Original Research Article

Basics of Cyclical sequences theory

Abstract

This paper introduces two recursive algorithms that create a new subject of numerical sequences which are called **cyclical sequences** and the Collatz sequence is classified in them. They also divide integers into tree partitions. This theory studies the structure of cyclical sequences. One of the obstacles to proving Collatz's conjecture has been the lack of sufficient knowledge about the Collatz sequence. This article provides a new and complete understanding of Collatz's conjecture, in which, we have also redefined the collatz sequence with a parameter.

Keywords: Regular algorithm, cyclical sequences, tree partition, isocyclic partition

MSC Classification: 11B50 , 11Y16 , 11K31 , 11Y55

1 Introduction

In the reaserch, we introduce a new subject of numerical sequences called cyclical sequences, in which, the Collatz sequence is classified. This theory studies the structure of cyclical sequences.

One of the important obstacles to prove Collatz's conjecture[1] was the complete lack of knowledge of the Collatz sequence.[1] study of this paper is necessary to understand Collatz's conjecture. We also redefine the Collatz sequence by a parameter, based on which, the collatz conjecture is proved.

In the section of numerical cycles, by extending the division algorithm, sequences similar to the Collatz sequence are created, which also reach a cycle. Such sequences and the Collatz sequences are placed in a new mathematical subject which are called cyclical sequences . We also create new partitions of rational number and integers by recursive algorithms.

We have used the following sets of numbers in this article:

[1] An unsolved open problem in mathematics which Lothar Collatz introduced in 1937 and during research that lasted 6 years, we completed its proof on May 12, 2024 after 87 years.

Naturals $\mathbb{N}=\{1,2,\ldots\}$, arithmatics $\mathbb{W}=\mathbb{N}-1=\{0,1,2,\ldots\}$, integers $\mathbb{Z}=\{\ldots,-2,-1,0,1,2,\ldots\}$, rationals $\mathbb{Q}=\{\frac{a}{b}:\ a,b\in\mathbb{Z},\ b\neq 0\}$.

2 Cyclical sequences

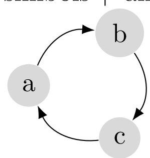
You should be familiar with properties of sequences. In a sequence, repetition is allowed and order is important. in view of these evidences, it is better a sequence is denoted by symbols [and }.

Definition 1 (Periodic sequence and cycle)**.** *A sequence consists of a finite and ordered set of numbers that repeats indefinitely.*

Each Periodic sequence is denoted by all members respectively as follows: $C_p=[c_t\}_{t=1}^{+\infty}=[c_1,c_2,\ldots,c_p,c_{p+1},c_{p+2},\ldots,c_{2p},\ldots\}$, $\quad p\in\mathbb{N}$, *is the* ***base period****,* $c_{t+k\cdot p}=c_t$, $\quad k\in\mathbb{W}$.

Thus by definition, a Periodic sequence can be represented by a **cycle**[2]. Each cycle is also denoted with all element, respectively, between two simbols $\rceil$ and $\lfloor$.

Let $C_3=[a,b,c,a,b,c,\ldots\}$ be a periodic sequence which has period $p=3$ then, it is represented with these forms: by simbols of cycle $\rceil a,b,c\lfloor$, and by directed cycle graph, $a \rightleftarrows b \rightarrow c$ or like this figur:



Definition 2. *The sequence that eventually reaches (falls into) a cycle are called a* ***cyclical sequence****, the final cycle is called* ***stop cycle****.*

Cyclical sequences are created by ***special functions*** *or* ***recursive algorithms****.*

Let a be the starting term,[3] then a cyclical sequence is represented as follows:

$$[a,r_m\}_{m=0}^{+\infty}=[a,r_0,r_1,r_2,\ldots,r_m,r_{m+1},r_{m+2},\ldots,r_{m+p-1},r_{m+p},\ldots\},$$

Where, $r_m=r_{m+k\cdot p}:k\in\mathbb{W}$, therefore, its stop cycle is as follow:

$$C_p=\rceil r_m,r_{m+1},r_{m+2},\ldots,r_{m+p-1}\lfloor,\ \text{in which, its period is } p.$$

Definition 3. *The cyclical sequence from a point onwards are placed in a cycle, the value of the first stop from which the cycle begins, is called* ***stop value****. The number of generated terms up to the stop point is called* ***stop time*** *which is denoted by* $\mathtt{M}$. *Thus, the stop time* $\mathtt{M}=m+p-1$*, the* ***stop value*** $r_{\mathtt{M}}=r_{m+p-1}$. *See example 1, 2.*

Thus by definition, final cycle with some different starting point is as follows:

$$C_p=\rceil r_{\mathtt{M}-p+1},r_{\mathtt{M}-p+2},r_{\mathtt{M}-p+3},\ldots,r_{\mathtt{M}}\lfloor=\rceil r_{\mathtt{M}},r_{\mathtt{M}+1},r_{\mathtt{M}+2},\ldots,r_{\mathtt{M}+p-1}\lfloor.$$

3 Regular algorithms

Definition 4 (Regular algorithm)**.** *A relation that has a specific mathematical formula and its output variables have one or more conditions is called a* ***regular algorithm****.*

[2]A cycle is a path in which only the first and last vertice is equal.
[3]The starting term a may not be the same as the initial term r_0.

Theorem 1 (Arithmetic algorithm and congruence)**.** *For any two integer a and $d \geq 2$, we can find $q \in \mathbb{W}$ and integer r such that:*

$$a \pm d \times q = r \Leftrightarrow a \overset{d}{\equiv} r : \quad q \in \mathbb{W},\ |r| < d,\ \textit{where } d \textit{ is modulo.}$$

Theorem 2 (Geometric algorithm)**.** *Let $\nmid$ be the indivisibility symbol. The geometric algorithm states that given any two integers or fractionals a and $d \geq 2$, we can find $q \in \mathbb{W}$ and integer or fractional number r such that:*

$$\frac{a}{d^q} = r \quad \textit{or} \quad a \times d^q = r : \quad q \in \mathbb{W},\ d \nmid r, \quad \textit{where } d \textit{ is modulo.}$$

3.1 Recursive form of numerical algorithms

To expand an algorithm, we rewrite its relation recursively, then we change the previous remainder with a suitable function, so that the value of the function is outside the condition of the remainder. The expansion function can be any form and must be such that using it one or more times violates the remainder condition.

Cyclical sequences can be created by the recursive algorithms. For any integer r_0, two cyclical sequences are created to gether by a recursive algorithm.

Definition 5 (Sequences created by recursive arithmetic algorithm)**.** *For any two integer a and $d \geq 1$, two sequences are created by the mapping $A : \mathbb{N} \rightarrow \mathbb{Z}$ and the recursive arithmatic algorithm as follow:*

$$r_m = \mu(r_{m-1}) \pm d \times q_m : \quad q_m \in \mathbb{W},\ |r_m| < d, \quad \forall m \in \mathbb{N},$$
$$\textit{If } |a| \geq d \quad \textit{then, } r_0 = a \pm d \times q_0 : \ q_0 \in \mathbb{W},\ |r_0| < d, \quad \textit{else } r_0 = a.$$

*where d is called **quoteint modulo**, $\mu(r) \in \mathbb{Z}$ is called **expansion function** and $|r| < d$ is called **remainder condition**.*

The expansion function must be such that using it one or more times violates the remainder condition.

*In this algorithm, q is the number of steps "minus or plus d" which is **quoteint** of modulo and m is the number of steps to use $\mu(r)$, which is the indice value of the next term.*

Thus by the recursive arithmatic algorithm, two cyclical sequences are created together, $[r_m\}_{m=0}^{+\infty}$, which is called **remainders sequence** and $[q_m\}_{m=0}^{+\infty}$, which is called **quotients sequence**.

Two cyclical sequences $[q_m\}$ and $[r_m\}$ are created by recursive algorithm can be represent by a **weight path**[4], such that its vertics are elements of the sequence $[r_m\}$ and its edges weight are element of $[q_m\}$.

Each cyclical sequence can be denoted with start term as the first vertice, between two simbols as $\upharpoonright a, r_{\mathtt{M}} \upharpoonleft$ when its algorithm is known. the last vertice is always in the

[4] A weight path is a walk in which all vertices and edges are distinct and every its edge has a weight, where a value is relayed to a vertex.

final cycle. Hence, we can also represent both resulting sequences by a matrix. See the graph 3.1 and example 2.

Thus by definitions, in the graph 3.1, the path $\upharpoonleft a \upharpoonright = \begin{bmatrix} q_m \\ r_m \end{bmatrix}_{m=0}^{+\infty}$, the stop value $r_{\mathtt{M}} = r_6$ and the stop time $\mathtt{M} = 6$, since $r_7 = r_2$, then the period $p = 7 - 2 = 5$, therefore, the final cycles are $C_5 = \upharpoonleft r_2, r_3, r_4, r_5, r_6 \downharpoonleft$, $C'_5 = \upharpoonleft q_3, q_4, q_5, q_6, q_7 \downharpoonleft$.

$$a \xrightarrow{q_0} r_0 \xrightarrow{q_1} r_1 \xrightarrow{q_2} r_2 \xrightarrow{q_3} r_3 \xrightarrow{q_4} r_4 \xrightarrow{q_5} r_5 \xrightarrow{q_6} r_6 \quad \left(r_6 \xrightarrow[q_7]{r_7=\mu(r_6)-d\times q_7=r_2} r_2 \right) . \tag{3.1}$$

The **division algorithm** is a special case of the recursive arithmetic algorithm in which the expantion function is $\mu(r) = b \times r$, where $b \in \mathbb{N}$, $b \geq 2$, is the **numerical base**, modulo $d \in \mathbb{N}$ and remainder condition is as: $0 \leq r < d$.

A familiar example for us in this case, the decimal expansion of a numerical fraction is done by division algorithm.

The decimal expansions of rational numbers always reaches periodic sequences which is a numerical cycle.

Example 1. *If a division algorithm is started whith $a = 81, 59$, modulo $d = 13$, and $\mu(r) = b \times r = 10 \times r$ then the result sequences and their path are as follows:*

$$a = 81 \xrightarrow{6} r_0 = 3 \xrightarrow{2} 4 \xrightarrow{3} 1 \xrightarrow{q_3=0} 10 \xrightarrow{7} 9 \xrightarrow{6} r_5 = 12 \quad \left(r_5 \xrightarrow[9]{r_6=10\times 12-13\times 9=3} r_0 \right),$$

Thus by the path graph, remainders and quotients sequence are as follow:

$$[81, r_m\} = [81, 3, 4, 1, 10, 9, 12, 3, 4, 1, 10, 9, 12, \ldots\} \overset{fall\ into}{\longrightarrow} C_6 = \upharpoonleft 3, 4, 1, 10, 9, 12 \downharpoonleft,$$
$$[q_m\} = [0, 6, 2, 3, 0, 7, 6, 9, 2, 3, 0, 7, 6, 9, \ldots\} \overset{final\ cycle}{\longrightarrow} C'_6 = \upharpoonleft 2, 3, 0, 7, 6, 9 \downharpoonleft,$$

therefore, the stop value $r_{\mathtt{M}} = r_5 = 12$, the stop time $\mathtt{M} = 5$, since $r_6 = r_0 = 3$, then the period $p = 6 - 0 = 6$.

$$a = 59 \xrightarrow{4} r_0 = 7 \xrightarrow{5} 5 \xrightarrow{3} 11 \xrightarrow{q_4=8} 6 \xrightarrow{4} 8 \xrightarrow{6} r_5 = 2 = r_{\mathtt{M}} \quad \left(r_5 \xrightarrow[1]{r_6=10\times 2-13\times 1=7} r_0 \right).$$

by matrix form, $\upharpoonleft 59, r_{\mathtt{M}} \upharpoonright = \begin{bmatrix} q \\ r \\ m \end{bmatrix}_{m=0}^{+\infty} = \begin{bmatrix} 0 & 4 & 5 & 3 & 8 & 4 & 6 & 1 & 5 & 3 & 8 & 4 & 6 & 1 & \ldots \\ 59 & 7 & 5 & 11 & 6 & 8 & 2 & 7 & 5 & 11 & 6 & 8 & 2 & 7 & \ldots \\ 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \ldots \end{bmatrix}$

Thus, the stop time $\mathtt{M} = 5$, the stop value $r_{\mathtt{M}} = 2$, $q_{\mathtt{M}} = 2$, since $r_6 = r_0 = 7$, then the

period $p = 6 - 0 = 6$, *the final cycles (stop cycles) are as follow:*
remainders stop cycle $C_6 = \upharpoonleft 7, 5, 11, 6, 8, 2 \downharpoonleft$,
quotients stop cycle $C'_6 = \upharpoonleft 5, 3, 8, 4, 6, 1 \downharpoonleft$.

Corollary 1. *The quoteints sequence is the same decimal expansion of a fraction so that its terms are decimal digits of the fraction.*

For example: $\frac{81}{13} = 6.\overbrace{230769}\overbrace{230769}\ldots$, $\frac{43}{13} = 3.\overbrace{307692}\overbrace{307692}\ldots$,
$\frac{59}{13} = 4.\overbrace{538461}\overbrace{538461}\ldots$, $\frac{97}{13} = 7.\overbrace{461538}\overbrace{461538}\ldots$, $\frac{-19}{13} = -1.\overbrace{461538}\overbrace{461538}\ldots$

Corollary 2. *All cyclical sequences that is creared by a division algorithm with start value* $a \in \mathbb{Z}$, *modulo* $d = 13$, *and expansion function* $\mu(r) = 10 \times r$, *can only reach one of the following quocient cycles in the figur A1.*

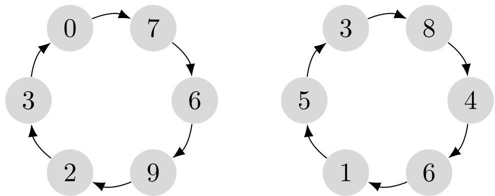
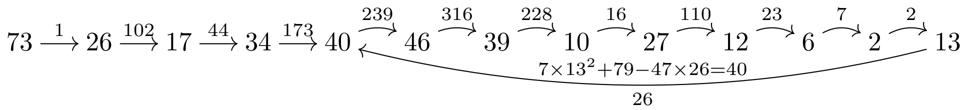


Fig. 1: The quocient stop cycles in the form of a cycle graph for $\frac{a}{13}$, where, $a \in \mathbb{Z}$.

Example 2. *If we start with:* $a = 73$, $d = 47$ *and* $\mu(r) = 7 \times r^2 + 79$, *then the remainders and quotients sequences and its path graph are as follows:*



$$\upharpoonleft 73, r_{\mathtt{M}} \upharpoonleft = \begin{bmatrix} q \\ r \\ m \end{bmatrix}_{m=0}^{+\infty} = \begin{bmatrix} 0 & 1 & 102 & 44 & 173 & 239 & 316 & 228 & 16 & 110 & 23 & 7 & 2 & 26 & \ldots \\ 73 & 26 & 17 & 34 & 40 & 46 & 39 & 10 & 27 & 12 & 6 & 2 & 13 & 40 & \ldots \\ 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \ldots \end{bmatrix}$$

Thus, the stop time $\mathtt{M} = 11$, *the stop value* $r_{\mathtt{M}} = 13$, $q_{\mathtt{M}} = 2$, *since* $r_3 = r_{12} = 40$, *then the period* $p = 12 - 3 = 9$, *the final cycles are as follow:*

$$C_9 = \upharpoonleft 40, 46, 39, 10, 27, 12, 6, 2, 13 \downharpoonleft, \quad C'_9 = \upharpoonleft 239, 316, 228, 16, 110, 23, 7, 2, 26 \downharpoonleft.$$

Definition 6 (sequences created by recursive geometric algorithm)**.** *For any two integer* $a \in \mathbb{Z}$, $d \geq 2$, *two sequence are created by the mapping* $G : \mathbb{N} \to \mathbb{Z}$ *and the recursive geometric algorithm as follows:*

$$r_m = \frac{\mu(r_{m-1})}{d^{q_m}} : \quad q_m \in \mathbb{W}, \quad d \nmid r_m, \quad \forall m \in \mathbb{N}.$$

$$\textit{If } d \mid a \textit{ then, } r_0 = \frac{a}{d^{q_0}} : \quad q_0 \in \mathbb{W},\ d \nmid r_0, \quad \textit{else } r_0 = a.$$

where $d \in \mathbb{N}$ is called ***power modulo*** *and $\mu(r) \in \mathbb{Z}$ is called* ***expansion function****. Here we can assume that $d \nmid 0$.*

In this algorithm, q is the number of steps "divided by d" which is the power of the modulo for one step and m is the number of steps to use $\mu(r)$, which is the indice value of the next term. Thus by the recursive geometric algorithm, two cyclical sequences are created, $[r_m\}_{m=0}^{+\infty}$, that is called ***remainders sequence*** *and $[q_m\}_{m=0}^{+\infty}$, which is called* ***powers sequence****.*

Definition 7. *The number of steps "divided by d", to generate the term r_m is called its* ***shrinkage parameter*** *and denoted by s_m.*

Let $[r_m\}_{m=0}^{+\infty}$ be a Collatz sequence then, all shrinkage parameters create the sequence $[s_m\}_{m=0}^{+\infty}$ so that $s_m = \sum_{i=0}^{m} q_i$. In a stop cycle $s_p = \sum_{i=1}^{p} q_i$.

Definition 8. *the shrinkage parameter of the first stop is called* ***stop parameter of cyclical sequence****, and denoted by $\mathtt{S}$. If $\mathtt{M}$ is the* ***stop time****; then $s_{\mathtt{M}} = \mathtt{S}$. The stop value $r_{\mathtt{M}}$ doesn't depend on starting value a.*

Example 3. *If we start the recursive giometric algorithm by $r_0 = 93$, with modulo $d = 2$ and expantion function $\mu(r) = r - 7$, then the cyclical sequences and its weight path are as follows:*

$$93 \xrightarrow{1} 43 \xrightarrow{2} 9 \xrightarrow{1} 1 \xrightarrow{q_4=1} -3 \underset{2}{\overset{1}{\rightleftarrows}} -5 \qquad r_6 = \frac{-5-7}{2^2} = -3$$

$$\rceil 93, r_{\mathtt{M}} \lceil = \begin{bmatrix} q_m \\ r_m \\ m \end{bmatrix}_{m=0}^{+\infty} = \begin{bmatrix} 0 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 2 & \dots \\ 93 & 43 & 9 & 1 & -3 & -5 & -3 & -5 & -3 & \dots \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \dots \end{bmatrix}$$

Thus, the stop time $\mathtt{M} = 5$, the stop values $r_{\mathtt{M}} = -5, q_{\mathtt{M}} = 2$, since $r_4 = r_6 = -3$, then the period $p = 6 - 4 = 2$, the stop cycles are: $C_2 = \rceil -3, -5 \lfloor$, $C_2' = \rceil 1, 2 \lfloor$, $s_p = 3$.

In the subject of cyclical sequences, the following theorem is proposed, which must be proved in its special cases.

Corollary 3. *Both Sequences created by recursive arithmetic algorithm in integers Z and created by recursive geometric algorithm in rationals Q eventually reach (fall into) a cycle from a term onwards. Therefore, Collatz's sequence is a special case of sequences created by geometric algoritm with $\mu(r) = 3r + 1$ and $d = 2$ in naturals $\mathbb{N}$, which reaches to stop cycle $\rceil 4, 2, 1 \lfloor$.*

Definition 9. *The final part of any cyclical sequence is a periodic sequence in which the points with index $m = \mathtt{M} + k \cdot p$ lie on a straight line, here, we call it* ***extension line of the sequence****. Since $r_{\mathtt{M}} = r_{\mathtt{M}+k \cdot p}$, then its equation is $y = r_{\mathtt{M}}$. see figure A3*

3.2 Partition by recursive algorithms

By runnig a recursive algorithm with known expantion function and modulo on integers, each time it is used, it creates a remainder and after a few uses it reaches the stop remainder and from this remainder onward, the recursive algorithm falls into the final cycle.

For all integers, a finite number of cycles is created, so we can part the integers based on.

The set of integers that reach the same stop remainder form a **tree**[5], so each stop remainder is equivalant to a tree and denoted by $[r_{\mathtt{M}}]$.

Eeach cyclical sequence[6] is a path of the tree so that this path is from vertice r_0 to the final cycle, and is denoted by $\upharpoonleft r_0, r_{\mathtt{M}} \upharpoonleft$.

Eeach stop remainder belong to a final cycle so the cycle C_p has precisely p tree, hence, the set of all trees is a partition on Z.

Definition 10. *We can define an isocyclic on $\mathbb{Z}$, when Two elements $x, y \in \mathbb{Z}$ create the same cycle by a recursive algorithm. Thus isocyclic class of a is called the set of all element of Z which are isocycle to a, this class is denoted by $\lceil a \rfloor$. Two elements $x, y \in \mathbb{Z}$ are isocycle if and only if they belong to the same isocyclic class, thus we can write: $\lceil x \rfloor = \lceil y \rfloor$.*

The set of isocyclic classes is a partition of $\mathbb{Z}$. The smallest element of an cycle may be chosen as a representative of the isocyclic class as $\lceil c_{min} \rfloor$.

The stop value of the arithmatic sequences resulting from the division algorithm with modulo d is precisely one of the integers in the complete set of stop remainders F_n as follows: the remaining condition of the algorithm tell us that,

$$0 \leq r < d \Rightarrow F_{n=d} = [0, 1, 2, \ldots, d-1\}.$$

This means that each integer is the vertex of precisely one of the trees in the following partition. Let n be the number of the subset in partirion P_n then,

$$P_{n=d} = \{[0], [1], [2], \ldots, [d-1]\}.$$

In example 1, division algorithm with modulo $d = 13$ and $\mu(r) = 10r$ create two partitions on Z as follows:

$$\text{the set of stop remainders } F_{13} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\},$$
$$\text{thus, the tree partition is } P_{13} = \{[0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12]\}.$$

Let H_n be the complete set of final cycles which are created by the division algorithm then,

$$H_3 = \{\upharpoonleft 0 \downharpoonleft, \upharpoonleft 1, 10, 9, 12, 3, 4 \downharpoonleft, \upharpoonleft 2, 7, 5, 11, 6, 8 \downharpoonleft\},$$

[5]a tree is a graph in which any two vertices are conected by exactly one path.
[6]that is created by a recursive algoritm

thus with composition sizes $1+6+6$, the isocyclic partition is as follows:

$$P_3 = \{[0], \{[1], [10], [9], [12], [3], [4]\}, \{[2], [7], [5], [11], [6], [8]\}\} = \{\lceil 0 \rfloor, \lceil 1 \rfloor, \lceil 2 \rfloor\} = \mathbb{Z}.$$

Hence, there are 3 isocyclic classes $\lceil 0 \rfloor$, $\lceil 1 \rfloor$ and $\lceil 2 \rfloor$ in the partition P_3 so that:

I. $\lceil 0 \rfloor \cup \lceil 1 \rfloor \cup \lceil 2 \rfloor = \mathbb{Z}$.

II. $\lceil 0 \rfloor \cap \lceil 1 \rfloor = \emptyset$, $\lceil 0 \rfloor \cap \lceil 2 \rfloor = \emptyset$, $\lceil 1 \rfloor \cap \lceil 2 \rfloor = \emptyset$.

Corollary 4. *Infact, the isocyclic partition P_3 is a three membred subset from the complete set of final remainder, with modulo d*

Remark 1. *Composition sizes depend on expantion function and modulo of the recursive algorithm.*

in the example 3 by the recursive geometric sequence with $\mu(r) = r - 7$, $d = 2$, two partitions on Z with the composition sizes[7] $1+1+1+1$, $1+2+1$ are created as follows: Since, $-7 \leq r \leq -1$, $2 \nmid r$,
thus, the complete set of stop remainder $F_4 = \{-1, -3, -5, -7\}$,
hence, the tree partition $P_4 = \{[-1], [-3], [-5], [-7]\} = \mathbb{Z}$.
The complete set of the stop cycles $H_3 = \{\upharpoonleft -1 \downharpoonleft, \upharpoonleft -3, -5 \downharpoonleft, \upharpoonleft -7 \downharpoonleft\}$, thus,
the isocyclic partition $P_3 = \{[-1], \{[-3], [-5]\}, [-7]\} = \{\lceil -1 \rfloor, \lceil -3 \rfloor, \lceil -7 \rfloor\} = \mathbb{Z}$.

4 Forms of the Collatz sequense

The collatz sequence is a cyclical sequence that is created by special modes of recursive geometric algorithm with expantion function $\mu(r) = 3r + 1$ and modulo $d = 2$.

In its old definition, the mode of its algorithm is two-piecewise, in which the even and odd result of each step was assumed a term of Collatz's sequence, but in the new definition, only odd result appear, and the mode of the algorithm used is one piecewise.

both sequences poceeds in pulse form, such that in old definition, the $3r + 1$ piece maginfies the only term of ascending edge, then the $\frac{r}{2}$ piece shrinks each term of descending edge of a pulse once, but in new definithion, the ascending edge of any pulses can ascend one or more in a row by $\frac{3r+1}{2}$, and its descending edge consists of several terms, each term can be reduced one or more times in a row by $\frac{r}{2}$.

Definition 11 (Old definition of Collatz's sequence)**.** *This sequence is defined by the mapping $L : \mathbb{N} \to \mathbb{Z}$ and created by the following recursive algorithm as follows:*

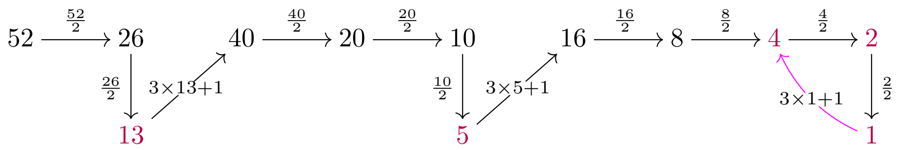
$$r_t = L(r_{t-1}) = \begin{cases} \frac{r_{t-1}}{2} & 2 \mid r_{t-1} \\ 3r_{t-1} + 1 & 2 \nmid r_{t-1}, \end{cases} \qquad \forall\ t \in \mathbb{N},\ r_0 \in \mathbb{Z},\ r_0 \neq 0.$$

According to Collatz's conjecture in 1937, all created sequences with $r_0 \in \mathbb{N}$, $\mu(r) = 3r + 1$ and modulo $d = 2$, eventually reaches the cycle $\upharpoonleft 4, 2, 1 \downharpoonleft$, which is the same **periodic sequence** $C_3 = [4, 2, 1, 4, 2, 1, \ldots\}$ that can be denoted by the cycle graph, $4 \rightarrow 2 \rightarrow 1 \rightarrow 4$. see the graph(A) in figure A2.

[7]The sizes is also the period of cycles in the second partition.

We can denote it by path symbols as: $\upharpoonleft r_0, r_{\mathtt{T}} = 1 \upharpoonright$ where $\mathtt{T}$ is its stop time and $r_{\mathtt{T}}$ is its stop remainder (stop value).

Example 4. *By the old definition, Collatz's sequence and its pulsed graph*[8], *where start by* $r_0 = 52$ *is as follows:*



$$\upharpoonleft r_0 = 52, r_{\mathtt{T}} \upharpoonright = [52, r_t\}_{t=0}^{+\infty} = [52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, \ldots\}.$$

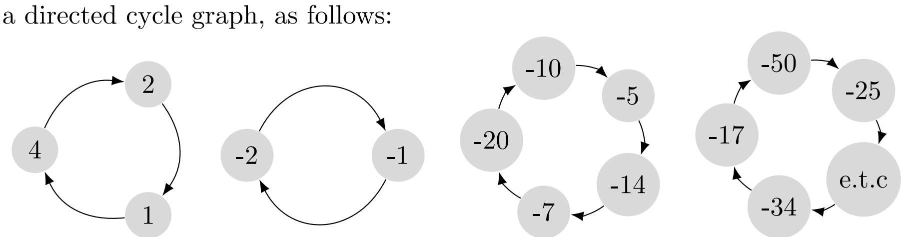
Thus , the stop value $r_{\mathtt{T}} = r_{11} = 1$ *then the stop time* $\mathtt{T} = 11$ *and the stop cycle* $C_3 = \upharpoonleft 4, 2, 1 \downharpoonleft$.

Example 5. *By the old definition, Collatz's sequence with* $r_0 = -19$ *is as follows:*

$$\upharpoonleft -19, r_{\mathtt{T}} \upharpoonright = [-19, -56, -28, -14, -7, -20, -10, -5, -14, -7, -20, -10, -5, \ldots\},$$

thus by this path, the stop value $r_{\mathtt{T}} = r_7 = -5$, *then the stop time* $\mathtt{T} = 7$ *and the stop cycle* $C_5 = \upharpoonleft -14, -7, -20, -10, -5 \downharpoonleft$.

All cycles created by Collatz's old algorithm on the integers are shown in the form of a directed cycle graph, as follows:



The uncertainty of the number of steps "divided by two" versus one step, "triple plus one" made it very difficult to prove and also was the reason for the ambiguity of this conjecture.

4.1 Redefinition of Collatz's sequence

The collatz sequence can be generalized to rational numbers $\mathbb{Q}$, that each of them is separated into several subsets based on the stop cycle.

After leaving the remainder, from its condition using the expansion function $\mu(r) = 3 \times r + 1$, dividing by 2 is performed several times, so that the condition is established again. If we assume the number of dividing by 2 is q, then the Collatz sequence is redefined as follows which is created a sequence with odd terms, and we call it Collatz's jump sequence.

Definition 12. *A sequence is created by the followsing recursive geometric algorithm on the mapping* $L : \mathbb{N} \to \mathbb{Z}$ *is called* ***Collatz's jump sequence***.
Let $0 \neq a \in \mathbb{Z}$ *be the starting value of the algorithm, then,*

---

[8] pulsed graph is a path that is represented to a waveform.

$$r_m = L(r_{m-1}) = \frac{3r_{m-1}+1}{2^{q_m}} \;:\quad q_m \in \mathbb{N},\ 2 \nmid r_m,\ \forall m \in \mathbb{N}.$$
If $2 \mid a$, *then initial term* $r_0 = \frac{a}{2^{q_0}} : q_0 \in \mathbb{N},\ 2 \nmid r_0,$ *else* $r_0 = a.$

If you are not surprised, the values of m, q must be zero for the starting value a, in order for the sequence graph conform to its formula, i.e, we always assume that $m = 0, q = 0$ for the starting value a, which is also $m = 0$ for the initial term r_0.

Remark 2. *In view of the new definition, for proof of Collatz's conjecture we have to prove which any Collatz sequence with $a \in \mathbb{N}$ eventually reches the stop cycle $\upharpoonleft 1 \downharpoonleft$, in other word, it converges to $r_{\mathtt{M}} = 1$. see the graph(B) in figure A2.*

Example 6. *The Collatz jump sequence and its **weight path** with $a = 28$ is as follows: we can use program code in the listing 1,*

$$28 \xrightarrow[\frac{28}{2^2}=7]{2} 7 \xrightarrow[\frac{3\times7+1}{2^1}=11]{1} 11 \xrightarrow{1} 17 \xrightarrow{2} 13 \xrightarrow{3} 5 \xrightarrow[\frac{3\times5+1}{2^4}=1]{q_5=4} 1 \circlearrowleft^{2}_{\frac{3\times1+1}{2^2}=1}$$

Thus by this graph, $\upharpoonleft 28, r_{\mathtt{M}} \upharpoonright = [28, r_m\}_{m=0}^{+\infty} = [28, 7, 11, 17, 13, 5, 1, 1, \ldots\}$, $r_5 = 1$, $\mathtt{M} = 5$, *then the stop cycle* $C_1 = \upharpoonleft 1 \downharpoonleft$. $[q_m\} = [0, 2, 1, 1, 2, 3, 4, 2, 2, \ldots\} \Rightarrow C_1' = \upharpoonleft 2 \downharpoonleft$.

By runing program code in the listing 1 for $a \in \mathbb{Z}$, all terms of the Collatz sequence, $[r_m\}$ and their parameters sequences, $[q_m\}$, $[s_m\}$ are calculated.

In the table A1, the stop parameters $\mathtt{M}, \mathtt{S}, \mathtt{T}$ of collatz's sequence with $a \in \mathbb{Z}$, so that $r_{\mathtt{M}} = \pm 1$, which are created by program code in the listing 1. Always $\mathtt{T} = \mathtt{M} + \mathtt{S}$.

Lemma 3. *Let $[r_m\}_{m=0}^{+\infty}$ be a Collatz sequence with $r_0 \in \mathbb{Z}$. If $m \geq 1 \Rightarrow 3 \nmid r_m$.*

Proof. $3 \times r_{m-1} + 1 = 2^{q_m} r_m \Rightarrow 2^{q_m} r_m \overset{3}{\equiv} 1,\ 2 \overset{3}{\equiv} -1 \Rightarrow r_m \overset{3}{\equiv} \pm 1 \Rightarrow 3 \nmid r_m$. □

By the Collatz algorithm with $\mu(r) = 3r + 1$, $d = 2$, two partitions on Z with the composition sizes 1^{11}, $1^2 + 2^1 + 7^1$ are craeted.

The complete set of stop remainder on integer $\mathbb{Z}$ is as follows:

$$F_{11} = \{1, -1, -5, -7, -17, -25, -37, -41, -55, -61, -91\},$$

hence, the tree partition by assuming that zero is odd, as follows:

$$P_{11} = \{[1], [-1], [-5], [-7], [-17], [-25], [-37], [-41], [-55], [-61], [-91]\} = \mathbb{Z}.$$

The complete set of the stop cycles on $\mathbb{Z}$ is as follows:

$$H_4 = \{\upharpoonleft 1 \downharpoonleft, \upharpoonleft -1 \downharpoonleft, \upharpoonleft -5, -7 \downharpoonleft, \upharpoonleft -17, -25, -37, -55, -41, -61, -91 \downharpoonleft\},$$

thus, the isocyclic partition $P_4 = \{\lceil 1 \rfloor, \lceil -1 \rfloor, \lceil -5 \rfloor, \lceil -17 \rfloor\} = \mathbb{Z}$,
in which $P_1 = \{\lceil 1 \rfloor\} = \mathbb{Z}^+$, $P_3 = \{\lceil -1 \rfloor, \lceil -5 \rfloor, \lceil -17 \rfloor\} = \mathbb{Z}^-$.

Lemma 4. *Let $[r_m\}_{m=0}^{+\infty}$ be Collatz's sequence, If r_0 is positive, negative or fractional, so is r_m.*
In other words, $r_0 \in \mathbb{Z}^+ \Rightarrow r_m \in \mathbb{Z}^+$, $r_0 \in \mathbb{Z}^- \Rightarrow r_m \in \mathbb{Z}^-$, $r_0 \in \mathbb{Q} \Rightarrow r_m \in \mathbb{Q}$.

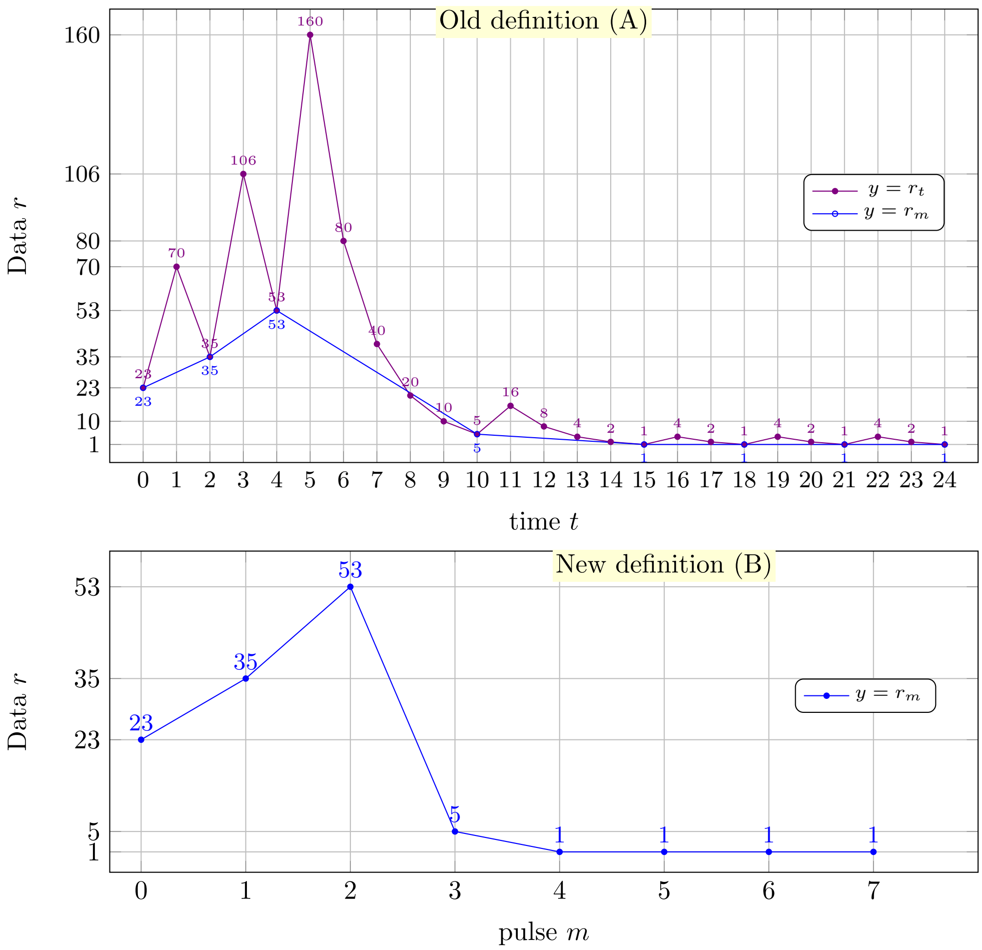


Fig. 2: Graphs of the Collatz sequence with $r_0 = 23$ on based both definition.

4.2 Symmetric sequences

The difference of the symmetric sequence algorithm is in their expansion function, But the parameters of both are the same.

Sequence with $\mu(r) = 3r + 1$ and Sequence with $\mu(r) = 3r - 1$ are symmetrical.

Let $\mu(r) = 3r - 1,\ d = 2$ then its recursive algorithm is as follows:

$$r_m = L(r_{m-1}) = \frac{3r_{m-1} - 1}{2^{q_m}} \ :\ q_m \in \mathbb{N},\ 2 \nmid r_m,\ \forall m \in \mathbb{N};\ r_0 = \frac{a}{2^{q_0}} \ :\ q_0 \in \mathbb{W},\ 2 \nmid r_m.$$

Example 7. *If we start a recursive geometric algorithm with:* $r_0 = 63,\ d = 2$ *and* $\mu(r) = 3 \times r - 1$*, then the remainders and parameters sequences are as follows:*

we can use program code in the listing 1.

$$\upharpoonleft 63, r_{\mathtt{M}} \upharpoonright = \begin{bmatrix} q_m \\ r_m \\ m \end{bmatrix}_{m=0}^{+\infty} = \begin{bmatrix} 0 & 2 & 2 & 3 & 1 & 3 & 2 & 1 & \dots \\ 63 & 47 & 35 & 13 & 19 & 7 & 5 & 7 & \dots \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \dots \end{bmatrix}, \textit{see figure A3},$$

therefore, the stop time $\mathtt{M} = 6$, *the stop remainder* $r_{\mathtt{M}} = 5$, $s_{\mathtt{M}} = 13$, *since,* $r_5 = r_7 = 7$, *then the period* $p = 7 - 5 = 2$, *the final cycles is* $C_2 = \upharpoonleft 7, 5 \downharpoonleft$.

By this algorithm with $\mu(r) = 3r - 1$, $d = 2$, two partitions on Z with the composition sizes 1^{11}, $1^2 + 2^1 + 7^1$ are craeted.

The complete set of the stop remainder is as follows:

$$F_{11} = \{-1, 1, 5, 7, 17, 25, 37, 41, 55, 61, 91\},$$

hence, the tree partition is as follows:

$$P_{11} = \{[-1], [1], [5], [7], [17], [25], [37], [41], [55], [61], [91]\} = \mathbb{Z}.$$

The complete set of the stop cycles is as follows:

$$H_4 = \{\upharpoonleft -1 \downharpoonleft, \upharpoonleft 1 \downharpoonleft, \upharpoonleft 5, 7 \downharpoonleft, \upharpoonleft 17, 25, 37, 55, 41, 61, 91 \downharpoonleft\},$$

thus, the isocyclic partition $P_4 = \{\lceil -1 \rfloor, \lceil 1 \rfloor, \lceil 5 \rfloor, \lceil 17 \rfloor\} = \mathbb{Z}$,
in which $P_1 = \{\lceil -1 \rfloor\} = \mathbb{Z}^-$, $P_3 = \{\lceil 1 \rfloor, \lceil 5 \rfloor, \lceil 17 \rfloor\} = \mathbb{Z}^+$.

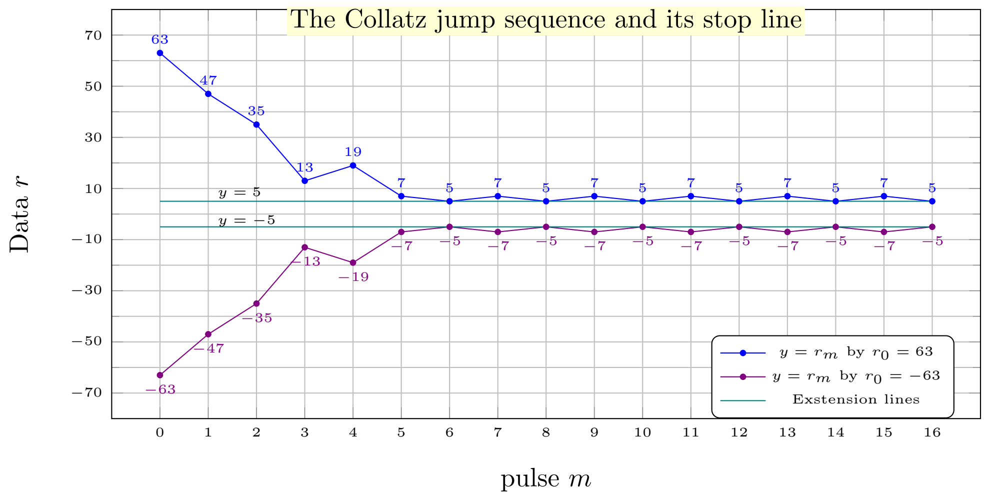


Fig. 3: Symmetric sequences by $\mu(r) = 3r - 1, r_0 = 63$ and $\mu(r) = 3r + 1, r_0 = -63$.

5 Conclusion

In the reaserch, we introduce a new subject of numerical sequences in which the Collatz sequence is classified. We also create new tree partitions of rational number and integers by recursive algorithms.

The uncertainly of the value of the stop parameter S versus the stop time M has been a main reason for the difficulty of proving the Collatz's conjecture. One of these result is the proof of Collatz's conjecture after 87 years.

data availability. Data sharing is not applicable to this article as no datasets were generayed or analysed in the current paper.

References

[1] *R. K. Guy*, Unsolved problems in number theory. 3rd ed. New York, NY: Springer-Verlag (2004; Zbl 1058.11001)

Appendix A Calculater codes and tables

Here, we used a program code that can be executed in Texas Ti-84 calculator.

Listing 1: Calculate the Collatz sequence r_m and its parameters q_m, s_m for $a \in \mathbb{Z}$.

```
a=?; m=0; q=0; s=0;  Print(a," ",m," ",q," ",s);
r=a; If(Mod(r,2)==0, While(Mod(r,2)==0,r/=2;q++;s++);
Print(", ",r," ",m," ",q," ",s));
While(r!=1&&r!=-1&&r=-5&&r!=-17, m++; r=3*r+1; q=0;
While(Mod(r,2)==0, r/=2; q++; s++);
Print(", ",r," ",m," ",q," ",s))
```

Table A1: The stop parameters of a Collatz sequence for $a \in \mathbb{Z}$, so that the stop value $r_{\mathtt{M}} = \pm 1$.

starting value	Stop parameters		
a	M	S	T
27	41	70	111
63728127	357	592	949
670617279	370	616	986
9780657630	424	707	1131
9780657631	425	707	1132
75128138247	461	767	1228
989345275647	506	842	1348
-989345275637	50	119	169
-63728119	60	121	181
-7512813821	91	177	268