of the remainder. The expansion function can be any form and must be such that using it one or more times violates the remainder condition.

Cyclical sequences can be created by the recursive algorithms. For any integer r_0, two cyclical sequences are created to gether by a recursive algorithm.

4.1 Recursive arithmetic algorithm with positive remainder

Definition 4.1 (Sequences created by recursive arithmetic algorithm)**.** For any two integer r_{00} = a and d 6= 0, two sequences are created by the mapping A : N ! Z and the recursive arithmatic algorithm as follow:

r_m = _($r_{m□1}$) _ q_m _ d : q_m 2 W; 0 _ r_m < jdj; 8m 2 N;

If a < 0 or a _ jdj then, r_0 = a_q_0_d : q_0 2 W; 0 _ r_0 < jdj; else r_0 = a; i.e. 0 _ a < jdj:

where d is called **quoteint modulo of the algorithm**, _(r) 2 Z is called **expansion function**.

The inequality 0 _ r < jdj is called **remainder condition**.

The expansion function must be such that using it one or more times violates the remainder condition, i.e. 9n 2 N : _$_{(n)}$(r) < 0 or _$_{(n)}$(r) _ jdj, which is its condition

In this algorithm, q is the number of steps ”minus or plus d” which is **quoteint** of modulo.

The q value and its sign must be such that the result of _($r_{m□1}$) _ q _ d is in the interval [0; jdj).

The m value is the number of steps to use _(r), which is the indice of the next term.

Thus by the recursive arithmatic algorithm, two cyclical sequences are created together, [r_mg^{+1}
$_{m=0}$,
which is called **remainders sequence** and [q_mg^{+1}
$_{m=0}$, which is called **quotients sequence**.

Two cyclical sequences [q_mg and [r_mg are created by recursive algorithm can be represent by a **weight path**7, such that its vertics are elements of the sequence [r_mg and its edges weight are element of [q_mg.

Each cyclical sequence can be denoted by its starter term and last vertice, between two simbols

as _a; r_M _, when its algorithm is known. the last vertice is often in the final cycle. Hence, we can also represent both resulting sequences by a matrix. See the graph 4.1 and example 4.1.
Thus by definitions, in the graph 4.1, the path _a; r_M _ =
_
q_m
r_m
$\_^{+1}$
m=00
; the stop value $r_M = r_6$ and
the stop time M = 6, since $r_7 = r_2$, then the period p = 7 □ 2 = 5, therefore, the final cycles are
C_5 = _r_2; r_3; r_4; r_5; r_6 _; C_0
$_5$ = _q_3; q_4; q_5; q_6; q_7 _:
r_{00}
q_0
/ r_0
q_1
/ r_1
q_2
/ r_2
q_3
) r_3
q_4
) r_4
q_5
) r_5
q_6
) r_6
q_7
r_7=_(r_6)□d_q_7=r_2
i : (4.1)

Example 4.1. If we start with: r_0 = 73; d = 47 and _(r) = 7 _ r^2 + 79, then the remainders and quotients sequences and its path graph are as follows:
73 1
/ 26 102
/ 17 44
/ 34 173
/ 40
239
)
46
316
)
39
228
)
10
16
)
27
110
)
12
23
' 6
7
& 2
2
)
13
26
7_13^2+79□47_26=40
k
_73; r_M _ =
2
4
q

r
m
9=
;
+1
m=0
=
2
4
0 1 102 44 173 239 316 228 16 110 23 7 2 26 : : :
73 26 17 34 40 46 39 10 27 12 6 2 13 40 : : :
0 0 1 2 3 4 5 6 7 8 9 10 11 12 : : :
9=
;

---

[7] A weight path is a walk in which all vertices and edges are distinct and every its edge has a weight, where a value is relayed to a vertex.

Thus, the stop time M = 11, the stop value r_M = 13; q_M = 2, since r_3 = r_{12} = 40, then the period p = 12 □ 3 = 9, the final cycles are as follow:
C_9 = _40; 46; 39; 10; 27; 12; 6; 2; 13_; C_0
9 = _239; 316; 228; 16; 110; 23; 7; 2; 26_:

4.1.1 Recursive division algorithm

The recursive division algorithm is a special case of the recursive arithmetic algorithm with positive remainder, in which the expantion function is _(r) = b _ r, where b 2 N; b _ 2 is the **numerical base**.

Modulo d 2 Z; d 6= 0 and remainder condition is 0 _ r < jdj.

A familiar example for us in this case, the decimal expansion of a numerical fraction is done by division algorithm.

The decimal expansions of rational numbers always reaches periodic sequences which is a numerical cycle.

Example 4.2. If a division algorithm is started whith r_{00} = 81; 59, modulo d = 13, and _(r) = b _ r = 10 _ r then the result sequences and their path graph are as follows:

r_{00} = 81 6
/ r_0 = 3
2
) 4
3
' 1
q_3=0
) 10
7
(9
6
-
r_5 = 12
9
r_6=10_12□13_9=3
k ;

Thus by the path graph, remainders and quotients sequence are as follow:
[81; r_mg = [81; 3; 4; 1; 10; 9; 12; 3; 4; 1; 10; 9; 12; : : :g
fall into
□! C_6 = _3; 4; 1; 10; 9; 12_;
[q_mg = [0; 6; 2; 3; 0; 7; 6; 9; 2; 3; 0; 7; 6; 9; : : :g
final cycle
□! C_0
6 = _2; 3; 0; 7; 6; 9_;

therefore, the stop value r_M = r_5 = 12, the stop time M = 5, since r_6 = r_0 = 3, then the period p = 6 □ 0 = 6.

r_{00} = 59
4
/ r_0 = 7
5
) 5
3
) 11

$q_4=8$
(6
4
(8
6
.
$r_5 = 2 = r_M$
1
$r_6=10\_2□13\_1=7$
k :
By matrix form: _59; r_M _ =
2
4
q
r
m
9=
;
+1
m=0
=
2
4
0 4 5 3 8 4 6 1 5 3 8 4 6 1 : : :
59 7 5 11 6 8 2 7 5 11 6 8 2 7 : : :
0 0 1 2 3 4 5 6 7 8 9 10 11 12 : : :
9=
;
Thus, the stop time M = 5, the stop value $r_M = 2$; $q_M = 2$, since $r_6 = r_0 = 7$, then the period p = 6 □ 0 = 6, the final cycles (stop cycles) are as follow:
remainders stop cycle C_6 = _7; 5; 11; 6; 8; 2_, quotients stop cycle C_0
$_6$ = _5; 3; 8; 4; 6; 1_.

Corollary 4.3. The quoteints sequence is the same decimal expansion of a fraction so that its terms are decimal digits of the fraction.

For example: 81
13 = 6:
z }| {
230769
z }| {
230769 : : :, 43
13 = 3:
z }| {
307692
z }| {
307692 : : :, 59
13 = 4:
z }| {
538461
z }| {
538461 : : :,
97
13 = 7:
z }| {
461538
z }| {
461538 : : :, □19
13 = □1:
z }| {
461538
z }| {
461538 : : :, see the figure 1.

Corollary 4.4. All cyclical sequences that is creared by a division algorithm with r_{00} = a 2 Z, the modulo d = 13, and the expansion function _(r) = 10_r, can only reach one of the quocient cycles in the figure 1.

3
0 7
6
2 9
5
3 8
4
1 6

Figure 1: The quocient stop cycles in the form of a cycle graph for a
13 , where, a 2 Z.

4.2 Recursive geometric algorithm

Definition 4.2 (sequences created by recursive geometric algorithm)**.** For any two integer r_{00} = a, d _ 2, two sequence are created by the mapping G : N ! Z and the recursive geometric algorithm as follows:

r_m =
_($r_{m□1}$)
d^{q_m}
: q_m 2 W; d - r_m; 8m 2 N:
If d j a then, r_0 =
a
d^{q_0}
: q_0 2 W; d - r_0; else r_0 = a:

Similarly, d 2 N is called **power modulo** and _(r) 2 Z is called **expansion function** and 9n 2 N : d j $\_{(n)}$(r) is its condithon. d - r is the remainder condition.

In this algorithm, q is the number of steps ”divided by d” which is the power of the modulo for one step and m is the number of steps to use _(r), which is the indice value of the next term. Here we can assume that d - 0.

Thus by the recursive geometric algorithm, two cyclical sequences are created, [r_mg$^{+1}_{m=0}$, that is called **remainders sequence** and [q_mg$^{+1}_{m=0}$, which is called **powers sequence**.

Definition 4.3. The number of steps ”divided by d”, to generate the term r_m is called its **shrinkage parameter** and denoted by s_m.

Let [r_mg$^{+1}_{m=0}$ be a Collatz sequence then, all shrinkage parameters create the sequence [s_mg$^{+1}_{m=0}$ so that s_m =
P_m
$_{i=0}$ q_i. In a stop cycle s_p =
P_p
$_{i=1}$ q_i.

Definition 4.4. the shrinkage parameter of the first stop is called **stop parameter of cyclical sequence**, and denoted by S. If M is the **stop time**; then s_M = S. The stop value r_M doesn’t depend on starter value r_{00} = a.

Example 4.5. If we start the recursive giometric algorithm by r_0 = 93, with modulo d = 2 and expantion function _(r) = r □ 7, then the cyclical sequences and its weight path are as follows:

r_0 = 93
1
/ 43 2
/ 9 1
/ 1
q_4=1
/ □3
1
* 2 □5
r_6=□5□7
22 =□3
j
_93; r_M _ =

2
4
q_m
r_m
m
9=
;
+1
m=0
=
2
4
0 1 2 1 1 1 2 1 2 : : :
93 43 9 1 □3 □5 □3 □5 □3 : : :
0 1 2 3 4 5 6 7 8 : : :
9=
;

Thus, the stop time M = 5, the stop values r_M = □5; q_M = 2, since r_4 = r_6 = □3, then the period p = 6 □ 4 = 2, the stop cycles are: C_2 = _□3;□5_; C_0
$_2$ = _1; 2_; s_p = 3:

Corollary 4.6. In the subject of cyclical sequences, both Sequences created by recursive arithmetic algorithm in integers Z and created by recursive geometric algorithm in rationals Q eventually reach (fall into) a cycle from a term onwards.

Therefore, Collatz's sequence is a special case of sequences created by geometric algoritm with _(r) = 3r + 1 and d = 2 in naturals N, which reaches to stop cycle _4; 2; 1_.

Definition 4.5. The final part of any cyclical sequence is a periodic sequence in which the terms with index m = M + k _ p lie on a straight line, here, we call it **extension line of the sequence**. Since r_M = r_{M+k_p}, then its equation is y = r_M. see figure 3

4.3 Partition by recursive algorithms

By runnig a recursive algorithm with known expantion function and modulo on integers, each time it is used, it creates a remainder and after a few uses it reaches the stop remainder and from this remainder onward, the recursive algorithm falls into the final cycle.

For all integers, a finite number of cycles is created, so we can part the integers based on.

The set of integers that reach the same stop remainder form a **tree**[8], so each stop remainder is equivalant to a tree and denoted by [r_M].

Eeach cyclical sequence[9] is a path of the tree so that this path is from vertice r_0 to the final cycle, and is denoted by _r_0; r_M _.

Eeach stop remainder belong to a final cycle so the cycle C_p has precisely p tree, hence, the set of all trees is a partition on Z.

Definition 4.6. We can define an isocyclic on Z, when Two elements x; y 2 Z create the same cycle by a recursive algorithm. Thus isocyclic class of a is called the set of all element of Z which are isocycle to a, this class is denoted by dac. Two elements x; y 2 Z are isocycle if and only if they belong to the same isocyclic class, thus we can write: dxc = dyc.

The set of isocyclic classes is a partition of Z. The smallest element of an cycle may be chosen as a representative of the isocyclic class as dc_{min}c.

The stop value of the arithmatic sequences resulting from the division algorithm with modulo d is precisely one of the integers in the complete set of stop remainders F_n as follows: the remaining condition of the algorithm tell us that,

0 _ r < d) $F_{n=d}$ = [0; 1; 2; : : : ; d □ 1g:

This means that each integer is the vertex of precisely one of the trees in the following partition. Let n be the number of the subset in partirion P_n then,

$P_{n=d}$ = f[0]; [1]; [2]; : : : ; [d □ 1]g:

In example 4.2, division algorithm with modulo d = 13 and _(r) = 10r create two partitions on Z as follows:

the set of stop remainders F_{13} = f0; 1; 2; 3; 4; 5; 6; 7; 8; 9; 10; 11; 12g;

thus, the tree partition is P_{13} = f[0]; [1]; [2]; [3]; [4]; [5]; [6]; [7]; [8]; [9]; [10]; [11]; [12]g:

Let H_n be the complete set of final cycles which are created by the division algorithm then,

H_3 = f_0_; _1; 10; 9; 12; 3; 4_; _2; 7; 5; 11; 6; 8_g;

---

[8] a tree is a graph in which any two vertices are conected by exactly one path.

[9] that is created by a recursive algoritm

thus with composition sizes 1 + 6 + 6, the isocyclic partition is as follows:
P_3 = f[0]; f[1]; [10]; [9]; [12]; [3]; [4]g; f[2]; [7]; [5]; [11]; [6]; [8]gg = fd0c; d1c; d2cg = Z:
Hence, there are 3 isocyclic classes d0c, d1c and d2c in the partition P_3 so that:
I. d0c [d1c [d2c = Z.
II. d0c \ d1c = ;; d0c \ d2c = ;; d1c \ d2c = ;.

Corollary 4.7. Infact, the isocyclic partition P_3 is a three membred subset from the complete set of final remainder, with modulo d

Remark 4.1. Composition sizes depend on expantion function and modulo of the recursive algorithm.
In the example 4.5 by the recursive geometric sequence with _(r) = r □ 7; d = 2, two partitions on Z with the composition sizes[10] 1 + 1 + 1 + 1; 1 + 2 + 1 are created as follows: Since, □7 _ r _ □1; 2 - r,
thus, the complete set of stop remainder F_4 = f□1;□3;□5;□7g,
hence, the tree partition P_4 = f[□1]; [□3]; [□5]; [□7]g = Z.
The complete set of the stop cycles H_3 = f_□1_; _□3;□5_; _□7_g, thus,
the isocyclic partition P_3 = f[□1]; f[□3]; [□5]g; [□7]g = fd□1c; d□3c; d□7cg = Z.

5 Forms of the Collatz sequense

The collatz sequence is a cyclical sequence that is created by special modes of recursive geometric algorithm with expantion function _(r) = 3r + 1 and modulo d = 2.

In its old definition, the mode of its algorithm is two-piecewise, in which the even and odd result of each step was assumed a term of Collatz's sequence, but in the new definition, only odd result appear, and the mode of the algorithm used is one piecewise.

Both sequences poceeds in pulse form, such that in old definition, the 3r +1 piece maginfies the only term of ascending edge, then the $\frac{r}{2}$ piece shrinks each term of descending edge of a pulse once, but in new definithion, the ascending edge of any pulses can ascend one or more in a row by $\frac{3r+1}{2}$, and its descending edge consists of several terms, each term can be reduced one or more times in a row by $\frac{r}{2}$.

Definition 5.1 (Old definition of Collatz's sequence)**.** This sequence is defined by the mapping L : N ! Z and created by the following recursive algorithm as follows:

r_t = L($r_{t□1}$) =
(
$\frac{r_{t□1}}{2}$ 2 j $r_{t□1}$
3$r_{t□1}$ + 1 2 - $r_{t□1}$;
8 t 2 N; r_0 2 Z; r_0 6= 0:

According to Collatz's conjecture in 1937, all created sequences with r_0 2 N, _(r) = 3r + 1 and modulo d = 2, eventually reaches the cycle _4; 2; 1_, which is the same **periodic sequence** C_3 = [4; 2; 1; 4; 2; 1; : : :g that can be denoted by the cycle graph, 4 ' 2 ' i 1 . see the graph(A) in the figure 2.

We can denote it by path symbols as: _r_0; r_T = 1_ where T is its stop time and r_T is its stop remainder (stop value).

[10]The sizes is also the period of cycles in the second partition.

Example 5.1. By the old definition, Collatz's sequence and its pulsed graph[11], where start by r_0 = 52 is as follows:

52
$\frac{52}{2}$
/ 26
$\frac{26}{2}$
_
40
$\frac{40}{2}$
/ 20
$\frac{20}{2}$
/ 10
$\frac{10}{2}$
_

16
16
2
/ 8
82
/ 4
42
/ 2
22
_
13
3_13+1
=
5
3_5+1
=
1
3_1+1
V

_r_0 = 52; r_T _ = [52; r_tg$_{+1}$
$_{t=0}$ = [52; 26; 13; 40; 20; 10; 5; 16; 8; 4; 2; 1; 4; 2; 1; : : :g:
Thus , the stop value $r_T = r_{11}$ = 1 then the stop time T = 11 and the stop cycle C_3 = _4; 2; 1_.

Example 5.2. By the old definition, Collatz's sequence with r_0 = □19 is as follows:
_□19; r_T _ = [□19;□56;□28;□14;□7;□20;□10;□5;□14;□7;□20;□10;□5; : : :g;
thus by this path, the stop value $r_T = r_7$ = □5 = r_{12}, then the stop time T = 7, the base period
p = 12 □ 7 = 5 and the stop cycle C_5 = _□14;□7;□20;□10;□5_.

All cycles created by Collatz's old algorithm on the integers are shown in the form of a directed cycle graph as follows:

4
2
1
-2 -1 -20
-10
-5
-14
-7
-17
-50
-25
e.t.c
-34

The uncertainty of the number of steps "divided by two" versus one step, "triple plus one" made it very difficult to prove and also was the reason for the ambiguity of Collatz's conjecture.

5.1 Redefinition of Collatz's sequence

The collatz sequence can be generalized to rational numbers Q, that each of them is separated into several subsets based on the stop cycle.

After leaving the remainder, from its condition using the expansion function _(r) = 3 _ r + 1, dividing by 2 is performed several times, so that the condition is established again. If we assume the number of dividing by 2 is q, then the Collatz sequence is redefined as follows which is created a sequence with odd terms, and we call it Collatz's jump sequence.

Definition 5.2. A sequence is created by the followsing recursive geometric algorithm on the mapping L : N ! Z is called **Collatz's jump sequence**. Let 0 6= a 2 Z be the starter value of the Collatz algorithm, then we can get the sequence as follows:

r_m = L($r_{m□1}$) =
3$r_{m□1}$ + 1
2$_{qm}$
: q_m 2 N; 2 - r_m; 8m 2 N:
If 2 j a, then initial term r_0 = $_a$
$_{2q0}$: q_0 2 N; 2 - r_0, else r_0 = a:

^{11}pulsed graph is a path that is represented to a waveform.

If you are not surprised, the values of m; q must be zero for the starter term r_{00} = a, in order for the sequence graph conform to its formula, i.e, we always assume that m = 0; q = 0 for the starter value a, which is also m = 0 for the initial term r_0.

Remark 5.1. In view of the new definition, for proof of Collatz's conjecture we have to prove which any Collatz sequence with a 2 N eventually reches the stop cycle _1_, in other word, it converges to r_M = 1. see the graph(B) in figure 2.

Example 5.3. The Collatz jump sequence and its **weight path** with r_{00} = 28 is as follows: we can use program code in the listing 1,

r_{00} = 28
q_0=2
28
22 =7
/ 7 $_1$
3_7+1
21 =11
/ 11 $_1$
/ 17 $_2$
/ 13 $_3$
/ 5
q_5=4
3_5+1
24 =1
/ 1 >
3_1+1
22 =1
g $_2$

Thus by this graph, _28; r_M _ = [28; r_mg+1
m=0 = [28; 7; 11; 17; 13; 5; 1; 1; 1; : : :g; r_5 = 1;
M = 5, then the stop cycle C_1 = _1_. [q_mg = [0; 2; 1; 1; 2; 3; 4; 2; 2; 2; : : :g) C_0
$_1$ = _2_.

By runing program code in the listing 1 for a 2 Z, all terms of the Collatz sequence, [r_mg and their parameters sequences, [q_mg, [s_mg are calculated.

In the table 1, the stop parameters M; S; T of collatz's sequence with a 2 Z, so that r_M = _1, which are created by program code in the listing 1. Always T = M + S.

Lemma 5.4. Let [r_mg+1
m=0 be a Collatz sequence with r_0 2 Z. If m _ 1) 3 - r_m.

Proof. 3 _ $r_{m□1}$ + 1 = $2_{q_m}r_m$) $2_{q_m}r_m$
3_
1; 2
3_
□1) r_m
3_
_1) 3 - r_m.

By the Collatz algorithm with _(r) = 3r +1; d = 2, two partitions on Z with the composition sizes 1_{11}; 1_2 + 2_1 + 7_1 are craeted.

The complete set of stop remainder on integer Z is as follows:
F_{11} = f1;□1;□5;□7;□17;□25;□37;□41;□55;□61;□91g;
hence, the tree partition by assuming that zero is odd, as follows:
P_{11} = f[1]; [□1]; [□5]; [□7]; [□17]; [□25]; [□37]; [□41]; [□55]; [□61]; [□91]g = Z:
The complete set of the stop cycles on Z is as follows:
H_4 = f_1_; _□1_; _□5;□7_; _□17;□25;□37;□55;□41;□61;□91_g;
thus, the isocyclic partition P_4 = fd1c; d□1c; d□5c; d□17cg = Z,
in which P_1 = fd1cg = Z_+; P_3 = fd□1c; d□5c; d□17cg = $Z_□$____________.

Lemma 5.5. Let [r_mg+1
m=0 be Collatz's sequence, If r_0 is positive, negative or fractional, so is r_m.
In other words, r_0 2 Z_+) r_m 2 Z_+; r_0 2 $Z_□$) r_m 2 $Z_□$; r_0 2 Q) r_m 2 Q.

5.2 Symmetric sequences

The difference of the symmetric sequence algorithm is in their expansion function, But the parameters of both are the same.

A sequence with _(r) = 3r + 1, r_0 2 $Z_□$ and A sequence with _(r) = 3r □ 1, r_0 2 $Z_□$ are symmetrical. Let _(r) = 3r □ 1; d = 2 then its recursive algorithm is as follows:

r_m = L($r_{m□1}$) =
3$r_{m□1}$ □ 1
2_{q_m}
: q_m 2 N; 2 - r_m; 8m 2 N; r_0 =
a
2_{q_0}

: q_0 2 W; 2 - r_m:

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
1
10
23
35
53
70
80
106
160
23
70
35
106
53
160
80
40
20
10
5
16
8
4 2 1
4 2 1
4 2 1
4 2 1
23
35
53
5
1 1 1 1
time t
Data r
Old definition (A)
y = r_t
y = r_m
0 1 2 3 4 5 6 7
1 5
23
35
53
23
35
53
5
1 1 1 1
pulse m
Data r
New definition (B)
y = r_m

Figure 2: Graphs of the Collatz sequence with $r_0 = 23$ on based both definition.

Example 5.6. If we start a recursive geometric algorithm with: $r_0 = 63$; d = 2 and _(r) = 3 _ r □ 1, then the remainders and parameters sequences are as follows:
we can use program code in the listing 1.

_63; r_M _ =
2
4
q_m
r_m
m
9=
;
+1
m=0
=

2
4
0 2 2 3 1 3 2 1 2 1 : : :
63 47 35 13 19 7 5 7 5 7 : : :
0 1 2 3 4 5 6 7 8 9 : : :
9=
;; see figure 3,
therefore, the stop time M = 6, the stop remainder $r_M = 5$; $s_M = 13$, since, $r_5 = r_7 = 7$, then the period p = 7 □ 5 = 2, the final cycles are C_2 = _7; 5_; C_0
$_2$ = _2; 1_.

By this algorithm with _(r) = 3r □ 1; d = 2, two partitions on Z with the composition sizes 1_{11}; $1_2 + 2_1 + 7_1$ are craeted.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
□70
□50
□30
□10
10
30
50
70
y = 5
y = □5
63
47
35
13
19
7 5 7 5 7 5 7 5 7 5 7 5
□63
□47
□35
□13
□19
□7 □5 □7 □5 □7 □5 □7 □5 □7 □5 □7 □5
pulse m
Data r
The Collatz jump sequence and its extension line
y = r_m by r_0 = 63
y = r_m by r_0 = □63
Exstension lines

Figure 3: Symmetric sequences by _(r) = 3r □ 1; r_0 = 63 and _(r) = 3r + 1; r_0 = □63.

The complete set of the stop remainder is as follows:
F_{11} = f□1; 1; 5; 7; 17; 25; 37; 41; 55; 61; 91g;
hence, the tree partition is as follows:
P_{11} = f[□1]; [1]; [5]; [7]; [17]; [25]; [37]; [41]; [55]; [61]; [91]g = Z:
The complete set of the stop cycles is as follows:
H_4 = f_□1_; _1_; _5; 7_; _17; 25; 37; 55; 41; 61; 91_g;
thus, the isocyclic partition P_4 = fd□1c; d1c; d5c; d17cg = Z,
in which P_1 = fd□1cg = $Z_□$; P_3 = fd1c; d5c; d17cg = Z_+.

6 Conclusion

This reaserch introduces a new subject of number theory called **cyclical sequences theory**, which is still in its early stages of development and new concepts and theorems can be discovered by continuing the research.

Collatz's conjecture falls within the scope of this theory. Two of its results are the proof of Collatz's conjecture[12] after 87 years and new tree partitions of rational number and integers.

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[12] I myself am the author of the paper on the proof of Collatz's conjecture and will submit it for publication soon.

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Appendix A Calculater codes and tables

Here, we used a program code that can be executed in Texas Ti-84 calculator.

Listing 1: Calculate the Collatz sequence r_m and its parameters q_m; s_m for a 2 Z.

```
a=?; m=0; q=0; s=0; Print (a ," ",m ," ",q ," ",s);
r=a; If(Mod(r ,2)==0 , While (Mod(r ,2)==0 , r /=2; q++;s ++);
Print (", ",r ," ",m ," ",q ," ",s));
While (r !=1&& r!= -1&&r= -5&&r!= -17 , m++; r=3*r+1; q=0;
While (Mod(r ,2)==0 , r /=2; q++; s ++);
Print (", ",r ," ",m ," ",q ," ",s))
```

Table 1: The stop parameters of a Collatz sequence for a 2 Z, so that the stop value r_M = _1.

starter value	Stop parameters		
a	M	S	T
27	41	70	111
63728127	357	592	949
670617279	370	616	986
9780657630	424	707	1131
9780657631	425	707	1132
75128138247	461	767	1228
989345275647	506	842	1348
-989345275637	50	119	169
-63728119	60	121	181
-7512813821	91	177	268

Appendix B Non-stop geometric algorithms

Theorem B.1 (Logarithm algorithm theorem)**.** The logarithm theorim states that given any two real numbers a; d > 0 so that d 6= 1,we can find two uniche numbers q 2 W and real number r such that:

a _ d_q = r or a
d_q = r : q 2 W; 1 _ r <_d_ :

Definition B.1 (sequences created by recursive logarithm algorithm)**.** For any two real numbers r_{00} = a; d > 0, d 6= 1, two sequence are created by the mapping G : N ! R and the recursive logarithm algorithm as follows:

r_m = _($r_{m□1}$) _ d_{q_m} or r_m =
_($r_{m□1}$)
d_{q_m}
: q_m 2 W; 1 _ r_m <_d_; 8m 2 N:

If a __d_; then, r_0 =
a
d_{q_0}
: q_0 2 W; 1 _ r_0 <_d_;

and If a < 1; then, r_0 = a_d_{q_0} : q_0 2 W; 1 _ r_0 <_d_; else r_0 = a; i.e. 1 _ a <_d_ :

Similarly, d 2 R is called **power base** and _(r) 2 Z is called **expansion function** and 9n 2 N : $_{(n)}$(r) < 1 or $_{(n)}$(r) __d_ is its condithon. 1 _ r <_d_ is the remainder condition.

In this algorithm, q is the number of steps ”divided by d” or ”multiplied by d” which is the power value in one step. The m value is the number of steps to use _(r), which is the indice value of the next term. The condition $\frac{1}{\_d\_}$ < r _ 1 is also an option for this algorithm.

By logarithm recursive algorithm, we can calculate $\log_b(a)$ up to billions of decimal digits, without using series or operator factorial !. This algorithm created non-stop sequences. We will provide its full article in the near future.

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