On Dual Hyperbolic Generalized Woodall Numbers

Abstract. In this work, we introduce the generalized dual hyperbolic Woodall numbers. As special cases, we study with dual hyperbolic Woodall, dual hyperbolic modified Woodall, dual hyperbolic Cullen numbers and dual hyperbolic modified Cullen numbers. Also, we present Binet's formulas, generating functions, some identities, linear sums and matrices related with these sequences. In addition, we give Catalan's and Cassini's identities.

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1. Introduction

The generalized Woodall sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, 5, -8, 4)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = 5W_{n-1} - 8W_{n-2} + 4W_{n-3} \tag{1.1}$$

with the initial values W_0, W_1, W_2 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-1)} - \frac{5}{4}W_{-(n-2)} + \frac{1}{4}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

Next, we can list some important properties of generalized Woodall numbers that are needed.

Now, we give Binet formula of generalized Woodall numbers.

Theorem 1. *[29, Theorem 1.1] Binet formula of generalized Woodall numbers can be given as*

$$W_n = (A_1 + A_2 n) \times 2^n + A_3$$

where

$$A_1 = -W_2 + 4W_1 - 3W_0,$$
$$A_2 = \frac{W_2 - 3W_1 + 2W_0}{2},$$
$$A_3 = W_2 - 4W_1 + 4W_0,$$

that is,

$$W_n = ((-W_2 + 4W_1 - 3W_0) + \frac{W_2 - 3W_1 + 2W_0}{2} n) \times 2^n + (W_2 - 4W_1 + 4W_0). \tag{1.2}$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - 5x^2 + 8x - 4 = (x-2)^2 (x-1) = 0,$$

where $\alpha = \beta = 2$, $\gamma = 1$.

Now, we define four specific cases of the sequence $\{W_n\}$.

(1) The Woodall numbers $\{R_n\}$, sometimes called Riesel numbers, and also called Cullen numbers of the second kind, are numbers of the form

$$R_n = n \times 2^n - 1.$$

The first few Woodall numbers are:

$$1, 7, 23, 63, 159, 383, 895, 2047, 4607, 10239, 22527, 49151, 106495, 229375, 491519, 1048575, \ldots$$

(sequence A003261 in the OEIS [24]). Woodall numbers were first studied by Allan J. C. Cunningham and H. J. Woodall in [8] in 1917, inspired by James Cullen's earlier study of the similarly-defined Cullen numbers.

(2) The Cullen numbers $\{C_n\}$ are numbers of the form

$$C_n = n \times 2^n + 1.$$

The first few Cullen numbers are:

$$1, 3, 9, 25, 65, 161, 385, 897, 2049, 4609, 10241, 22529, 49153, 106497, 229377, 491521, ...$$

(sequence A002064 in the OEIS). Woodall and Cullen sequences have been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [3,4,8,13,12,15,18,19,20,21,22] and references therein. Note that $\{R_n\}$ and $\{C_n\}$ hold the following relations:

$$R_n = 4R_{n-1} - 4R_{n-2} - 1,$$
$$C_n = 4C_{n-1} - 4C_{n-2} + 1.$$

Note also that the sequences $\{R_n\}$ and $\{C_n\}$ satisfy the following third order linear recurrences:

$$R_n = 5R_{n-1} - 8R_{n-2} + 4R_{n-3}, \qquad R_0 = -1, R_1 = 1, R_2 = 7, \tag{1.3}$$

$$C_n = 5C_{n-1} - 8C_{n-2} + 4C_{n-3}, \qquad C_0 = 1, C_1 = 3, C_2 = 9. \tag{1.4}$$

(3) The modified Woodall numbers $\{G_n\}$ are numbers of the form

$$G_n = (n-1)\,2^n + 1 \text{ (using initial conditions in (1.2).}$$

The modified Woodall sequence $\{G_n\}_{n\geq 0}$ is defined, respectively, by the third order recurrence relation:

$$G_n = 5G_{n-1} - 8G_{n-2} + 4G_{n-3}, \qquad G_0 = 0, G_1 = 1, G_2 = 5, \tag{1.5}$$

(4) The modified Cullen numbers $\{H_n\}$ are numbers of the form

$$H_n = 2^{n+1} + 1 \text{ (using initial conditions in (1.2).}$$

The modified Cullen sequence $\{H_n\}_{n\geq 0}$ is defined, respectively, by the third order recurrence relation:

$$H_n = 5H_{n-1} - 8H_{n-2} + 4H_{n-3}, \qquad H_0 = 3, H_1 = 5, H_2 = 9, \tag{1.6}$$

Then, the sequences $\{G_n\}_{n\geq 0}, \{H_n\}_{n\geq 0}, \{R_n\}$ and $\{C_n\}$ can be extended to negative subscripts by defining,

$$\begin{aligned}
G_{-n} &= 2G_{-(n-1)} - \frac{5}{4}G_{-(n-2)} + \frac{1}{4}G_{-(n-3)}, \\
H_{-n} &= 2H_{-(n-1)} - \frac{5}{4}H_{-(n-2)} + \frac{1}{4}H_{-(n-3)}, \\
R_{-n} &= 2R_{-(n-1)} - \frac{5}{4}R_{-(n-2)} + \frac{1}{4}R_{-(n-3)}, \\
C_{-n} &= 2C_{-(n-1)} - \frac{5}{4}C_{-(n-2)} + \frac{1}{4}C_{-(n-3)},
\end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.3), (1.4), (1.5) and (1.6) hold for all integer n.

Now, we recall the generating function and the Cassini identity for generalized Woodall numbers.

The generating function for generalized Woodall numbers is:

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 5W_0)x + (W_2 - 5W_1 + 8W_0)x^2}{1 - 5x + 8x^2 - 4x^3}. \tag{1.7}$$

The Cassini identity for generalized Woodall numbers is:

$$\begin{aligned}
W_{n+1}W_{n-1} - W_n^2 &= \frac{1}{4}2^n(A + B2^n + Cn). \\
A &= 4W_1^2 + W_2^2 - 4W_0W_1 + 4W_0W_2 - 5W_1W_2. \\
B &= -4W_0^2 - 9W_1^2 - W_2^2 + 12W_0W_1 - 4W_0W_2 + 6W_1W_2. \\
C &= 8W_0^2 + 12W_1^2 + W_2^2 - 20W_0W_1 + 6W_0W_2 - 7W_1W_2.
\end{aligned}$$

For further information about generalized Woodall numbers, see [29]. For an application of generalized Woodall numbers to Gaussian number, see [10].

Now, we should state about hypercomplex number systems. The hypercomplex number systems were studied by Kantor and Solodovnikov in 1989, [17]. These number systems are extensions of real numbers. Some of the commutative ones of these number systems; complex numbers, hyperbolic (double, split-complex) numbers [25] and dual numbers [11] are given below in order.

$$\begin{aligned} \mathbb{C} &= \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\}, \\ \mathbb{H} &= \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\}, \\ \mathbb{D} &= \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}. \end{aligned}$$

Some non-commutative examples of hypercomplex number systems are quaternions, [14],

$$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\},$$

octonions [2] and sedenions [26]. The algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) are real algebras obtained from the real numbers $\mathbb{R}$ by a doubling procedure called the Cayley-Dickson Process. This doubling process can be extended beyond the sedenions to form what are known as the 2^n-ions (see for example [5], [16], [23]).

Quaternions were invented by Irish mathematician W. R. Hamilton (1805-1865) [14] as an extension to the complex numbers. Hyperbolic numbers with complex coefficients are introduced by J. Cockle in 1848, [7]. H. H. Cheng and S. Thompson [6] introduced dual numbers with complex coefficients and called complex dual numbers. Akar, Yüce and Şahin [1] introduced dual hyperbolic numbers.

A dual hyperbolic number is a hyper-complex number and is defined by

$$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3$$

where a_0, a_1, a_2 and a_3 are real numbers.

The set of all dual hyperbolic numbers are denoted by

$$\mathbb{H}_{\mathbb{D}} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3 : a_0, a_1, a_2, a_3 \in \mathbb{R},\ j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The base elements $\{1, j, \varepsilon, \varepsilon j\}$ of dual hyperbolic numbers satisfy the following properties (commutative multiplications):

$$\begin{aligned} 1.\varepsilon &= \varepsilon, 1.j = j,\ \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0,\ j^2 = j.j = 1 \\ \varepsilon.j &= j.\varepsilon,\ \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0,\ j(\varepsilon j) = (\varepsilon j)j = \varepsilon \end{aligned}$$

where ε denotes the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), j denotes the hyperbolic unit ($j^2 = 1$), and εj denotes the dual hyperbolic unit ($(j\varepsilon)^2 = 0$).

Let m and n two dual numbers as $m = a_0 + ja_1 + \varepsilon a_2 + j\varepsilon a_3$ and $n = b_0 + jb_1 + \varepsilon b_2 + j\varepsilon b_3$; The addition and substraction of two dual numbers as m and n is

$$m \mp n = a_0 \mp b_0 + j(a_1 \mp b_1) + \varepsilon(a_2 \mp b_2) + j\varepsilon(a_3 \mp b_3),$$

then, the multiplication of two dual numbers as m and n is

$$mn = a_0b_0 + a_1b_1 + j(a_0b_1 + a_1b_0) + \varepsilon(a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1) + j\varepsilon(a_0b_3 + a_1b_2 + a_2b_1 + b_0a_3)$$

The dual hyperbolic numbers form a commutative ring, real vector space and an algebra. But $\mathbb{H}_{\mathbb{D}}$ is not field because every dual hyperbolic numbers doesn't have an inverse. For more information on the dual hyperbolic numbers, see [1].

Next, we provide details about dual hyperbolic and some information related to dual hyperbolic sequences from the literature.

- Akar, Yüce and Şahin [1] presented the dual hyperbolic numbers.
- Cheng and Thompson [6] introduced dual numbers with complex coefficients.
- Cockle [7] studied the Hyperbolic numbers with complex coefficients.
- Cihan, Azak, Güngör, Tosun, [9] studied dual hyperbolic Fibonacci and Lucas numbers given by,

$$\begin{aligned} DHF_n &= F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3}, \\ DHL_n &= L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3} \end{aligned}$$

  where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$, $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$.
- Soykan, Taşdemir, Okumuş, [31] studied on dual hyperbolic numbers with generalized Jacobsthal numbers components given by,

$$\begin{aligned} \widehat{J}_n &= Jn + jJ_{n+1} + \varepsilon J_{n+2} + 2 + j\varepsilon J_{n+3}, \\ \widehat{K}_n &= K_n + jK_{n+1} + \varepsilon K_{n+2} + j\varepsilon K_{n+3} \end{aligned}$$

  where Jacobsthal and Jacobsthal-Lucas numbers, respectively, given by $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = 0$, $J_1 = 1$, $K_n = K_{n-1} + 2K_{n-2}$, $K_0 = 2$, $K_1 = 1$.
- Soykan, Gümüş, Göcen [28] presented dual hyperbolic generalized Pell numbers given by

$$\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3}$$

  where generalized Pell numbers are given by $V_n = 2V_{n-1} + V_{n-2}$, $V_0 = a$, $V_1 = b$ $(n \geq 2)$ with the initial values V_0, V_1 not all being zero.

In this paper, we define the dual hyperbolic generalized Woodall numbers in the next section and give some properties of them.

2. Dual Hyperbolic Generalized Woodall Numbers

In this section, we define dual hyperbolic generalized Woodall numbers and present generating functions and Binet's formulas for them.

We now define dual hyperbolic generalized Woodall numbers over $\mathbb{H}_{\mathbb{D}}$. The nth dual hyperbolic generalized Woodall number is

$$\widehat{W}_n = W_n + jW_{n+1} + \varepsilon W_{n+2} + j\varepsilon W_{n+3}. \tag{2.1}$$

with the initial values $\widehat{W}_0$, $\widehat{W}_1$, $\widehat{W}_2$. (2.1) can be written to negative subscripts by defining,

$$\widehat{W}_{-n} = W_{-n} + jW_{-n+1} + \varepsilon W_{-n+2} + j\varepsilon W_{-n+3}.$$

so identity (2.1) holds for all integers n.

For four special cases of the nth dual hyperbolic generalized Woodall numbers are given as

$$\begin{aligned}
\widehat{G}_n &= G_n + jG_{n+1} + \varepsilon G_{n+2} + j\varepsilon G_{n+3}, \\
\widehat{H}_n &= H_n + jH_{n+1} + \varepsilon H_{n+2} + j\varepsilon H_{n+3}, \\
\widehat{R}_n &= R_n + jR_{n+1} + \varepsilon R_{n+2} + j\varepsilon R_{n+3}, \\
\widehat{C}_n &= C_n + jC_{n+1} + \varepsilon C_{n+2} + j\varepsilon C_{n+3}.
\end{aligned}$$

It is clear that

$$\widehat{W}_n = 5\widehat{W}_{n-1} - 8\widehat{W}_{n-2} + 4\widehat{W}_{n-3}. \tag{2.2}$$

The sequence $\{\widehat{W}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\widehat{W}_{-n} = -2\widehat{W}_{-(n-1)} - \frac{5}{4}\widehat{W}_{-(n-2)} + \frac{1}{4}\widehat{W}_{-(n-3)}.$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.2) holds for all integer n.

The initial several dual hyperbolic generalized Woodall numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few dual hyperbolic generalized Woodall numbers.

n	$\widehat{W}_n$	$\widehat{W}_{-n}$
0	$\widehat{W}_0$	$\widehat{W}_0$
1	$\widehat{W}_1$	$\frac{1}{4}(8\widehat{W}_0 - 5\widehat{W}_1 + \widehat{W}_2)$
2	$\widehat{W}_2$	$\frac{1}{4}(11\widehat{W}_0 - 9\widehat{W}_1 + 2\widehat{W}_2)$
3	$4\widehat{W}_0 - 8\widehat{W}_1 + 5\widehat{W}_2$	$\frac{1}{16}(52\widehat{W}_0 - 47\widehat{W}_1 + 11\widehat{W}_2)$
4	$20\widehat{W}_0 - 36\widehat{W}_1 + 17\widehat{W}_2$	$\frac{1}{16}(57\widehat{W}_0 - 54\widehat{W}_1 + 13\widehat{W}_2)$
5	$68\widehat{W}_0 - 116\widehat{W}_1 + 49\widehat{W}_2$	$\frac{1}{64}(240\widehat{W}_0 - 233\widehat{W}_1 + 57\widehat{W}_2)$

Note that

$$
\begin{aligned}
\widehat{W}_0 &= W_0 + jW_1 + \varepsilon W_2 + j\varepsilon W_3 = W_0 + jW_1 + \varepsilon W_2 + j\varepsilon(4W_0 - 8W_1 + 5W_2), \\
\widehat{W}_1 &= W_1 + jW_2 + \varepsilon W_3 + j\varepsilon W_4 = W_1 + jW_2 + \varepsilon(4W_0 - 8W_1 + 5W_2) \\
&\quad + j\varepsilon(20W_0 - 36W_1 + 17W_2), \\
\widehat{W}_2 &= W_2 + jW_3 + \varepsilon W_4 + j\varepsilon W_5 = W_2 + j(4W_0 - 8W_1 + 5W_2) \\
&\quad + \varepsilon(20W_0 - 36W_1 + 17W_2) + j\varepsilon(68W_0 - 116W_1 + 49W_2).
\end{aligned}
$$

For four special cases of dual hyperbolic generalized Woodall numbers, we obtain the following initial conditions.

$$
\begin{aligned}
\widehat{G}_0 &= G_0 + jG_1 + \varepsilon G_2 + j\varepsilon G_3 = j + 5\varepsilon + 17j\varepsilon, \\
\widehat{G}_1 &= G_1 + jG_2 + \varepsilon G_3 + j\varepsilon G_4 = 1 + 5j + 17\varepsilon + 49j\varepsilon, \\
\widehat{G}_2 &= G_2 + jG_3 + \varepsilon G_4 + j\varepsilon G_5 = 5 + 17j + 49\varepsilon + 129j\varepsilon.
\end{aligned}
$$

$$
\begin{aligned}
\widehat{H}_0 &= H_0 + jH_1 + \varepsilon H_2 + j\varepsilon H_3 = 3 + 5j + 9\varepsilon + 17j\varepsilon, \\
\widehat{H}_1 &= H_1 + jH_2 + \varepsilon H_3 + j\varepsilon H_4 = 5 + 9j + 17\varepsilon + 33j\varepsilon, \\
\widehat{H}_2 &= H_2 + jH_3 + \varepsilon H_4 + j\varepsilon H_5 = 9 + 17j + 33\varepsilon + 65j\varepsilon.
\end{aligned}
$$

$$
\begin{aligned}
\widehat{R}_0 &= R_0 + jR_1 + \varepsilon R_2 + j\varepsilon R_3 = -1 + j + 7\varepsilon + 23j\varepsilon, \\
\widehat{R}_1 &= R_1 + jR_2 + \varepsilon R_3 + j\varepsilon R_4 = 1 + 7j + 23\varepsilon + 63j\varepsilon, \\
\widehat{R}_2 &= R_2 + jR_3 + \varepsilon R_4 + j\varepsilon R_5 = 7 + 23j + 63\varepsilon + 159j\varepsilon.
\end{aligned}
$$

$$
\begin{aligned}
\widehat{C}_0 &= C_0 + jC_1 + \varepsilon C_2 + j\varepsilon C_3 = 1 + 3j + 9\varepsilon + 25j\varepsilon, \\
\widehat{C}_1 &= C_1 + jC_2 + \varepsilon C_3 + j\varepsilon C_4 = 3 + 9j + 25\varepsilon + 65j\varepsilon, \\
\widehat{C}_2 &= C_2 + jC_3 + \varepsilon C_4 + j\varepsilon C_5 = 9 + 25j + 65\varepsilon + 161j\varepsilon.
\end{aligned}
$$

A few $\widehat{G}_n$, $\widehat{H}_n$, $\widehat{R}_n$ and $\widehat{C}_n$ with positive subscript and negative subscript are given in the following Table 2, Table 3, Table 4 and Table 5.

Table 2. Dual hyperbolic modified Woodall numbers

n	$\widehat{G}_n$	$\widehat{G}_{-n}$
0	$j+5\varepsilon+17j\varepsilon$	$j+5\varepsilon+17j\varepsilon$
1	$1+5j+17\varepsilon+49j\varepsilon$	$\varepsilon+5j\varepsilon$
2	$5+17j+49\varepsilon+129j\varepsilon$	$\frac{1}{4}+j\varepsilon$
3	$17+49j+129\varepsilon+321j\varepsilon$	$\frac{1}{2}+\frac{1}{4}j$
4	$49+129j+321\varepsilon+769j\varepsilon$	$\frac{11}{16}+\frac{1}{2}j+\frac{1}{4}\varepsilon$
5	$129+321j+769\varepsilon+1793j\varepsilon$	$\frac{13}{16}+\frac{11}{16}j+\frac{1}{2}\varepsilon+\frac{1}{4}j\varepsilon$

Table 3. Dual hyperbolic modified Cullen numbers

n	$\widehat{H}_n$	$\widehat{H}_{-n}$
0	$3+5j+9\varepsilon+17j\varepsilon$	$3+5j+9\varepsilon+17j\varepsilon$
1	$5+9j+17\varepsilon+33j\varepsilon$	$2+3j+5\varepsilon+9j\varepsilon$
2	$9+17j+33\varepsilon+65j\varepsilon$	$\frac{3}{2}+2\varepsilon+3j+5j\varepsilon$
3	$17+33j+65\varepsilon+129j\varepsilon$	$\frac{5}{4}+\frac{3}{2}j+2\varepsilon+3j\varepsilon$
4	$33+65j+129\varepsilon+257j\varepsilon$	$\frac{9}{8}+\frac{5}{4}\varepsilon+\frac{3}{2}j+2j\varepsilon$
5	$65+129j+257\varepsilon+513j\varepsilon$	$\frac{17}{16}+\frac{9}{8}j+\frac{5}{4}\varepsilon+\frac{3}{2}j\varepsilon$

Table 4. Dual hyperbolic Woodall numbers

n	$\widehat{R}_n$	$\widehat{R}_{-n}$
0	$-1+j+7\varepsilon+23j\varepsilon$	$-1+j+7\varepsilon+23j\varepsilon$
1	$1+7j+23\varepsilon+63j\varepsilon$	$-\frac{3}{2}-j+\varepsilon+7j\varepsilon$
2	$7+23j+63\varepsilon+159j\varepsilon$	$-\frac{3}{2}-\frac{3}{2}j-\varepsilon+j\varepsilon$
3	$23+63j+159\varepsilon+383j\varepsilon$	$-\frac{11}{8}-\frac{3}{2}j-\frac{3}{2}\varepsilon-j\varepsilon$
4	$63+159j+383\varepsilon+895j\varepsilon$	$-\frac{5}{4}-\frac{11}{8}j-\frac{3}{2}\varepsilon-\frac{3}{2}j\varepsilon$
5	$159+383j+895\varepsilon+2047j\varepsilon$	$-\frac{37}{32}-\frac{5}{4}j-\frac{11}{8}\varepsilon-\frac{3}{2}j\varepsilon$

Table 5. Dual hyperbolic Cullen numbers

n	$\widehat{C}_n$	$\widehat{C}_{-n}$
0	$1+3j+9\varepsilon+25j\varepsilon$	$1+3j+9\varepsilon+25j\varepsilon$
1	$3+9j+25\varepsilon+65j\varepsilon$	$\frac{1}{2}+j+3\varepsilon+9j\varepsilon$
2	$9+25j+65\varepsilon+161j\varepsilon$	$\frac{1}{2}+\frac{1}{2}\varepsilon+j+3j\varepsilon$
3	$25+65j+161\varepsilon+385j\varepsilon$	$\frac{5}{8}+\frac{1}{2}j+\frac{1}{2}\varepsilon+j\varepsilon$
4	$65+161j+385\varepsilon+897j\varepsilon$	$\frac{3}{4}+\frac{5}{8}\varepsilon+\frac{1}{2}j+\frac{1}{2}j\varepsilon$
5	$161+385j+897\varepsilon+2049j\varepsilon$	$\frac{27}{32}+\frac{3}{4}j+\frac{5}{8}\varepsilon+\frac{1}{2}j\varepsilon$

Now, we will state Binet's formula for the dual hyperbolic generalized Woodall numbers and in the rest of the paper, we fix the following notations:

$$
\begin{aligned}
\widehat{\alpha} &= 1+2j+4\varepsilon+8j\varepsilon, \\
\widehat{\beta} &= 2j+8\varepsilon+24j\varepsilon, \\
\widehat{\gamma} &= 1+j+\varepsilon+j\varepsilon.
\end{aligned}
$$

Note that we have the following identities:

$$\begin{aligned}
\widehat{\alpha}^2 &= 5+4j+40\varepsilon++32j\varepsilon, \\
\widehat{\beta}^2 &= 4+96\varepsilon+32j\varepsilon, \\
\widehat{\gamma}^2 &= 2+2j+4\varepsilon+4j\varepsilon, \\
\widehat{\alpha}\widehat{\beta} &= (1+2j+4\varepsilon+8j\varepsilon)(2j+8\varepsilon+24j\varepsilon), \\
\widehat{\alpha}\widehat{\gamma} &= 3+3j+15\varepsilon+15j\varepsilon, \\
\widehat{\beta}\widehat{\gamma} &= 2+2j+34\varepsilon+34j\varepsilon, \\
\widehat{\alpha}\widehat{\beta}\widehat{\gamma} &= 6+6j+126\varepsilon+126j\varepsilon.
\end{aligned}$$

Now, we present Binet's formula in the following.

2.1. Binet's Formula of Dual Hyperbolic Generalized Woodall Number.

THEOREM 2. *(Binet's Formula) For any integer n, the nth dual hyperbolic generalized Woodall number is*

$$\widehat{W}_n = (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2 n\widehat{\alpha})2^n + A_3\widehat{\gamma}. \tag{2.3}$$

Proof. Using Binet's formula

$$W_n = (A_1 + A_2 n)2^n + A_3$$

of the generalized Woodall numbers, we obtain

$$\begin{aligned}
\widehat{W}_n &= W_n + jW_{n+1} + \varepsilon W_{n+2} + j\varepsilon W_{n+3} \\
&= (A_1 + A_2 n)2^n + A_3 + j((A_1 + A_2(n+1))2^{n+1} + A_3) + \varepsilon((A_1 + A_2(n+2))2^{n+2} + A_3) \\
&\quad + j\varepsilon((A_1 + A_2(n+3))2^{n+3} + A_3) \\
&= A_1 2^n + A_2 n 2^n + A_3 \\
&\quad + jA_1 2^{n+1} + jA_2 n 2^{n+1} + jA_2 2^{n+1} + jA_3 \\
&\quad + \varepsilon A_1 2^{n+2} + \varepsilon A_2 n 2^{n+2} + 2\varepsilon A_2 2^{n+2} + \varepsilon A_3 \\
&\quad + j\varepsilon A_1 2^{n+3} + j\varepsilon A_2 n 2^{n+3} + 3j\varepsilon A_2 2^{n+3} + j\varepsilon A_3 \\
&= A_1 2^n(1+2j+4\varepsilon+8j\varepsilon) + A_2 n 2^n(1+2j+4\varepsilon+8j\varepsilon) + A_2 2^n(2j+8\varepsilon+24j\varepsilon) + A_3(1+j+\varepsilon+j\varepsilon) \\
&= A_1 2^n\widehat{\alpha} + A_2 n 2^n\widehat{\alpha} + A_2 2^n\widehat{\beta} + A_3\widehat{\gamma} \\
&= (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2 n\widehat{\alpha})2^n + A_3\widehat{\gamma}.
\end{aligned}$$

This proves (2.3). □

As special cases, for any integer n, the Binet's Formula of nth dual hyperbolic modified Woodall number, dual hyperbolic modified Cullen number, dual hyperbolic Woodall number and dual hyperbolic Cullen number are

- $\widehat{G}_n = (-\widehat{\alpha} + \widehat{\beta} + n\widehat{\alpha})2^n + \widehat{\gamma}$,
  $\widehat{G}_n = 1 + (n-1)2^n + j(1 + n2^{n+1}) + \varepsilon(1 + 2^{n+2} + n2^{n+2}) + j\varepsilon(1 + 2^{n+4} + n2^{n+3})$.
- $\widehat{H}_n = (2\widehat{\alpha})2^n + \widehat{\gamma}$,
  $\widehat{H}_n = 1 + 2^{n+1} + j(1 + 2^{n+2}) + \varepsilon(1 + 2^{n+3}) + j\varepsilon(1 + 2^{n+4})$.
- $\widehat{R}_n = (\widehat{\beta} + n\widehat{\alpha})2^n - \widehat{\gamma}$,
  $\widehat{R}_n = -1 + n2^n + j(-1 + 2^{n+1} + n2^{n+1}) + \varepsilon(-1 + 2^{n+3} + n2^{n+2}) + j\varepsilon(-1 + 3 \times 2^{n+3} + n2^{n+3})$.
- $\widehat{C}_n = (\widehat{\beta} + n\widehat{\alpha})2^n + \widehat{\gamma}$,
  $\widehat{C}_n = 1 + n2^n + j(1 + 2^{n+1} + n2^{n+1}) + \varepsilon(1 + 2^{n+3} + n2^{n+2}) + j\varepsilon(1 + 3 \times 2^{n+3} + n2^{n+3})$.

Next, we present generating function for dual hyperbolic generalized Woodall numbers.

2.2. Generating Functions of Dual Hyperbolic Generalized Woodall Numbers.

THEOREM 3. *The generating function for the dual hyperbolic generalized Woodall numbers is*

$$\sum_{n=0}^{\infty} \widehat{W}_n x^n = \frac{\widehat{W}_0 + (\widehat{W}_1 - 5\widehat{W}_0)x + (\widehat{W}_2 - 5\widehat{W}_1 + 8\widehat{W}_0)x^2}{1 - 5x + 8x^2 - 4x^3}. \tag{2.4}$$

Proof. Let

$$g(x) = \sum_{n=0}^{\infty} \widehat{W}_n x^n$$

be generating function of the dual hyperbolic generalized Woodall numbers. Then, using the definition of the dual hyperbolic generalized Woodall numbers, and substracting $xg(x)$, $x^2g(x)$ and $x^3g(x)$ from $g(x)$, we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
(1 - 5x + 8x^2 - 4x^3)g(x) &= \sum_{n=0}^{\infty} \widehat{W}_n x^n - 5x\sum_{n=0}^{\infty} \widehat{W}_n x^n + 8x^2\sum_{n=0}^{\infty} \widehat{W}_n x^n - 4x^3\sum_{n=0}^{\infty} \widehat{W}_n x^n \\
&= \sum_{n=0}^{\infty} \widehat{W}_n x^n - 5\sum_{n=0}^{\infty} \widehat{W}_n x^{n+1} + 8\sum_{n=0}^{\infty} \widehat{W}_n x^{n+2} - 4\sum_{n=0}^{\infty} \widehat{W}_n x^{n+3} \\
&= \sum_{n=0}^{\infty} \widehat{W}_n x^n - 5\sum_{n=1}^{\infty} \widehat{W}_{n-1} x^n + 8\sum_{n=2}^{\infty} \widehat{W}_{n-2} x^n - 4\sum_{n=3}^{\infty} \widehat{W}_{n-3} x^n \\
&= (\widehat{W}_0 + \widehat{W}_1 x + \widehat{W}_2 x^2) - 5(\widehat{W}_0 x + \widehat{W}_1 x^2) + 8\widehat{W}_0 x^2 \\
&\quad + \sum_{n=3}^{\infty} (\widehat{W}_n - 5\widehat{W}_{n-1} + 8\widehat{W}_{n-2} - 4\widehat{W}_{n-3})x^n \\
&= \widehat{W}_0 + (\widehat{W}_1 - 5\widehat{W}_0)x + (\widehat{W}_2 - 5\widehat{W}_1 + 8\widehat{W}_0)x^2.
\end{aligned}$$

Note that we used the recurrence relation $\widehat{W}_n = 5\widehat{W}_{n-1} - 8\widehat{W}_{n-2} + 4\widehat{W}_{n-3}$. Rearranging above equation, we get

$$g(x) = \frac{\widehat{W}_0 + (\widehat{W}_1 - 5\widehat{W}_0)x + (\widehat{W}_2 - 5\widehat{W}_1 + 8\widehat{W}_0)x^2}{1 - 5x + 8x^2 - 4x^3}.$$

The proof is finished. □

As special cases, the generating functions for the dual hyperbolic modified Woodall, dual hyperbolic modified Cullen, dual hyperbolic Woodall and dual hyperbolic Cullen numbers are

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{G}_n x^n &= \frac{j+5\varepsilon+17j\varepsilon+(1-36j\varepsilon-8\varepsilon)\,x+(4\varepsilon+20j\varepsilon)\,x^2}{1-5x+8x^2-4x^3},\\ \sum_{n=0}^{\infty}\widehat{H}_n x^n &= \frac{5j+9\varepsilon+17j\varepsilon+3+(-16j-28\varepsilon-52j\varepsilon-10)\,x+(12j+20\varepsilon+36j\varepsilon+8)\,x^2}{1-5x+8x^2-4x^3},\\ \sum_{n=0}^{\infty}\widehat{R}_n x^n &= \frac{-1+j+7\varepsilon+23j\varepsilon+(2j-12\varepsilon-52j\varepsilon+6)x+(4\varepsilon-4j+28j\varepsilon-6)x^2}{1-5x+8x^2-4x^3}\end{aligned}$$

and

$$\sum_{n=0}^{\infty}\widehat{C}_n x^n = \frac{3j+9\varepsilon+25j\varepsilon+1+(-6j-20\varepsilon-60j\varepsilon-2)\,x+(4j+12\varepsilon+36j\varepsilon+2)\,x^2}{1-5x+8x^2-4x^3}$$

respectively.

Now, we give obtaining Binet's formula from generating function.

2.3. Obtaining Binet's Formula From Generating Function. We obtain Binet's formula of dual hyperbolic generalized Woodall number $\{\widehat{W}_n\}$ by the use of generating function for $\widehat{W}_n$.

THEOREM 4. *(Binet's formula of dual hyperbolic generalized Woodall numbers)*

$$\widehat{W}_n = (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2 n\widehat{\alpha})2^n + A_3\widehat{\gamma}. \tag{2.5}$$

Proof. Let

$$\sum_{n=0}^{\infty}\widehat{W}_n x^n = \frac{\widehat{W}_0+(\widehat{W}_1-5\widehat{W}_0)x+(\widehat{W}_2-5\widehat{W}_1+8\widehat{W}_0)x^2}{1-5x+8x^2-4x^3}.$$

Then we write

$$\frac{\widehat{W}_0+(\widehat{W}_1-5\widehat{W}_0)x+(\widehat{W}_2-5\widehat{W}_1+8\widehat{W}_0)x^2}{(1-x)\,(1-2x)^2} = \frac{d_1}{(1-x)}+\frac{d_2}{(1-2x)}+\frac{d_3}{(1-2x)^2}. \tag{2.6}$$

So

$$\widehat{W}_0+(\widehat{W}_1-5\widehat{W}_0)x+(\widehat{W}_2-5\widehat{W}_1+8\widehat{W}_0)x^2 = (d_1+d_2+d_3)+(-4d_1-3d_2-d_3)x+(4d_1+2d_2)x^2.$$

We get

$$\begin{aligned}\widehat{W}_0 &= d_1+d_2+d_3,\\ \widehat{W}_1-5\widehat{W}_0 &= -4d_1-3d_2-d_3,\\ \widehat{W}_2-5\widehat{W}_1+8\widehat{W}_0 &= 4d_1+2d_2.\end{aligned}$$

If we solve these simultanious equation,

$$\begin{aligned} d_1 &= 4\widehat{W}_0 - 4\widehat{W}_1 + \widehat{W}_2, \\ d_2 &= -4\widehat{W}_0 + \frac{11}{2}\widehat{W}_1 - \frac{3}{2}\widehat{W}_2, \\ d_3 &= \widehat{W}_0 - \frac{3}{2}\widehat{W}_1 + \frac{1}{2}\widehat{W}_2. \end{aligned}$$

Thus (2.6) can be written as

$$\begin{aligned} \sum_{n=0}^{\infty} \widehat{W}_n x^n &= d_1 \frac{1}{(1-x)} + d_2 \frac{1}{(1-2x)} + d_3 \frac{1}{(2x-1)^2}, \\ &= d_1 \sum_{n=0}^{\infty} x^n + d_2 \sum_{n=0}^{\infty} 2^n x^n + d_3 \sum_{n=0}^{\infty} 2^n (n+1) x^n, \\ &= \sum_{n=0}^{\infty} (d_1 + d_2 2^n + d_3 2^n (n+1)) x^n, \\ &= \sum_{n=0}^{\infty} (4\widehat{W}_0 - 4\widehat{W}_1 + \widehat{W}_2 + (-4\widehat{W}_0 + \frac{11}{2}\widehat{W}_1 - \frac{3}{2}\widehat{W}_2)2^n + (\widehat{W}_0 - \frac{3}{2}\widehat{W}_1 + \frac{1}{2}\widehat{W}_2)2^n (n+1)) x^n, \\ &= \sum_{n=0}^{\infty} (4\widehat{W}_0 - 4\widehat{W}_1 + \widehat{W}_2 + (-4\widehat{W}_0 + \frac{11}{2}\widehat{W}_1 - \frac{3}{2}\widehat{W}_2)2^n + (\widehat{W}_0 - \frac{3}{2}\widehat{W}_1 + \frac{1}{2}\widehat{W}_2)2^n \\ &\quad + (\widehat{W}_0 - \frac{3}{2}\widehat{W}_1 + \frac{1}{2}\widehat{W}_2)2^n n) x^n, \\ &= \sum_{n=0}^{\infty} (4\widehat{W}_0 - 4\widehat{W}_1 + \widehat{W}_2 + (\widehat{W}_0 - \frac{3}{2}\widehat{W}_1 + \frac{1}{2}\widehat{W}_2) n 2^n + (-3\widehat{W}_0 + 4\widehat{W}_1 - \widehat{W}_2)2^n) x^n, \\ &= \sum_{n=0}^{\infty} ((-3\widehat{W}_0 + 4\widehat{W}_1 - \widehat{W}_2) + (\widehat{W}_0 - \frac{3}{2}\widehat{W}_1 + \frac{1}{2}\widehat{W}_2) n) 2^n + 4\widehat{W}_0 - 4\widehat{W}_1 + \widehat{W}_2) x^n. \end{aligned}$$

This gives

$$\widehat{W}_n = (\widehat{A_1} + \widehat{A_2} n) 2^n + \widehat{A_3}$$

where

$$\begin{aligned} \widehat{A_1} &= -3\widehat{W}_0 + 4\widehat{W}_1 - \widehat{W}_2, \\ \widehat{A_2} &= \widehat{W}_0 - \frac{3}{2}\widehat{W}_1 + \frac{1}{2}\widehat{W}_2, \\ \widehat{A_3} &= 4\widehat{W}_0 - 4\widehat{W}_1 + \widehat{W}_2. \end{aligned}$$

Note that the following equalities are true:

$$\begin{aligned} A_1\widehat{\alpha} + A_2\widehat{\beta} &= (-W_2 + 4W_1 - 3W_0)(1 + 2j + 4\varepsilon + 8j\varepsilon) + (\frac{W_2 - 3W_1 + 2W_0}{2})(2j + 8\varepsilon + 24j\varepsilon) \\ &= -3W_0 + 4W_1 - W_2 + j(-4W_0 + 5W_1 - W_2) + \varepsilon(-4W_0 + 4W_1) + j\varepsilon(-4W_1 + 4W_2). \end{aligned}$$

$$\begin{aligned} A_2\widehat{\alpha} &= \frac{W_2 - 3W_1 + 2W_0}{2}(1 + 2j + 4\varepsilon + 8j\varepsilon) \\ &= W_0 - \frac{3}{2}W_1 + \frac{1}{2}W_2 + j(2W_0 - 3W_1 + W_2) + \varepsilon(4W_0 - 6W_1 + 2W_2) + j\varepsilon(8W_0 - 12W_1 + 4W_2). \end{aligned}$$

$$A_3\widehat{\gamma} = W_2 - 4W_1 + 4W_0 + j(W_2 - 4W_1 + 4W_0) + \varepsilon(W_2 - 4W_1 + 4W_0) + j\varepsilon(W_2 - 4W_1 + 4W_0).$$

Therefore, we can write the following equalition:

$$\widehat{W}_n = (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2 n\widehat{\alpha})2^n + A_3\widehat{\gamma}.$$

The proof is finished. □

Next, using Theorem 4, we present the Binet's formulas of dual hyperbolic modified Woodall, dual hyperbolic modified Cullen, dual hyperbolic Woodall and dual hyperbolic Cullen numbers.

3. Some Identities For Dual Hyperbolic Generalized Woodall Numbers

We now present a few special identities for the dual hyperbolic generalized Woodall sequence $\{\widehat{W}_n\}$. The following theorem presents the Simpson's identity for the dual hyperbolic generalized Woodall numbers.

THEOREM 5. *(Simpson's formula for dual hyperbolic generalized Woodall sequence)For all integers n we have*

$$\begin{vmatrix} \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n \\ \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} \\ \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} \end{vmatrix} = 4^n \begin{vmatrix} \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \end{vmatrix}.$$

Proof. For the proof we use mathematical induction. We suppose that $n \geq 0$. For $n = 0$ identity is true. Now we obtain is true for $n = k$. Hence we write the following identity

$$\begin{vmatrix} \widehat{W}_{k+2} & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_{k+1} & \widehat{W}_k & \widehat{W}_{k-1} \\ \widehat{W}_k & \widehat{W}_{k-1} & \widehat{W}_{k-2} \end{vmatrix} = 4^k \begin{vmatrix} \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \end{vmatrix}.$$

For $n = k + 1$, we get

$$\begin{aligned} \begin{vmatrix} \widehat{W}_{k+3} & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_{k+2} & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_{k+1} & \widehat{W}_k & \widehat{W}_{k-1} \end{vmatrix} &= \begin{vmatrix} 5\widehat{W}_{k+2} - 8\widehat{W}_{k+1} + 4\widehat{W}_k & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ 5\widehat{W}_{k+1} - 8\widehat{W}_k + 4\widehat{W}_{k-1} & \widehat{W}_{k+1} & \widehat{W}_k \\ 5\widehat{W}_k - 8\widehat{W}_{k-1} + 4\widehat{W}_{k-2} & \widehat{W}_k & \widehat{W}_{k-1} \end{vmatrix} \\ &= 5\begin{vmatrix} \widehat{W}_{k+2} & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_{k+1} & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_k & \widehat{W}_k & \widehat{W}_{k-1} \end{vmatrix} - 8\begin{vmatrix} \widehat{W}_{k+1} & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_k & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_{k-1} & \widehat{W}_k & \widehat{W}_{k-1} \end{vmatrix} \\ &\quad +4\begin{vmatrix} \widehat{W}_k & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_{k-1} & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_{k-2} & \widehat{W}_k & \widehat{W}_{k-1} \end{vmatrix} \\ &= 4\begin{vmatrix} \widehat{W}_{k+2} & \widehat{W}_{k+1} & \widehat{W}_k \\ \widehat{W}_{k+1} & \widehat{W}_k & \widehat{W}_{k-1} \\ \widehat{W}_k & \widehat{W}_{k-1} & \widehat{W}_{k-2} \end{vmatrix} = 4^{k+1}\begin{vmatrix} \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \end{vmatrix}. \end{aligned}$$

Thus, the proof is finished. $n < 0$ can be proved similarly. □

From prewious theorem, we get following corollary.

COROLLARY 6. *(Simpson's formula for dual hyperbolic generalized Woodall sequence's special cases)*

(a): $$\begin{vmatrix} \widehat{G}_{k+2} & \widehat{G}_{k+1} & \widehat{G}_{k} \\ \widehat{G}_{k+1} & \widehat{G}_{k} & \widehat{G}_{k-1} \\ \widehat{G}_{k} & \widehat{G}_{k-1} & \widehat{G}_{k-2} \end{vmatrix} = -4^{n-1}(9+9j+9\varepsilon+153j\varepsilon).$$

(b): $$\begin{vmatrix} \widehat{H}_{k+2} & \widehat{H}_{k+1} & \widehat{H}_{k} \\ \widehat{H}_{k+1} & \widehat{H}_{k} & \widehat{H}_{k-1} \\ \widehat{H}_{k} & \widehat{H}_{k-1} & \widehat{H}_{k-2} \end{vmatrix} = 0.$$

(c): $$\begin{vmatrix} \widehat{R}_{k+2} & \widehat{R}_{k+1} & \widehat{R}_{k} \\ \widehat{R}_{k+1} & \widehat{R}_{k} & \widehat{R}_{k-1} \\ \widehat{R}_{k} & \widehat{R}_{k-1} & \widehat{R}_{k-2} \end{vmatrix} = 4^{n-1}(9+9j+9\varepsilon+153j\varepsilon).$$

(d): $$\begin{vmatrix} \widehat{C}_{k+2} & \widehat{C}_{k+1} & \widehat{C}_{k} \\ \widehat{C}_{k+1} & \widehat{C}_{k} & \widehat{C}_{k-1} \\ \widehat{C}_{k} & \widehat{C}_{k-1} & \widehat{C}_{k-2} \end{vmatrix} = -4^{n-1}(9+9j+9\varepsilon+153j\varepsilon).$$

THEOREM 7. *(Catalan's identity) For all integers n and m, the following identity holds:*
$\widehat{W}_{n+m}\widehat{W}_{n-m} - \widehat{W}_n^2 = 2^{n-m}(-2^{m+n}m^2\widehat{\alpha}^2A_2^2 + A_2A_3(-2^{m+1}\widehat{\beta}\widehat{\gamma} + \widehat{\beta}\widehat{\gamma} + 2^{2m}\widehat{\beta}\widehat{\gamma} - m\widehat{\alpha}\widehat{\gamma} + n\widehat{\alpha}\widehat{\gamma} - 2^{m+1}n\widehat{\alpha}\widehat{\gamma} + 2^{2m}m\widehat{\alpha}\widehat{\gamma} + 2^{2m}n\widehat{\alpha}\widehat{\gamma}) + A_1A_3(\widehat{\alpha}\widehat{\gamma} - 2^{m+1}\widehat{\alpha}\widehat{\gamma} + 2^{2m}\widehat{\alpha}\widehat{\gamma}))$.

Proof. Using the Binet's formula $\widehat{W}_n = (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2n\widehat{\alpha})2^n + A_3\widehat{\gamma}$, we get the required identity. □

As special cases of the above theorem, we give Catalan's identity of dual hyperbolic modified Woodall, dual hyperbolic modified Cullen, dual hyperbolic Woodall and dual hyperbolic Cullen numbers. Firstly, we present Catalan's identity of dual hyperbolic Woodall numbers.

COROLLARY 8. *(Catalan's identity for the dual hyperbolic modified Woodall numbers) For all integers n and m, the following identity holds:*

$$\begin{aligned} \widehat{G}_{n+m}\widehat{G}_{n-m} - \widehat{G}_n^2 &= -2^{n-m}(\widehat{\alpha}\widehat{\gamma} - \widehat{\beta}\widehat{\gamma} + 2^{2m}\widehat{\alpha}\widehat{\gamma} - 2^{2m}\widehat{\beta}\widehat{\gamma} - 2^{m+1}\widehat{\alpha}\widehat{\gamma} + 2^{m+1}\widehat{\beta}\widehat{\gamma} + m\widehat{\alpha}\widehat{\gamma} - n\widehat{\alpha}\widehat{\gamma} \\ &\quad + 2^{m+n}m^2\widehat{\alpha}^2 - 2^{2m}m\widehat{\alpha}\widehat{\gamma} - 2^{2m}n\widehat{\alpha}\widehat{\gamma} + 2^{m+1}n\widehat{\alpha}\widehat{\gamma}). \end{aligned}$$

Proof. Take $W_n = G_n$ in Theorem 7. □

Secondly, we give Catalan's identity of dual hyperbolic modified Cullen numbers.

COROLLARY 9. *(Catalan's identity for the dual hyperbolic modified Cullen numbers) For all integers n and m, the following identity holds:*

$$\widehat{H}_{n+m}\widehat{H}_{n-m} - \widehat{H}_n^2 = 2^{n-m}(2\widehat{\alpha}\widehat{\gamma} + 2\times 2^{2m}\widehat{\alpha}\widehat{\gamma} - 2\times 2^{m+1}\widehat{\alpha}\widehat{\gamma}).$$

Proof. Take $W_n = H_n$ in Theorem 7. □

Thirdly, we give Catalan's identity of dual hyperbolic Woodall numbers.

Corollary 10. *(Catalan's identity for the dual hyperbolic Woodall numbers) For all integers n and m, the following identity holds:*

$$\widehat{R}_{n+m}\widehat{R}_{n-m}-\widehat{R}_n^2 = -2^{n-m}(\widehat{\beta}\widehat{\gamma}+2^{2m}\widehat{\beta}\widehat{\gamma}-2^{m+1}\widehat{\beta}\widehat{\gamma}-m\widehat{\alpha}\widehat{\gamma}+n\widehat{\alpha}\widehat{\gamma}+2^{m+n}m^2\widehat{\alpha}^2+2^{2m}m\widehat{\alpha}\widehat{\gamma}+2^{2m}n\widehat{\alpha}\widehat{\gamma}-2^{m+1}n\widehat{\alpha}\widehat{\gamma}).$$

Proof. Take $W_n = R_n$ in Theorem 7. □

Fourthly, we give Catalan's identity of dual hyperbolic Cullen numbers.

Corollary 11. *(Catalan's identity for the dual hyperbolic Cullen numbers) For all integers n and m, the following identity holds:*

$$\widehat{C}_{n+m}\widehat{C}_{n-m}-\widehat{C}_n^2 = 2^{n-m}(\widehat{\beta}\widehat{\gamma}+2^{2m}\widehat{\beta}\widehat{\gamma}-2^{m+1}\widehat{\beta}\widehat{\gamma}-m\widehat{\alpha}\widehat{\gamma}+n\widehat{\alpha}\widehat{\gamma}-2^{m+n}m^2\widehat{\alpha}^2+2^{2m}m\widehat{\alpha}\widehat{\gamma}+2^{2m}n\widehat{\alpha}\widehat{\gamma}-2^{m+1}n\widehat{\alpha}\widehat{\gamma}).$$

Proof. Take $W_n = C_n$ in Theorem 7. □

Note that for $m = 1$ in Catalan's identity, we get the Cassini's identity for the dual hyperbolic generalized Woodall sequence.

Corollary 12. *(Cassini's identity) For all integers n, the following identity holds:*

$$\widehat{W}_{n+1}\widehat{W}_{n-1} - \widehat{W}_n^2 = 2^{n-1}(A_2A_3(3\widehat{\alpha}\widehat{\gamma} + \widehat{\beta}\widehat{\gamma} + n\widehat{\alpha}\widehat{\gamma}) - 2^{n+1}A_2^2\widehat{\alpha}^2 + A_1A_3\widehat{\alpha}\widehat{\gamma}).$$

As special cases of Cassini's identity, we give Cassini's identity of dual hyperbolic modified Woodall, dual hyperbolic modified Cullen, dual hyperbolic Woodall and dual hyperbolic Cullen numbers. Firstly, we present Cassini's identity of dual hyperbolic modified Woodall numbers.

Corollary 13. *(Cassini's identity of dual hyperbolic modified Woodall numbers) For all integers n, the following identity holds:*

$$\widehat{G}_{n+1}\widehat{G}_{n-1} - \widehat{G}_n^2 = 2^{n-1}(2\widehat{\alpha}\widehat{\gamma} + \widehat{\beta}\widehat{\gamma} - 2^{n+1}\widehat{\alpha}^2 + n\widehat{\alpha}\widehat{\gamma}).$$

Secondly, we give Cassini's identity of dual hyperbolic modified Cullen numbers.

Corollary 14. *(Cassini's identity of dual hyperbolic modified Cullen numbers) For all integers n, the following identity holds:*

$$\widehat{H}_{n+1}\widehat{H}_{n-1} - \widehat{H}_n^2 = 2^n\widehat{\alpha}\widehat{\gamma}.$$

Fourthly, we give Cassini's identity of dual hyperbolic Woodall numbers.

Corollary 15. *(Cassini's identity of dual hyperbolic Woodall numbers) For all integers n, the following identity holds:*

$$\widehat{R}_{n+1}\widehat{R}_{n-1} - \widehat{R}_n^2 = -2^{n-1}(3\widehat{\alpha}\widehat{\gamma} + \widehat{\beta}\widehat{\gamma} + 2^{n+1}\widehat{\alpha}^2 + n\widehat{\alpha}\widehat{\gamma}).$$

Thirdly, we give Cassini's identity of dual hyperbolic Cullen numbers.

Corollary 16. *(Cassini's identity of dual hyperbolic Cullen numbers) For all integers n, the following identity holds:*

$$\widehat{C}_{n+1}\widehat{C}_{n-1}-\widehat{C}_{n}^{2}=2^{n-1}(3\widehat{\alpha}\widehat{\gamma}+\widehat{\beta}\widehat{\gamma}-2^{n+1}\widehat{\alpha}^{2}+n\widehat{\alpha}\widehat{\gamma}).$$

Theorem 17. *For all integers m, n, G_n is woodall numbers, the following identity is true:*

$$\widehat{W}_{n+m}=\widehat{W}_{n}G_{m+1}+\widehat{W}_{n-1}(-8G_{m}+4G_{m-1})+4\widehat{W}_{n-2}G_{m}.$$

Proof. The identity (17) can be proved by mathematical induction on m. Firstly, we assume that $m\geq 0$ and $n\geq 0$. If $m=0$ we get

$$\widehat{W}_{n}=\widehat{W}_{n}G_{1}+\widehat{W}_{n-1}(-8G_{0}+4G_{-1})+4\widehat{W}_{n-2}G_{0}$$

which is true by seeing that $G_{-1}=0$, $G_{-2}=\frac{1}{4}$, $G_{-3}=\frac{1}{2}$. We assume that the identity given holds for $m=k$. For $m=k+1$, we get

$$\begin{aligned}
\widehat{W}_{(k+1)+n} &= 5\widehat{W}_{n+k}-8\widehat{W}_{n+k-1}+4\widehat{W}_{n+k-2}\\
&= 5(\widehat{W}_{n}G_{k+1}+\widehat{W}_{n-1}(-8G_{k}+4G_{k-1})+4\widehat{W}_{n-2}G_{k})\\
&\quad -8(\widehat{W}_{n}G_{k}+\widehat{W}_{n-1}(-8G_{k-1}+4G_{k-2})+4\widehat{W}_{n-2}G_{k-1})\\
&\quad +4(\widehat{W}_{n}G_{k-1}+\widehat{W}_{n-1}(-8G_{k-2}+4G_{k-3})+4\widehat{W}_{n-2}G_{k-2})\\
&= \widehat{W}_{n}(5G_{k+1}-8G_{k}+4G_{k-1})+\widehat{W}_{n-1}(-8(5G_{k}-8G_{k-1}+4G_{k-2})\\
&\quad +4(5G_{k-1}-8G_{k-2}+4G_{k-3}))+4\widehat{W}_{n-2}(5G_{k}-8G_{k-1}+4G_{k-2})\\
&= \widehat{W}_{n}G_{k+2}+\widehat{W}_{n-1}(-8G_{k+1}+4G_{k})+4\widehat{W}_{n-2}G_{k+1}\\
&= \widehat{W}_{n}G_{(k+1)+1}+\widehat{W}_{n-1}(-8G_{(k+1)}+4G_{(k+1)-1})+4\widehat{W}_{n-2}G_{(k+1)}.
\end{aligned}$$

Consequently, by mathematical induction on m, this proves (17). Similarly, we can show for the other cases. □

4. Linear Sums For Dual Hyperbolic Generalized Woodall Numbers

In this section, we give the summation formulas of the dual hyperbolic generalized Woodall numbers with positive and negatif subscripts. Now, we present the summation formulas of the generalized Woodall numbers.

Proposition 18. *For the generalized Woodall numbers, we have the following formulas:*

- $\sum_{k=0}^{n}W_{k}=\frac{1}{2}W_{2}(2n-2^{n+1}(n-1)+2^{n+2}(n-2)+6)-\frac{1}{2}W_{1}(8n-2^{n+1}(3n-5)+2^{n+2}(3n-8)+22)+W_{0}(4n-2^{n+1}(n-2)+2^{n+2}(n-3)+9).$
- $\sum_{k=0}^{n}W_{k+1}=\frac{1}{2}W_{2}(2n+2^{n+3}(n-1)-2^{n+2}n+8)-\frac{1}{2}W_{1}(8n-2^{n+2}(3n-2)+2^{n+3}(3n-5)+30)+W_{0}(4n-2^{n+2}(n-1)+2^{n+3}(n-2)+12).$

- $\sum_{k=0}^{n} W_{k+2} = \frac{1}{2}W_2(2n - 2^{n+3}(n+1) + 2^{n+4}n + 10) + W_0(4n + 2^{n+4}(n-1) - 2^{n+3}n + 16) - \frac{1}{2}W_1(8n - 2^{n+3}(3n+1) + 2^{n+4}(3n-2) + 40)$.
- $\sum_{k=0}^{n} W_{k+3} = W_0(4n - 2^{n+4}(n+1) + 2^{n+5}n + 20) - \frac{1}{2}W_1(8n + 2^{n+5}(3n+1) - 2^{n+4}(3n+4) + 48) + \frac{1}{2}W_2(2n - 2^{n+4}(n+2) + 2^{n+5}(n+1) + 10)$.

Proof. For the proof, see Soykan [27]. □

Proposition 19. *For the generalized Woodall numbers, we have the following formulas:*

- $\sum_{k=0}^{n} W_{2k} = \frac{1}{9}W_0(36n - 2^{2n+2}(2n-1) + 2^{2n+4}(2n-3) + 53) - \frac{1}{18}W_1(72n - 2^{2n+2}(6n-2) + 2^{2n+4}(6n-8) + 120) + \frac{1}{18}W_2(18n + 2^{2n+4}(2n-2) - 2 \times 2^{2n+2}n + 32)$.
- $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{18}W_2(18n - 2^{2n+3}(2n+1) + 2^{2n+5}(2n-1) + 40) - \frac{1}{18}W_1(72n - 2^{2n+3}(6n+1) + 2^{2n+5}(6n-5) + 150) + \frac{1}{9}W_0(36n + 2^{2n+5}(2n-2) - 2 \times 2^{2n+3}n + 64)$.
- $\sum_{k=0}^{n} W_{2k+2} = \frac{1}{9}W_0(36n - 2^{2n+4}(2n+1) + 2^{2n+6}(2n-1) + 80) - \frac{1}{18}W_1(72n - 2^{2n+4}(6n+4) + 2^{2n+6}(6n-2) + 192) + \frac{1}{18}W_2(18n - 2^{2n+4}(2n+2) + 2 \times 2^{2n+6}n + 50)$.
- $\sum_{k=0}^{n} W_{2k+3} = \frac{1}{18}W_2\left((18n - 2^{2n+5}(2n+3) + 2^{2n+7}(2n+1) + 58\right) - \frac{1}{18}W_1(72n + 2^{2n+7}(6n+1) - 2^{2n+5}(6n+7) + 240) + \frac{1}{9}W_0(36n - 2^{2n+5}(2n+2) + 2 \times 2^{2n+7}n + 100)$.
- $\sum_{k=0}^{n} W_{2k+4} = \frac{1}{18}W_2(18n - 2^{2n+6}(2n+4) + 2^{2n+8}(2n+2) + 50) + \frac{1}{9}W_0(36n - 2^{2n+6}(2n+3) + 2^{2n+8}(2n+1) + 116) - \frac{1}{18}W_1(72n + 2^{2n+8}(6n+4) - 2^{2n+6}(6n+10) + 264)$.

Proof. For the proof, see Soykan [27]. □

Proposition 20. *For the generalized Woodall numbers, we have the following formulas:*

- $\sum_{k=0}^{n} W_{-k} = 4W_0(n + \frac{1}{2^{n+1}}(n+4) - \frac{1}{2^{n+2}}(n+3) - 1) + 2W_1(\frac{1}{2^{n+2}}(3n+8) - 2n - \frac{1}{2^{n+1}}(3n+11) + \frac{7}{2}) + 2W_2(\frac{1}{2}n + \frac{1}{2^{n+1}}(n+3) - \frac{1}{2^{n+2}}(n+2) - 1)$.
- $\sum_{k=0}^{n} W_{-k+1} = 2W_2(\frac{1}{2}n + \frac{1}{2^n}(n+2) - \frac{1}{2^{n+1}}(n+1) - \frac{3}{2}) + 4W_0(n + \frac{1}{2^n}(n+3) - \frac{1}{2^{n+1}}(n+2) - 2) + 2W_1(\frac{1}{2^{n+1}}(3n+5) - 2n - \frac{1}{2^n}(3n+8) + 6)$.
- $\sum_{k=0}^{n} W_{-k+2} = 2W_2(\frac{1}{2}n + 2^{1-n}(n+1) - \frac{1}{2^n}n - \frac{3}{2}) + 4W_0(n - \frac{1}{2^n}(n+1) + 2^{1-n}(n+2) - 3) - 2W_1(2n + 2^{1-n}(3n+5) - \frac{1}{2^n}(3n+2) - 8)$.
- $\sum_{k=0}^{n} W_{-k+3} = 2W_2(\frac{1}{2}n + 2^{2-n}n - 2^{1-n}(n-1) + \frac{1}{2}) + 2W_1(2^{1-n}(3n-1) - 2n - 2^{2-n}(3n+2) + 6) + 4W_0(n - 2^{1-n}n + 2^{2-n}(n+1) - 3)$.

Proof. For the proof, see Soykan [27]. □

Proposition 21. *For the generalized Woodall numbers, we have the following formulas:*

- $\sum_{k=0}^{n} W_{-2k} = \frac{8}{9}W_1(\frac{1}{2^{2n+4}}(6n+8) - \frac{9}{2}n - \frac{1}{2^{2n+2}}(6n+14) + 3) + \frac{16}{9}W_0(\frac{9}{4}n + \frac{1}{2^{2n+2}}(2n+5) - \frac{1}{2^{2n+4}}(2n+3) - \frac{1}{2}) + \frac{8}{9}W_2(\frac{9}{8}n + \frac{1}{2^{2n+2}}(2n+4) - \frac{1}{2^{2n+4}}(2n+2) - \frac{7}{8})$.
- $\sum_{k=0}^{n} W_{-2k+1} = \frac{8}{9}W_1(\frac{1}{2^{2n+3}}(6n+5) - \frac{9}{2}n - \frac{1}{2^{2n+1}}(6n+11) + 6) + \frac{16}{9}W_0(\frac{9}{4}n + \frac{1}{2^{2n+1}}(2n+4) - \frac{1}{2^{2n+3}}(2n+2) - \frac{7}{4}) + \frac{8}{9}W_2(\frac{9}{8}n + \frac{1}{2^{2n+1}}(2n+3) - \frac{1}{2^{2n+3}}(2n+1) - \frac{11}{8})$.

- $\sum_{k=0}^{n} W_{-2k+2} = \frac{8}{9}W_2(\frac{9}{8}n - \frac{2}{2^{2n+2}}n + \frac{1}{2^{2n}}(2n+2) - \frac{7}{8}) - \frac{16}{9}W_0(\frac{1}{2^{2n+2}}(2n+1) - \frac{9}{4}n - \frac{1}{2^{2n}}(2n+3) + \frac{11}{4}) + \frac{8}{9} W_1(\frac{1}{2^{2n+2}}(6n+2) - \frac{9}{2}n - \frac{1}{2^{2n}}(6n+8) + \frac{15}{2})$.
- $\sum_{k=0}^{n} W_{-2k+3} = \frac{8}{9}W_1(\frac{1}{2^{2n+1}}(6n-1) - \frac{9}{2}n - 2^{1-2n}(6n+5) + \frac{3}{2}) + \frac{8}{9}W_2(\frac{9}{8}n - \frac{1}{2^{2n+1}}(2n-1) + 2^{1-2n}(2n+1) + \frac{25}{8}) + \frac{16}{9}W_0(\frac{9}{4}n + 2^{1-2n}(2n+2) - \frac{2}{2^{2n+1}}n - \frac{7}{4})$.
- $\sum_{k=0}^{n} W_{-2k+4} = \frac{8}{9}W_2(\frac{9}{8}n + 2\times 2^{2-2n}n - \frac{1}{2^{2n}}(2n-2) + \frac{137}{8}) + \frac{16}{9}W_0(\frac{9}{4}n + 2^{2-2n}(2n+1) - \frac{1}{2^{2n}}(2n-1) + \frac{25}{4}) - \frac{8}{9}W_1(\frac{9}{2}n + 2^{2-2n}(6n+2) - \frac{1}{2^{2n}}(6n-4) + \frac{57}{2})$.

Proof. For the proof, see Soykan [27]. □

Next, we give the formulas which give the summation of the dual hyperbolic generalized Woodall numbers in the following theorem.

THEOREM 22. *For $n \geq 0$, dual hyperbolic generalized Woodall numbers have the following formulas:*

(a): $\sum_{k=0}^{n} \widehat{W}_k = (3+n-3\times 2^n + 2^n n + 4j + jn - 2^{n+2}j + 2^{n+1}jn + 5\varepsilon + n\varepsilon - 2^{n+2}\varepsilon + 2^{n+2}n\varepsilon + 5j\varepsilon + jn\varepsilon + 2^{n+3}jn\varepsilon)W_2 + (-11 - 4n + 11\times 2^n - 3\times 2^n n - 15j - 4jn + 2^{n+4}j - 3\times 2^{n+1}jn - 20\varepsilon - 4n\varepsilon + 5\times 2^{n+2}\varepsilon - 3\times 2^{n+2}n\varepsilon - 24j\varepsilon - 4jn\varepsilon + 2^{n+4}j\varepsilon - 3\times 2^{n+3}jn\varepsilon)W_1 + (9 + 4n - 2^{n+3} + 2^{n+1}n + 12j + 4jn - 3\times 2^{n+2}j + 2^{n+2}jn + 16\varepsilon + 4n\varepsilon - 2^{n+4}\varepsilon + 2^{n+3}n\varepsilon + 20j\varepsilon + 4jn\varepsilon - 2^{n+4}j\varepsilon + 2^{n+4}jn\varepsilon)W_0$.

(b): $\sum_{k=0}^{n} \widehat{W}_{2k} = (\frac{16}{9} + n - \frac{1}{9}2^{2n+4} + \frac{1}{3}2^{2n+2}n + \frac{20}{9}j + jn - \frac{5}{9}2^{2n+2}j + \frac{1}{3}2^{2n+3}jn + \frac{25}{9}\varepsilon + n\varepsilon - \frac{1}{9}2^{2n+4}\varepsilon + \frac{1}{3}2^{2n+4}n\varepsilon + \frac{29}{9}j\varepsilon + \frac{1}{9}2^{2n+4}j\varepsilon + jn\varepsilon + \frac{32}{3}2^{2n}jn\varepsilon)W_2 + (-\frac{20}{3} - 4n + \frac{5}{3}2^{2n+2} - 2^{2n+2}n - \frac{25}{3}j + \frac{7}{3}2^{2n+2}j - 4jn - 2^{2n+3}jn - \frac{32}{3}\varepsilon + \frac{1}{3}2^{2n+5}\varepsilon - 4n\varepsilon - 2^{2n+4}n\varepsilon - \frac{40}{3}j\varepsilon + \frac{1}{3}2^{2n+4}j\varepsilon - 4jn\varepsilon - 2^{2n+5}jn\varepsilon)W_1 + (\frac{53}{9} - \frac{11}{9}2^{2n+2} + 4n + \frac{1}{3}2^{2n+3}n + \frac{64}{9}j - \frac{1}{9}2^{2n+6}j + 4jn + \frac{1}{3}2^{2n+4}jn + \frac{80}{9}\varepsilon - \frac{5}{9}2^{2n+4}\varepsilon + 4n\varepsilon + \frac{1}{3}2^{2n+5}n\varepsilon + \frac{100}{9}j\varepsilon - \frac{1}{9}2^{2n+6}j\varepsilon + 4jn\varepsilon + \frac{1}{3}2^{2n+6}jn\varepsilon)W_0$.

(c): $\sum_{k=0}^{n} \widehat{W}_{2k+1} = (\frac{20}{9} - \frac{5}{9}2^{2n+2} + n + \frac{1}{3}2^{2n+3}n + \frac{25}{9}j - \frac{1}{9}2^{2n+4}j + jn + \frac{1}{3}2^{2n+4}jn + \frac{29}{9}\varepsilon + n\varepsilon + \frac{1}{9}2^{2n+4}\varepsilon + \frac{1}{3}2^{2n+5}n\varepsilon + \frac{25}{9}j\varepsilon + \frac{1}{9}2^{2n+7}j\varepsilon + jn\varepsilon + \frac{1}{3}2^{2n+6}jn\varepsilon)W_2 + (-\frac{25}{3} + \frac{7}{3}2^{2n+2} - 4n - 2^{2n+3}n + \frac{1}{3}2^{2n+5}j - \frac{32}{3}j - 4jn - 2^{2n+4}jn - \frac{40}{3}\varepsilon + \frac{1}{3}2^{2n+4}\varepsilon - 4n\varepsilon - 2^{2n+5}n\varepsilon - 4jn\varepsilon - \frac{44}{3}j\varepsilon - \frac{1}{3}2^{2n+6}j\varepsilon - 2^{2n+6}jn\varepsilon)W_1 + (\frac{64}{9} - \frac{1}{9}2^{2n+6} + 4n + \frac{1}{3}2^{2n+4}n + \frac{80}{9}j - \frac{5}{9}2^{2n+4}j + 4jn + \frac{1}{3}2^{2n+5}jn + \frac{100}{9}\varepsilon - \frac{1}{9}2^{2n+6}\varepsilon + 4n\varepsilon + \frac{1}{3}2^{2n+6}n\varepsilon + \frac{116}{9}j\varepsilon + \frac{1}{9}2^{2n+6}j\varepsilon + 4jn\varepsilon + \frac{1}{3}2^{2n+7}jn\varepsilon)W_0$.

Proof. Proof can be obtained by using Proposition 21.

(a): We can derive the following using the formulas in Proposition 18.

$$\sum_{k=0}^{n} \widehat{W}_k = \sum_{k=0}^{n} W_k + j\sum_{k=0}^{n} W_{k+1} + \varepsilon\sum_{k=0}^{n} W_{k+2} + j\varepsilon\sum_{k=0}^{n} W_{k+3}.$$

$$\sum_{k=0}^{n}\widehat{W}_k = \frac{1}{2}W_2(2n-2^{n+1}(n-1)+2^{n+2}(n-2)+6)-\frac{1}{2}W_1(8n-2^{n+1}(3n-5)+2^{n+2}(3n-8)+22)$$
$$+W_0(4n-2^{n+1}(n-2)+2^{n+2}(n-3)+9)$$
$$+j(\frac{1}{2}W_2(2n+2^{n+3}(n-1)-2^{n+2}n+8)-\frac{1}{2}W_1(8n-2^{n+2}(3n-2)+2^{n+3}(3n-5)+30)$$
$$+W_0(4n-2^{n+2}(n-1)+2^{n+3}(n-2)+12))$$
$$+\varepsilon(\frac{1}{2}W_2(2n-2^{n+3}(n+1)+2^{n+4}n+10)+W_0(4n+2^{n+4}(n-1)-2^{n+3}n+16)$$
$$-\frac{1}{2}W_1(8n-2^{n+3}(3n+1)+2^{n+4}(3n-2)+40))$$
$$+j\varepsilon(W_0(4n-2^{n+4}(n+1)+2^{n+5}n+20)-\frac{1}{2}W_1(8n+2^{n+5}(3n+1)-2^{n+4}(3n+4)+48)$$
$$+\frac{1}{2}W_2(2n-2^{n+4}(n+2)+2^{n+5}(n+1)+10)).$$

$$\sum_{k=0}^{n}\widehat{W}_k = (3+n-3\times 2^n+2^n n+4j+jn-2^{n+2}j+2^{n+1}jn+5\varepsilon+n\varepsilon-2^{n+2}\varepsilon+2^{n+2}n\varepsilon+5j\varepsilon+jn\varepsilon$$
$$+2^{n+3}jn\varepsilon)W_2$$
$$+(-11-4n+11\times 2^n-3\times 2^n n-15j-4jn+2^{n+4}j-3\times 2^{n+1}jn-20\varepsilon-4n\varepsilon+5\times 2^{n+2}\varepsilon$$
$$-3\times 2^{n+2}n\varepsilon-24j\varepsilon-4jn\varepsilon+2^{n+4}j\varepsilon-3\times 2^{n+3}jn\varepsilon)W_1$$
$$+(9+4n-2^{n+3}+2^{n+1}n+12j+4jn-3\times 2^{n+2}j+2^{n+2}jn+16\varepsilon+4n\varepsilon-2^{n+4}\varepsilon+2^{n+3}n\varepsilon$$
$$+20j\varepsilon+4jn\varepsilon-2^{n+4}j\varepsilon+2^{n+4}jn\varepsilon)W_0.$$

The proof is finished. □

(b): We can derive the following using the formulas in Proposition 19.

$$\sum_{k=0}^{n}\widehat{W}_{2k}=\sum_{k=0}^{n}W_{2k}+j\sum_{k=0}^{n}W_{2k+1}+\varepsilon\sum_{k=0}^{n}W_{2k+2}+j\varepsilon\sum_{k=0}^{n}W_{2k+3}.$$

$$\sum_{k=0}^{n}\widehat{W}_{2k} = \frac{1}{9}W_0(36n-2^{2n+2}(2n-1)+2^{2n+4}(2n-3)+53)-\frac{1}{18}W_1(72n-2^{2n+2}(6n-2)$$
$$+2^{2n+4}(6n-8)+120)+\frac{1}{18}W_2(18n+2^{2n+4}(2n-2)-2\times 2^{2n+2}n+32)$$
$$+j(\frac{1}{18}W_2(18n-2^{2n+3}(2n+1)+2^{2n+5}(2n-1)+40)-\frac{1}{18}W_1(72n-2^{2n+3}(6n+1)$$
$$+2^{2n+5}(6n-5)+150)+\frac{1}{9}W_0(36n+2^{2n+5}(2n-2)-2\times 2^{2n+3}n+64))$$
$$+\varepsilon(\frac{1}{9}W_0(36n-2^{2n+4}(2n+1)+2^{2n+6}(2n-1)+80)-\frac{1}{18}W_1(72n-2^{2n+4}(6n+4)$$
$$+2^{2n+6}(6n-2)+192)+\frac{1}{18}W_2(18n-2^{2n+4}(2n+2)+2\times 2^{2n+6}n+50))$$
$$+j\varepsilon(\frac{1}{18}W_2\left((18n-2^{2n+5}(2n+3)+2^{2n+7}(2n+1)+58\right)-\frac{1}{18}W_1(72n+2^{2n+7}(6n+1)$$
$$-2^{2n+5}(6n+7)+240)+\frac{1}{9}W_0(36n-2^{2n+5}(2n+2)+2\times 2^{2n+7}n+100)).$$

$$\begin{aligned}
\sum_{k=0}^{n}\widehat{W}_{2k} &= (\frac{16}{9}+n-\frac{1}{9}2^{2n+4}+\frac{1}{3}2^{2n+2}n+\frac{20}{9}j+jn-\frac{5}{9}2^{2n+2}j+\frac{1}{3}2^{2n+3}jn+\frac{25}{9}\varepsilon+n\varepsilon-\frac{1}{9}2^{2n+4}\varepsilon \\
&+\frac{1}{3}2^{2n+4}n\varepsilon+\frac{29}{9}j\varepsilon+\frac{1}{9}2^{2n+4}j\varepsilon+jn\varepsilon+\frac{32}{3}2^{2n}jn\varepsilon)W_2 \\
&+(-\frac{20}{3}-4n+\frac{5}{3}2^{2n+2}-2^{2n+2}n-\frac{25}{3}j+\frac{7}{3}2^{2n+2}j-4jn-2^{2n+3}jn-\frac{32}{3}\varepsilon+\frac{1}{3}2^{2n+5}\varepsilon \\
&-4n\varepsilon-2^{2n+4}n\varepsilon-\frac{40}{3}j\varepsilon+\frac{1}{3}2^{2n+4}j\varepsilon-4jn\varepsilon-2^{2n+5}jn\varepsilon)W_1 \\
&+(\frac{53}{9}-\frac{11}{9}2^{2n+2}+4n+\frac{1}{3}2^{2n+3}n+\frac{64}{9}j-\frac{1}{9}2^{2n+6}j+4jn+\frac{1}{3}2^{2n+4}jn+\frac{80}{9}\varepsilon-\frac{5}{9}2^{2n+4}\varepsilon \\
&+4n\varepsilon+\frac{1}{3}2^{2n+5}n\varepsilon+\frac{100}{9}j\varepsilon-\frac{1}{9}2^{2n+6}j\varepsilon+4jn\varepsilon+\frac{1}{3}2^{2n+6}jn\varepsilon)W_0.
\end{aligned}$$

The proof is completed. □

(c): We can derive the following using the formulas in Proposition 21.

$$\sum_{k=0}^{n}\widehat{W}_{2k+1}=\sum_{k=0}^{n}W_{2k+1}+j\sum_{k=0}^{n}W_{2k+2}+\varepsilon\sum_{k=0}^{n}W_{2k+3}+j\varepsilon\sum_{k=0}^{n}W_{2k+4}.$$

$$\begin{aligned}
\sum_{k=0}^{n}\widehat{W}_{2k+1} &= \frac{1}{18}W_2(18n-2^{2n+3}(2n+1)+2^{2n+5}(2n-1)+40)-\frac{1}{18}W_1(72n-2^{2n+3}(6n+1) \\
&+2^{2n+5}(6n-5)+150)+\frac{1}{9}W_0(36n+2^{2n+5}(2n-2)-2\times 2^{2n+3}n+64) \\
&+j(\frac{1}{9}W_0(36n-2^{2n+4}(2n+1)+2^{2n+6}(2n-1)+80)-\frac{1}{18}W_1(72n-2^{2n+4}(6n+4) \\
&+2^{2n+6}(6n-2)+192)+\frac{1}{18}W_2(18n-2^{2n+4}(2n+2)+2\times 2^{2n+6}n+50)) \\
&+\varepsilon(\frac{1}{18}W_2\left((18n-2^{2n+5}(2n+3)+2^{2n+7}(2n+1)+58\right)-\frac{1}{18}W_1(72n+2^{2n+7}(6n+1) \\
&-2^{2n+5}(6n+7)+240)+\frac{1}{9}W_0(36n-2^{2n+5}(2n+2)+2\times 2^{2n+7}n+100)) \\
&+j\varepsilon(\frac{1}{18}W_2(18n-2^{2n+6}(2n+4)+2^{2n+8}(2n+2)+50)+\frac{1}{9}W_0(36n-2^{2n+6}(2n+3) \\
&+2^{2n+8}(2n+1)+116)-\frac{1}{18}W_1(72n+2^{2n+8}(6n+4)-2^{2n+6}(6n+10)+264)).
\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{n}\widehat{W}_{2k+1} &= (\frac{20}{9}-\frac{5}{9}2^{2n+2}+n+\frac{1}{3}2^{2n+3}n+\frac{25}{9}j-\frac{1}{9}2^{2n+4}j+jn+\frac{1}{3}2^{2n+4}jn+\frac{29}{9}\varepsilon+n\varepsilon+\frac{1}{9}2^{2n+4}\varepsilon \\
&+\frac{1}{3}2^{2n+5}n\varepsilon+\frac{25}{9}j\varepsilon+\frac{1}{9}2^{2n+7}j\varepsilon+jn\varepsilon+\frac{1}{3}2^{2n+6}jn\varepsilon)W_2 \\
&+(-\frac{25}{3}+\frac{7}{3}2^{2n+2}-4n-2^{2n+3}n+\frac{1}{3}2^{2n+5}j-\frac{32}{3}j-4jn-2^{2n+4}jn-\frac{40}{3}\varepsilon+\frac{1}{3}2^{2n+4}\varepsilon \\
&-4n\varepsilon-2^{2n+5}n\varepsilon-4jn\varepsilon-\frac{44}{3}j\varepsilon-\frac{1}{3}2^{2n+6}j\varepsilon-2^{2n+6}jn\varepsilon)W_1 \\
&+(\frac{64}{9}-\frac{1}{9}2^{2n+6}+4n+\frac{1}{3}2^{2n+4}n+\frac{80}{9}j-\frac{5}{9}2^{2n+4}j+4jn+\frac{1}{3}2^{2n+5}jn+\frac{100}{9}\varepsilon-\frac{1}{9}2^{2n+6}\varepsilon \\
&+4n\varepsilon+\frac{1}{3}2^{2n+6}n\varepsilon+\frac{116}{9}j\varepsilon+\frac{1}{9}2^{2n+6}j\varepsilon+4jn\varepsilon+\frac{1}{3}2^{2n+7}jn\varepsilon)W_0.
\end{aligned}$$

The proof is finished. □

As a first special case of the above theorem, we have the following summation formulas for dual hyperbolic Woodall numbers:

COROLLARY 23. *For $n \geq 0$, dual hyperbolic modified Woodall numbers have the following properties:*

(a): $\sum_{k=0}^{n} \widehat{G}_k = 4 + n + 2^{n+1}n - 2^{n+2} + j(5 - 5 \times 2^{n+2} + n + 2^{n+4} + 2^{n+2}n) + \varepsilon(5 + n + 2^{n+3}n) + j\varepsilon(1 + 2^{n+4} + n + 2^{n+4}n)$.

(b): $\sum_{k=0}^{n} \widehat{G}_{2k} = \frac{20}{9} + n + \frac{2}{3}2^{2n+2}n + \frac{5}{3}2^{2n+2} - \frac{5}{9}2^{2n+4} + j(\frac{25}{9} - \frac{4}{9}2^{2n+2} + n + \frac{2}{3}2^{2n+3}n) + \varepsilon(\frac{29}{9} + n - \frac{5}{9}2^{2n+4} + \frac{1}{3}2^{2n+5} + \frac{2}{3}2^{2n+4}n) + j\varepsilon(\frac{25}{9} + n + \frac{8}{9}2^{2n+4} + \frac{160}{3}2^{2n}n - 2^{2n+5}n)$.

(c): $\sum_{k=0}^{n} \widehat{G}_{2k+1} = \frac{25}{9} + n + \frac{2}{3}2^{2n+3}n - \frac{4}{9}2^{2n+2} + j(\frac{29}{9} - \frac{5}{9}2^{2n+4} + \frac{1}{3}2^{2n+5} + n + \frac{2}{3}2^{2n+4}n) + \varepsilon(\frac{25}{9} + n + \frac{8}{9}2^{2n+4} + \frac{2}{3}2^{2n+5}n) + j\varepsilon(-\frac{7}{9} + n - \frac{1}{3}2^{2n+6} + \frac{5}{9}2^{2n+7} + \frac{2}{3}2^{2n+6}n)$.

As a second special case of the above theorem, we have the following summation formulas for dual hyperbolic modified Cullen numbers:

COROLLARY 24. *For $n \geq 0$, dual hyperbolic modified Cullen numbers have the following properties:*

(a): $\sum_{k=0}^{n} \widehat{H}_k = -1 + n - 6 \times 2^{n}n - 3 \times 2^{n+3} + 3 \times 2^{n+1}n + 28 \times 2^{n} + j(-3 - 18 \times 2^{n+2} + 5 \times 2^{n+4} + n - 6 \times 2^{n+1}n + 3 \times 2^{n+2}n) + \varepsilon(-7 + 16 \times 2^{n+2} - 3 \times 2^{n+4} + n - 6 \times 2^{n+2}n + 3 \times 2^{n+3} n) + j\varepsilon(-15 + 2 \times 2^{n+4} + n - 6 \times 2^{n+3}n + 3 \times 2^{n+4}n)$.

(b): $\sum_{k=0}^{n} \widehat{H}_{2k} = \frac{1}{3} + n - 2^{2n+3}n + 2^{2n+3}n + \frac{14}{3}2^{2n+2} - 2^{2n+4} + j(-\frac{1}{3} + \frac{20}{3}2^{2n+2} - \frac{1}{3}2^{2n+6} + n - 2^{2n+4}n + 2^{2n+4}n) + \varepsilon(-\frac{5}{3} + n - \frac{8}{3}2^{2n+4} + \frac{5}{3}2^{2n+5} - 2^{2n+5}n + 2^{2n+5}n) + j\varepsilon(-\frac{13}{3} + n + \frac{8}{3}2^{2n+4} - \frac{1}{3}2^{2n+6} + 96 \times 2^{2n}n - 5 \times 2^{2n+5}n + 2^{2n+6}n)$.

(c): $\sum_{k=0}^{n} \widehat{H}_{2k+1} = -\frac{1}{3} + n - 2 \times 2^{2n+3}n + 2^{2n+4}n + \frac{20}{3}2^{2n+2} - \frac{1}{3}2^{2n+6} + j(-\frac{5}{3} - \frac{8}{3}2^{2n+4} + \frac{5}{3}2^{2n+5} + n - 2^{2n+5}n + 2^{2n+5}n) + \varepsilon(-\frac{13}{3} + n + \frac{8}{3}2^{2n+4} - \frac{1}{3}2^{2n+6} - 2^{2n+6}n + 2^{2n+6}n) + j\varepsilon(-\frac{29}{3} + n - \frac{4}{3}2^{2n+6} + 2^{2n+7} - 2^{2n+7}n + 2^{2n+7}n)$.

As a third special case of the above theorem, we have the following summation formulas for dual hyperbolic Woodall numbers:

COROLLARY 25. *For $n \geq 0$, dual hyperbolic Woodall numbers have the following properties:*

(a): $\sum_{k=0}^{n} \widehat{R}_k = 1 - n + 4 \times 2^{n}n + 2^{n+3} - 2^{n+1}n - 10 \times 2^{n} + j(1 - 2^{n+4} + 2^{n+4} - n + 2^{n+3}n - 2^{n+2}n) + \varepsilon(-1 - 2^{n+3} + 2^{n+4} - n + 2^{n+4}n - 2^{n+3}n) + j\varepsilon(-9 + 2^{n+5} - n + 2^{n+5}n - 2^{n+4}n)$.

(b): $\sum_{k=0}^{n} \widehat{R}_{2k} = -\frac{1}{9} - n + \frac{4}{3}2^{2n+2}n - \frac{1}{3}2^{2n+3}n + \frac{26}{9}2^{2n+2} - \frac{7}{9}2^{2n+4} + j(\frac{1}{9} - n - \frac{14}{9}2^{2n+2} + \frac{1}{9}2^{2n+6} + \frac{4}{3}2^{2n+3}n - \frac{1}{3}2^{2n+4}n) + \varepsilon(-\frac{1}{9} - n - \frac{2}{9}2^{2n+4} + \frac{1}{3}2^{2n+5} + \frac{4}{3}2^{2n+4}n - \frac{1}{3}2^{2n+5}n) + j\varepsilon(-\frac{17}{9} - n + \frac{10}{9}2^{2n+4} + \frac{1}{9}2^{2n+6} + \frac{224}{3}2^{2n}n - 2^{2n+5}n - \frac{1}{3}2^{2n+6}n)$.

(c): $\sum_{k=0}^{n} \widehat{R}_{2k+1} = \frac{1}{9} - n + \frac{4}{3}2^{2n+3}n - \frac{1}{3}2^{2n+4}n - \frac{14}{9}2^{2n+2} + \frac{1}{9}2^{2n+6} + j(-\frac{1}{9} + \frac{1}{3}2^{2n+5} - \frac{2}{9}2^{2n+4} - n + \frac{4}{3}2^{2n+4} n - \frac{1}{3}2^{2n+5}n) + \varepsilon(-\frac{17}{9} - n + \frac{10}{9}2^{2n+4} + \frac{1}{9}2^{2n+6} + \frac{4}{3}2^{2n+5}n - \frac{1}{3}2^{2n+6}n) + j\varepsilon(-\frac{73}{9} - n - \frac{4}{9}2^{2n+6} + \frac{7}{9}2^{2n+7} + \frac{4}{3}2^{2n+6}n - \frac{1}{3}2^{2n+7}n)$.

As a fourth special case of the above theorem, we have the following summation formulas for dual hyperbolic Cullen numbers:

COROLLARY 26. *For $n \geq 0$, dual hyperbolic Cullen numbers have the following properties.*

(a): $\sum_{k=0}^{n} \widehat{C}_k = 3+n-2^{n+3}+2^{n+1}n+6\times 2^n+j(3+n+2^{n+2}n)+\varepsilon(1+2^{n+3}+n+2^{n+3}n)+j\varepsilon(-7+2^{n+5}+n+2^{n+4}n)$.

(b): $\sum_{k=0}^{n} \widehat{C}_{2k} = \frac{17}{9}+n+\frac{1}{3}2^{2n+3}n-\frac{2}{9}2^{2n+2}+j(\frac{19}{9}+n+\frac{1}{9}2^{2n+3}+\frac{1}{3}2^{2n+4}n)+\varepsilon(\frac{17}{9}+n+\frac{4}{9}2^{2n+4}+\frac{1}{3}2^{2n+5}n)+j\varepsilon(\frac{1}{9}+n+\frac{7}{9}2^{2n+5}+\frac{1}{3}2^{2n+6}n)$.

(c): $\sum_{k=0}^{n} \widehat{C}_{2k+1} = \frac{19}{9}+n+\frac{1}{3}2^{2n+4}n+\frac{1}{9}2^{2n+3}+j(\frac{17}{9}+\frac{4}{9}2^{2n+4}+n+\frac{1}{3}2^{2n+5}n)+\varepsilon(\frac{1}{9}+n+\frac{7}{9}2^{2n+5}+\frac{1}{3}2^{2n+6}n)+j\varepsilon(-\frac{55}{9}+n+\frac{5}{9}2^{2n+7}+\frac{1}{3}2^{2n+7}n)$.

We next introduce the formulas which give the summation of the dual hyperbolic generalized Woodall numbers with negative subscripts in the following theorem.

THEOREM 27. *For $n \geq 0$, dual hyperbolic generalized Woodall numbers have the following formulas:*

(a): $\sum_{k=0}^{n} \widehat{W}_{-k} = (-2+\frac{2}{2^n}-3j+n-3\varepsilon+\frac{3}{2^n}j+\frac{1}{2\times 2^n}n+jn+\frac{4}{2^n}\varepsilon+j\varepsilon+n\varepsilon+\frac{1}{2^n}jn+\frac{4}{2^n}j\varepsilon+\frac{2}{2^n}n\varepsilon+jn\varepsilon+\frac{4}{2^n}jn\varepsilon)W_2+(7-\frac{7}{2^n}+12j-4n+16\varepsilon-\frac{11}{2^n}j-\frac{3}{2\times 2^n}n-4jn-\frac{16}{2^n}\varepsilon+12j\varepsilon-4n\varepsilon-\frac{3}{2^n}jn-\frac{20}{2^n}j\varepsilon-\frac{6}{2^n}n\varepsilon-4jn\varepsilon-\frac{12}{2^n}jn\varepsilon)W_1+(-4+\frac{5}{2^n}-8j+4n-12\varepsilon+\frac{8}{2^n}j+\frac{1}{2^n}n+4jn+\frac{12}{2^n}\varepsilon-12j\varepsilon+4n\varepsilon+\frac{2}{2^n}jn+\frac{16}{2^n}j\varepsilon+\frac{4}{2^n}n\varepsilon+4jn\varepsilon+\frac{8}{2^n}jn\varepsilon)W_0$.

(b): $\sum_{k=0}^{n} \widehat{W}_{-2k} = (-\frac{7}{9}+\frac{7}{9\times 2^{2n}}-\frac{11}{9}j+n-\frac{7}{9}\varepsilon+\frac{11}{9\times 2^{2n}}j+\frac{1}{3\times 2^{2n}}n+jn+\frac{16}{9\times 2^{2n}}\varepsilon+\frac{25}{9}j\varepsilon+n\varepsilon+\frac{2}{3\times 2^{2n}}jn+\frac{20}{9\times 2^{2n}}j\varepsilon+\frac{4}{3\times 2^{2n}}n\varepsilon+jn\varepsilon+\frac{8}{3\times 2^{2n}}jn\varepsilon)W_2+(\frac{8}{3}-\frac{8}{3\times 2^{2n}}+\frac{16}{3}j-4n+\frac{20}{3}\varepsilon-\frac{13}{3\times 2^{2n}}j-\frac{1}{2^{2n}}n-4jn-\frac{20}{3\times 2^{2n}}\varepsilon+\frac{4}{3}j\varepsilon-4n\varepsilon-\frac{2}{2^{2n}}jn-\frac{28}{3\times 2^{2n}}j\varepsilon-\frac{4}{2^{2n}}n\varepsilon-4jn\varepsilon-\frac{8}{2^{2n}}jn\varepsilon)W_1+(-\frac{8}{9}+\frac{17}{9\times 2^{2n}}-\frac{28}{9}j+4n-\frac{44}{9}\varepsilon+\frac{28}{9\times 2^{2n}}j+\frac{2}{3\times 2^{2n}}n+4jn+\frac{44}{9\times 2^{2n}}\varepsilon-\frac{28}{9}j\varepsilon+4n\varepsilon+\frac{4}{3\times 2^{2n}}jn+\frac{64}{9\times 2^{2n}}j\varepsilon+\frac{8}{3\times 2^{2n}}n\varepsilon+4jn\varepsilon+\frac{16}{3\times 2^{2n}}jn\varepsilon)W_0$.

(c): $\sum_{k=0}^{n} \widehat{W}_{-2k+1} = (-\frac{11}{9}+\frac{11}{9\times 2^{2n}}-\frac{7}{9}j+n+\frac{25}{9}\varepsilon+\frac{16}{9\times 2^{2n}}j+\frac{2}{3\times 2^{2n}}n+jn+\frac{20}{9\times 2^{2n}}\varepsilon+\frac{137}{9}j\varepsilon+n\varepsilon+\frac{4}{3\times 2^{2n}}jn+\frac{16}{9\times 2^{2n}}j\varepsilon+\frac{8}{3\times 2^{2n}}n\varepsilon+jn\varepsilon+\frac{16}{3\times 2^{2n}}jn\varepsilon)W_2+(\frac{16}{3}-\frac{13}{3\times 2^{2n}}+\frac{20}{3}j-4n+\frac{4}{3}\varepsilon-\frac{20}{3\times 2^{2n}}j-\frac{2}{2^{2n}}n-4jn-\frac{28}{3\times 2^{2n}}\varepsilon-\frac{76}{3}j\varepsilon-4n\varepsilon-\frac{4}{2^{2n}}jn-\frac{32}{3\times 2^{2n}}j\varepsilon-\frac{8}{2^{2n}}n\varepsilon-4jn\varepsilon-\frac{16}{2^{2n}}jn\varepsilon)W_1+(-\frac{28}{9}+\frac{28}{9\times 2^{2n}}-\frac{44}{9}j+4n-\frac{28}{9}\varepsilon+\frac{44}{9\times 2^{2n}}j+\frac{4}{3\times 2^{2n}}n+4jn+\frac{64}{9\times 2^{2n}}\varepsilon+\frac{100}{9}j\varepsilon+4n\varepsilon+\frac{8}{3\times 2^{2n}}jn+\frac{80}{9\times 2^{2n}}j\varepsilon+\frac{16}{3\times 2^{2n}}n\varepsilon+4jn\varepsilon+\frac{32}{3\times 2^{2n}}jn\varepsilon)W_0$.

Proof. Proof can be obtained by using Proposition 20.

(a): We can derive the following using the formulas in Proposition 20.

$$\sum_{k=0}^{n} \widehat{W}_{-k} = \sum_{k=0}^{n} W_{-k} + j\sum_{k=0}^{n} W_{-k+1} + \varepsilon\sum_{k=0}^{n} W_{-k+2} + j\varepsilon\sum_{k=0}^{n} W_{-k+3}. \sum_{k=0}^{n} \widehat{W}_{-k}.$$

$$\begin{aligned}\sum_{k=0}^{n}\widehat{W}_{-k} &= 4W_0(n+\frac{1}{2^{n+1}}(n+4)-\frac{1}{2^{n+2}}(n+3)-1)+2W_1(\frac{1}{2^{n+2}}(3n+8)-2n-\frac{1}{2^{n+1}}(3n+11)+\frac{7}{2})\\ &+2W_2(\frac{1}{2}n+\frac{1}{2^{n+1}}(n+3)-\frac{1}{2^{n+2}}(n+2)-1)\\ &+j(2W_2(\frac{1}{2}n+\frac{1}{2^{n}}(n+2)-\frac{1}{2^{n+1}}(n+1)-\frac{3}{2})+4W_0(n+\frac{1}{2^{n}}(n+3)-\frac{1}{2^{n+1}}(n+2)-2)\\ &+2W_1(\frac{1}{2^{n+1}}(3n+5)-2n-\frac{1}{2^{n}}(3n+8)+6))\\ &+\varepsilon(2W_2(\frac{1}{2}n+2^{1-n}(n+1)-\frac{1}{2^{n}}n-\frac{3}{2})+4W_0(n-\frac{1}{2^{n}}(n+1)+2^{1-n}(n+2)-3)\\ &-2W_1(2n+2^{1-n}(3n+5)-\frac{1}{2^{n}}(3n+2)-8))\\ &+j\varepsilon(2W_2(\frac{1}{2}n+2^{2-n}n-2^{1-n}(n-1)+\frac{1}{2})+2W_1(2^{1-n}(3n-1)-2n-2^{2-n}(3n+2)+6)\\ &+4W_0(n-2^{1-n}n+2^{2-n}(n+1)-3)).\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{n}\widehat{W}_{-k} &= (-2+\frac{2}{2^{n}}-3j+n-3\varepsilon+\frac{3}{2^{n}}j+\frac{1}{2\times 2^{n}}n+jn+\frac{4}{2^{n}}\varepsilon+j\varepsilon+n\varepsilon+\frac{1}{2^{n}}jn+\frac{4}{2^{n}}j\varepsilon+\frac{2}{2^{n}}n\varepsilon\\ &+jn\varepsilon+\frac{4}{2^{n}}jn\varepsilon)W_2\\ &+(7-\frac{7}{2^{n}}+12j-4n+16\varepsilon-\frac{11}{2^{n}}j-\frac{3}{2\times 2^{n}}n-4jn-\frac{16}{2^{n}}\varepsilon+12j\varepsilon-4n\varepsilon-\frac{3}{2^{n}}jn-\frac{20}{2^{n}}j\varepsilon\\ &-\frac{6}{2^{n}}n\varepsilon-4jn\varepsilon-\frac{12}{2^{n}}jn\varepsilon)W_1\\ &+(-4+\frac{5}{2^{n}}-8j+4n-12\varepsilon+\frac{8}{2^{n}}j+\frac{1}{2^{n}}n+4jn+\frac{12}{2^{n}}\varepsilon-12j\varepsilon+4n\varepsilon+\frac{2}{2^{n}}jn+\frac{16}{2^{n}}j\varepsilon\\ &+\frac{4}{2^{n}}n\varepsilon+4jn\varepsilon+\frac{8}{2^{n}}jn\varepsilon)W_0.\end{aligned}$$

This proves (a). We can be prove (b) and (c) similarly way using Proposition 21. □

As a first special case of the above theorem, we have the following summation formulas for dual hyperbolic modified Woodall numbers:

COROLLARY 28. *For $n\geq 0$, dual hyperbolic modified Woodall numbers have the following properties:*

- **(a):** $\sum_{k=0}^{n}\widehat{G}_{-k}=-3+n+\frac{n+3}{2^{n}}+j(-3+n+\frac{2n+4}{2^{n}})+\varepsilon(1+n+\frac{4+4n}{2^{n}})+j\varepsilon(17+n+\frac{8}{2^{n}}n)$.
- **(b):** $\sum_{k=0}^{n}\widehat{G}_{-2k}=-\frac{11}{9}+n+\frac{11+6n}{9\times 2^{2n}}+j(-\frac{7}{9}+n+\frac{16+12n}{9\times 2^{2n}})+\varepsilon(\frac{25}{9}+n+\frac{20+24n}{9\times 2^{2n}})+j\varepsilon(\frac{137}{9}+n+\frac{16+48n}{9\times 2^{2n}})$.
- **(c):** $\sum_{k=0}^{n}\widehat{G}_{-2k+1}=-\frac{7}{9}+n+\frac{16+12n}{9\times 2^{2n}}+j(\frac{25}{9}+n+\frac{20+24n}{9\times 2^{2n}})+\varepsilon(\frac{137}{9}+\frac{16+48n}{9\times 2^{2n}}+n)+j\varepsilon(\frac{457}{9}+n+\frac{-16+96n}{9\times 2^{2n}})$.

As a second special case of the above theorem, we have the following summation formulas for dual hyperbolic modified Cullen numbers:

COROLLARY 29. *For $n\geq 0$, dual hyperbolic modified Cullen numbers have the following properties:*

- **(a):** $\sum_{k=0}^{n}\widehat{H}_{-k}=5+n-\frac{2}{2^{n}}+j(9-\frac{4}{2^{n}}+n)+\varepsilon(17-\frac{8}{2^{n}}+n)+j\varepsilon(33-\frac{16}{2^{n}}+n)$.
- **(b):** $\sum_{k=0}^{n}\widehat{H}_{-2k}=\frac{11}{3}+n-\frac{2}{3\times 2^{2n}}+j(\frac{19}{3}-\frac{4}{3\times 2^{2n}}+n)+\varepsilon(\frac{35}{3}-\frac{8}{3\times 2^{2n}}+n)+j\varepsilon(\frac{67}{3}-\frac{16}{3\times 2^{2n}}+n)$.

(c): $\sum_{k=0}^{n}\widehat{H}_{-2k+1}=\frac{19}{3}+n-\frac{4}{3\times 2^{2n}}+j(\frac{35}{3}-\frac{8}{3\times 2^{2n}}+n)+\varepsilon(\frac{67}{3}-\frac{16}{3\times 2^{2n}}+n)+j\varepsilon(\frac{131}{3}-\frac{32}{3\times 2^{2n}}+n).$

As a third special case of the above theorem, we have the following summation formulas for dual hyperbolic Woodall numbers:

Corollary 30. *For $n\geq 0$, dual hyperbolic Woodall numbers have the following properties:*

(a): $\sum_{k=0}^{n}\widehat{R}_{-k}=-3-n+\frac{2+n}{2^{n}}+j(-1-n+\frac{2+2n}{2^{n}})+\varepsilon(7-n+\frac{4}{2^{n}}n)+j\varepsilon(31-\frac{8}{2^{n}}-n+\frac{8}{2^{n}}n).$

(b): $\sum_{k=0}^{n}\widehat{R}_{-2k}=-\frac{17}{9}-n+\frac{8+6n}{9\times 2^{2n}}+j(-\frac{1}{9}-n+\frac{10+12n}{9\times 2^{2n}})+\varepsilon(\frac{55}{9}-n+\frac{8+24n}{9\times 2^{2n}})+j\varepsilon(\frac{215}{9}-n+\frac{-8+48n}{9\times 2^{2n}}).$

(c): $\sum_{k=0}^{n}\widehat{R}_{-2k+1}=-\frac{1}{9}-n+\frac{10+12n}{9\times 2^{2n}}+j(\frac{55}{9}-n+\frac{8+24n}{9\times 2^{2n}})+\varepsilon(\frac{215}{9}-n+\frac{-8+48n}{9\times 2^{2n}})+j\varepsilon(\frac{631}{9}-n+\frac{-64+96n}{9\times 2^{2n}}).$

As a fourth special case of the above theorem, we have the following summation formulas for dual hyperbolic Cullen numbers:

Corollary 31. *For $n\geq 0$, dual hyperbolic Cullen numbers have the following properties:*

(a): $\sum_{k=0}^{n}\widehat{C}_{-k}=-1+n+\frac{2+n}{2^{n}}+j(1+\frac{2+2n}{2^{n}}+n)+\varepsilon(9+n+\frac{4}{2^{n}}n)+j\varepsilon(33+n+\frac{-8+8n}{2^{n}}).$

(b): $\sum_{k=0}^{n}\widehat{C}_{-2k}=\frac{1}{9}+n+\frac{8+6n}{9\times 2^{2n}}+j(\frac{17}{9}+\frac{10+12n}{9\times 2^{2n}}+n)+\varepsilon(\frac{73}{9}+\frac{8+24n}{9\times 2^{2n}}+n)+j\varepsilon(\frac{233}{9}-\frac{8-48n}{9\times 2^{2n}}+n).$

(c): $\sum_{k=0}^{n}\widehat{C}_{-2k+1}=\frac{17}{9}+n+\frac{10+12n}{9\times 2^{2n}}+j(\frac{73}{9}+\frac{8+24n}{9\times 2^{2n}}+n)+\varepsilon(\frac{233}{9}-\frac{8-48n}{9\times 2^{2n}}+n)+j\varepsilon(\frac{649}{9}-\frac{64-96n}{9\times 2^{2n}}+n).$

5. Matrices related with Dual Hyperbolic Generalized Woodall Numbers

In this section, we give matrices related with dual hyperbolic generalized Woodall numbers.

Now, we recall $\{G_n\}$ defined by the third-order recurrence relation as follows

$$G_n=5G_{n-1}-8G_{n-2}+4G_{n-3} \text{ with the initial conditions } G_0=0,\ G_1=1,\ G_2=5.$$

We present the square matrix A of order 3 as

$$A=\begin{pmatrix}5&-8&4\\1&0&0\\0&1&0\end{pmatrix}$$

such that $\det A=1$. Then, we give the following Lemma.

Lemma 32. *For all integers n the following identity is true.*

$$\begin{pmatrix}\widehat{W}_{n+2}\\\widehat{W}_{n+1}\\\widehat{W}_{n}\end{pmatrix}=\begin{pmatrix}5&-8&4\\1&0&0\\0&1&0\end{pmatrix}^{n}\begin{pmatrix}\widehat{W}_{2}\\\widehat{W}_{1}\\\widehat{W}_{0}\end{pmatrix}.$$

Proof. First, we suppose that $n\geq 0$. Lemma (32) can be given by mathematical induction on n. If $n=0$ we get

$$\begin{pmatrix}\widehat{W}_{2}\\\widehat{W}_{1}\\\widehat{W}_{0}\end{pmatrix}=\begin{pmatrix}5&-8&4\\1&0&0\\0&1&0\end{pmatrix}^{0}\begin{pmatrix}\widehat{W}_{2}\\\widehat{W}_{1}\\\widehat{W}_{0}\end{pmatrix}$$

which is true. We assume that the identity given holds for $n = k$. Thus the following identity is true.

$$\begin{pmatrix} \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \\ \widehat{W}_{k} \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} \widehat{W}_{2} \\ \widehat{W}_{1} \\ \widehat{W}_{0} \end{pmatrix}$$

For $n = k+1$, we get

$$\begin{aligned} \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} \widehat{W}_{2} \\ \widehat{W}_{1} \\ \widehat{W}_{0} \end{pmatrix} &= \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} \widehat{W}_{2} \\ \widehat{W}_{1} \\ \widehat{W}_{0} \end{pmatrix} \\ &= \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \\ \widehat{W}_{k} \end{pmatrix} \\ &= \begin{pmatrix} 5\widehat{W}_{k+2} - 8\widehat{W}_{k+1} + 4\widehat{W}_{k} \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \end{pmatrix} \\ &= \begin{pmatrix} \widehat{W}_{k+3} \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \end{pmatrix}. \end{aligned}$$

If we suppose that $n < 0$ the proof can be done similarly. Consequently, by mathematical induction on n, the proof is completed. □

Note that

$$A^{n} = \begin{pmatrix} G_{n+1} & -8G_{n} + 4G_{n-1} & 4G_{n} \\ G_{n} & -8G_{n-1} + 4G_{n-2} & 4G_{n-1} \\ G_{n-1} & -8G_{n-2} + 4G_{n-3} & 4G_{n-2} \end{pmatrix}$$

For the proof see [30].

THEOREM 33. *If we define the matrices $N_{\widehat{W}}$ and $E_{\widehat{W}}$ as follow.*

$$N_{\widehat{W}} = \begin{pmatrix} \widehat{W}_{2} & \widehat{W}_{1} & \widehat{W}_{0} \\ \widehat{W}_{1} & \widehat{W}_{0} & \widehat{W}_{-1} \\ \widehat{W}_{0} & \widehat{W}_{-1} & \widehat{W}_{-2} \end{pmatrix}, \; E_{\widehat{W}} = \begin{pmatrix} \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_{n} \\ \widehat{W}_{n+1} & \widehat{W}_{n} & \widehat{W}_{n-1} \\ \widehat{W}_{n} & \widehat{W}_{n-1} & \widehat{W}_{n-2} \end{pmatrix}.$$

then the following identity is true:

$$A^{n} N_{\widehat{W}} = E_{\widehat{W}}.$$

Proof. We can use the following identities for the proof.

$$\begin{aligned} A^n N_{\widehat{W}} &= \begin{pmatrix} G_{n+1} & -8G_n + 4G_{n-1} & 4G_n \\ G_n & -8G_{n-1} + 4G_{n-2} & 4G_{n-1} \\ G_{n-1} & -8G_{n-2} + 4G_{n-3} & 4G_{n-2} \end{pmatrix} \begin{pmatrix} \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \end{pmatrix}, \\ &= \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} b_{11} &= \widehat{W}_2 G_{n+1} + \widehat{W}_1 (-8G_n + 4G_{n-1}) + \widehat{W}_0 4G_n, \\ b_{12} &= \widehat{W}_1 G_{n+1} + \widehat{W}_0 (-8G_n + 4G_{n-1}) + \widehat{W}_{-1} 4G_n, \\ b_{13} &= \widehat{W}_0 G_{n+1} + \widehat{W}_{-1} (-8G_n + 4G_{n-1}) + \widehat{W}_{-2} 4G_n, \\ b_{21} &= \widehat{W}_2 G_n + \widehat{W}_1 (-8G_n + 4G_{n-1}) + \widehat{W}_0 4G_{n-1}, \\ b_{22} &= \widehat{W}_1 G_n + \widehat{W}_0 (-8G_n + 4G_{n-1}) + \widehat{W}_{-1} 4G_{n-1}, \\ b_{23} &= \widehat{W}_0 G_n + \widehat{W}_{-1} (-8G_n + 4G_{n-1}) + \widehat{W}_{-2} 4G_{n-1}, \\ b_{31} &= \widehat{W}_2 G_{n-1} + \widehat{W}_1 (-8G_n + 4G_{n-1}) + \widehat{W}_0 4G_{n-2}, \\ b_{32} &= \widehat{W}_1 G_{n-1} + \widehat{W}_0 (-8G_n + 4G_{n-1}) + \widehat{W}_{-1} 4G_{n-2}, \\ b_{33} &= \widehat{W}_0 G_{n-1} + \widehat{W}_{-1} (-8G_n + 4G_{n-1}) + \widehat{W}_{-2} 4G_{n-2}, \end{aligned}$$

Using the Theorem (17) the proof is done. □

From Theorem (33), we can write the following corollary.

Corollary 34. *We have the following identity.*

(a): *If we define $N_{\widehat{G}}$ and $E_{\widehat{G}}$ as follows.*

$$N_{\widehat{G}} = \begin{pmatrix} \widehat{G}_2 & \widehat{G}_1 & \widehat{G}_0 \\ \widehat{G}_1 & \widehat{G}_0 & \widehat{G}_{-1} \\ \widehat{G}_0 & \widehat{G}_{-1} & \widehat{G}_{-2} \end{pmatrix}, \quad E_{\widehat{G}} = \begin{pmatrix} \widehat{G}_{n+2} & \widehat{G}_{n+1} & \widehat{G}_n \\ \widehat{G}_{n+1} & \widehat{G}_n & \widehat{G}_{n-1} \\ \widehat{G}_n & \widehat{G}_{n-1} & \widehat{G}_{n-2} \end{pmatrix},$$

then we get

$$A^n N_{\widehat{G}} = E_{\widehat{G}}.$$

(b): *If we define $N_{\widehat{H}}$ and $E_{\widehat{H}}$ as follows.*

$$N_{\widehat{H}} = \begin{pmatrix} \widehat{H}_2 & \widehat{H}_1 & \widehat{H}_0 \\ \widehat{H}_1 & \widehat{H}_0 & \widehat{H}_{-1} \\ \widehat{H}_0 & \widehat{H}_{-1} & \widehat{H}_{-2} \end{pmatrix}, \quad E_{\widehat{H}} = \begin{pmatrix} \widehat{H}_{n+2} & \widehat{H}_{n+1} & \widehat{H}_n \\ \widehat{H}_{n+1} & \widehat{H}_n & \widehat{H}_{n-1} \\ \widehat{H}_n & \widehat{H}_{n-1} & \widehat{H}_{n-2} \end{pmatrix},$$

then we get

$$A^n N_{\widehat{H}} = E_{\widehat{H}}.$$

(c): *If we define* $N_{\widehat{R}}$ *and* $E_{\widehat{R}}$ *as follows.*

$$N_{\widehat{R}} = \begin{pmatrix} \widehat{R}_2 & \widehat{R}_1 & \widehat{R}_0 \\ \widehat{R}_1 & \widehat{R}_0 & \widehat{R}_{-1} \\ \widehat{R}_0 & \widehat{R}_{-1} & \widehat{R}_{-2} \end{pmatrix}, \; E_{\widehat{R}} = \begin{pmatrix} \widehat{R}_{n+2} & \widehat{R}_{n+1} & \widehat{R}_n \\ \widehat{R}_{n+1} & \widehat{R}_n & \widehat{R}_{n-1} \\ \widehat{R}_n & \widehat{R}_{n-1} & \widehat{R}_{n-2} \end{pmatrix}.$$

then we get

$$A^n N_{\widehat{R}} = E_{\widehat{R}}.$$

(d): *If we define* $N_{\widehat{C}}$ *and* $E_{\widehat{C}}$ *as follows.*

$$N_{\widehat{C}} = \begin{pmatrix} \widehat{C}_2 & \widehat{C}_1 & \widehat{C}_0 \\ \widehat{C}_1 & \widehat{C}_0 & \widehat{C}_{-1} \\ \widehat{C}_0 & \widehat{C}_{-1} & \widehat{C}_{-2} \end{pmatrix}, \; E_{\widehat{C}} = \begin{pmatrix} \widehat{C}_{n+2} & \widehat{C}_{n+1} & \widehat{C}_n \\ \widehat{C}_{n+1} & \widehat{C}_n & \widehat{C}_{n-1} \\ \widehat{C}_n & \widehat{C}_{n-1} & \widehat{C}_{n-2} \end{pmatrix},$$

then we get

$$A^n N_{\widehat{C}} = E_{\widehat{C}}.$$

References

[1] Akar, M., Yüce, S., Şahin, Ş., On the Dual Hyperbolic Numbers and the Complex Hyperbolic Numbers, Journal of Computer Science & Computational Mathematics, 8(1), 1-6, 2018.

[2] Baez, J., The octonions, Bull. Amer. Math. Soc. 39(2), 145-205, 2002.

[3] P. Berrizbeitia, J. G. Fernandes, M. J. González, F. Luca, V. J. M. Huguet, On Cullen numbers which are Both Riesel and Sierpiński numbers, Journal of Number Theory, 132, 2836-2841, 2012.

[4] Y. Bilu, D. Marques, A. Togbe, Generalized Cullen numbers in linear recurrence sequences, Journal of Number Theory, 202, 412-425, 2019.

[5] Biss, D.K., Dugger, D., Isaksen, D.C., Large annihilators in Cayley-Dickson algebras, Communication in Algebra, 36 (2), 632-664, 2008.

[6] Cheng, H.H., Thompson, S., Dual Polynomials and Complex Dual Numbers for Analysis of Spatial Mechanisms, Proc. of ASME 24th Biennial Mechanisms Conference, Irvine, CA, August, 19-22, 1996.

[7] Cockle, J., On a New Imaginary in Algebra, Philosophical magazine, London-Dublin-Edinburgh, 3(34), 37-47, 1849.

[8] A. J. C. Cunningham, H. J. Woodall, Factorisation of $Q = (2^q \mp q)$ and $(q.2^q \mp 1)$, Messenger of Mathematics, 47, 1–38, 1917.

[9] A. Cihan, A., Azak, A. Z., Güngör, M. A., Tosun, M., A Study on Dual Hyperbolic Fibonacci and Lucas Numbers, An. Şt. Univ. Ovidius Constanta, 27(1), 35–48, 2019.

[10] Eren O., Soykan Y., Gaussian Generalized Woodall Numbers, Archives of Current Research International, 23(8), 48-68, 2023.

[11] Fjelstad, P., Gal, S.G., n-dimensional Hyperbolic Complex Numbers, Advances in Applied Clifford Algebras, 8(1), 47-68, 1998.

[12] R. Guy, Unsolved Problems in Number Theory (2nd ed.), Springer-Verlag, New York, 1994.

[13] J. Grantham, H. Graves, The abc conjecture implies that only finitely many s-Cullen numbers are repunits, http://arxiv.org/abs/2009.04052v3, MathNT, 2021.

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[14] Hamilton, W.R., Elements of Quaternions, Chelsea Publishing Company, New York 1969.

[15] C. Hooley, Applications of the sieve methods to the theory of numbers, Cambridge University Press, Cambridge, 1976.

[16] Imaeda, K., M. Imaeda, Sedenions: algebra and analysis, Applied Mathematics and Computation, 115, 77-88, 2000.

[17] I. Kantor, A. Solodovnikov, Hypercomplex Numbers, Springer-Verlag, New York 1989.

[18] W. Keller, New Cullen primes, Math. Comput. 64, 1733-1741, 1995.

[19] F. Luca, P. Stanica, Cullen numbers in binary recurrent sequences, FT. Howard (ed.), Applications of Fibonacci Numbers : Proceedings of the Tenth International Research Conference on Fibonacci Numbers and their Applications, Kluwer Academic Publishers, 9, 167-175, 2004.

[20] D. Marques, On Generalized Cullen and Woodall numbers that are also Fibonacci numbers, Journal of Integer Sequences, 17, Article 14.9.4, 2014.

[21] D. Marques, Fibonacci s-Cullen and s-Woodall numbers, , Journal of Integer Sequences, 18, Article 15.1.4, 2015.

[22] N. K. Meher, S. S. Rout, Cullen numbers in sums of terms of recurrence sequence, http://arxiv.org/abs/2010.10014v1, MathNT, 2020.

[23] Moreno, G., The zero divisors of the Cayley-Dickson algebras over the real numbers, Bol. Soc. Mat. Mexicana 3(4), 13-28, 1998.

[24] N.J.A. Sloane, The on-line encyclopedia of integer sequences, http://oeis.org/.

[25] Sobczyk, G., The Hyperbolic Number Plane, The College Mathematics Journal, 26(4), 268-280, 1995.

[26] Y. Soykan, Tribonacci and Tribonacci-Lucas Sedenions. Mathematics 7(1), 74, 2019.

[27] Y. Soykan, On Summing Formulas For Generalized Fibonacci and Gaussian Generalized Fibonacci Numbers, Advances in Research, 20(2), 1-15, 2019.

[28] Soykan, Y., Gümüş, M., Göcen, M., A study on dual hyperbolic generalized Pell numbers, Malaya Journal Of Matematik, 09(03), 99-116, 2021.

[29] Soykan, Y., Generalized Woodall Numbers: An Investigation of Properties of Woodall and Cullen Numbers via Their Third Order Linear Recurrence Relations, Universal Journal of Mathematics and Applications, 5 : 2, 69-81, 2022.

[30] Soykan, Y., A Study On Generalized (r,s,t)-Numbers, MathLAB Journal, 7, 101-129, 2020.

[31] Soykan, Y., Taşdemir, E., Okumuş, İ., On dual hyperbolic numbers with generalized Jacobsthal numbers components, Indian J Pure Appl Math, 54, 824–840, 2022.