CORRESPONDENCE OF FIXED POINT THEOREM IN T_2, $T_3 - SPACE$

Abstract

Fixed-point theory (FPT) has lot of applications not only in the field of mathematics but also in various other disciplines. Fixed Point Theorem presents that if $T: X \rightarrow X$ is a contraction mapping on a complete metric space (X, d) then there exists a unique fixed point in X. FPT is also essential in game theory, in this case Brower Fixed Point has an application in game theory specifically in non-cooperative games and existence of Equilibrium. In particular, a game is a set of actions done by the participants defined by a set of rules. This is commonly described using mathematical concepts, which offers a concrete model to describe a variety of situations. On the other hand, the separation axioms $T_i, i = 0,1,2,3,4$ are vital properties that describes the topological spaces T_0, T_1, T_2, T_3 and T_4. It is noted that a $T_3 - space$ is a generalized version of T_2-space and since various results on application of fixed point theory in game theory on an arbitrary locally convex $T_2 -$ space has been established, in this study we sort to extend this concept to the general $T_3 - space$. The utilization of a symmetric property of Hausdorff space established that if two continuous commutative mappings are defined on a $T_3 - space$, then the two maps achieves unique fixed points.

Keywords: *Fixed-point theory, Game theory, Brower Fixed Point, Separation axiom*

1. Introduction

A topological space X is said to be a Hausdorff space($T_2 -$ space) if $\forall\, a, b \in X, a \neq b$, then $\exists$ an open sets U and V such that $a \in U, b \in V, and\ U \cap V = \phi$, whereas a $T_3 - space$ is a regular $T_1 - space$ (Kuratowski, K. (2014). Furthermore, a topological space X is regular if $\forall a \in X$ and any closed set A of X, $\exists$ open sets $say\ U\ and\ V$ such that $a \in U, A \subseteq U\ and\ U \cap V = \emptyset$, while a topological space $X\ is$ said to be $T_1 - space$ if for any two points $a, b \in X$, where $a \neq b$ there exist open sets $U\ and\ V$ such that $a \in U, b \notin U$ and $b \in V, a \notin V$ (Thron, W. J. (1962). In this study, we consider $T_i - spaces$ for $i = 0,1,2,3,4$ which are key separable topological spaces. In this case, $T_4 \rightarrow T_3 \rightarrow T_2 \rightarrow T_1 \rightarrow T_0$. A case of interest in our study is the result that a $T_3 -$ space implies $T_2 - space$ as illustrated by the following lemma.

Lemma1.1: Every topological space $T_3 -$ space is a $T_2 -$ space (Munkres et al., 2000)

We also present of an important property of a symmetric Hausdorff space as presented by (Gupta, V., Aydi, H., & Mani, N. (2015) whereby, it is illustrated that a Hausdorff space X with a continuous mapping H is said to be a symmetric if it satisfies the following axioms;

I. $\mathrm{H(x,y) = 0 \text{ iff } x = y}$

II. $\mathrm{H(x,y) = H(y,x)}$

III. It is a $\mathrm{T_1} -$ space

Having seen that $T_3 - space$ is a generalization of $T_2 - space$ and suppose that the $T_3 - space$ satisfies a symmetric conditions, then as a consequence of Lemma 1.1 the following result is evident,

Lemma 1.2: Let X be a $T_3 - space\ and\ R{:}X \rightarrow X$ be a continuous mapping such $\forall x, y \in X$, then

I. $R(x,y) = 0 \quad iff\ x = y.$

II. $It\ is\ a\ regular\ T_1 - space$

III. $R(x,y) = R(y,x)$

Thus R is said to be symmetric on $T_3 - space$.

From lemma 1.1 and lemma 1.2 we have established $T_2 - space\ and\ T_3 - space$ are symmetric .

(Popa, 1983) established a generalized results of Banach Fixed point theorem through Hausdorff topological space by considering the properties below:

1. $H(x,y) \neq 0$……………………………………………………..…………(1)
2. $H(Fx,Fy) \leq \alpha M(x,y) + \beta H(x,y)$……………………………………….…(2)
3. $H(Fx,Fy) \leq \alpha(\frac{H(x,Fx)H(y,Fy)}{H(x,y)} + \beta(H(x,y))$..(triangular inequality)…….……..(3)
4. $M(x,y) = max\left\{H(x,y), \frac{H(x,F(x)H(y,f(y)}{H(x,y)}\right\}$……………………………………….(4)

5. $H^2(x,y) \geq H(x,x)H(y,y)$……………………………………………………....(5)

Where $\alpha, \beta > 0$ *and* $\alpha + \beta < 1$ for some $x_0 \in X$ such that $Fx_0 = x_1$. By defining the sequence x_n in a Hausdorff space X such that $Fx_n = x_{n+1}$ and $Fx_{n-1} = x_n$, then through iteration process the sequence $x_n = \{F^n x_0\}$ has a convergent subsequent. As a result fixed point of F is attain.

Based on the results discussed above by (Popa, 1983) and utilizing the concept of symmetric Hausdorff space, we similarly make an effort of derivation of Fixed Point Theorem under $T_3 - space$ in the proceeding proposition.

2. Main Results

Proposition 2.1:

Let $T: X \to X$ be a continuous mapping on $T_3 - space\ X$ into itself and let $R: X \to X$ be a continuous mapping which commutes with T satisfying the following conditions;

1. $R(x) \subset T(x)\ \forall\ x \in X$
2. $d(Rx, Ry) \leq \alpha(Tx, Ty), \forall\ x, y \in X,$

Then T *and* R have a fixed point.

Proof:

Choosing $x_0 \in X$ such that $Rx_0 = Tx_1$. Based on this, we define a sequence x_n *in* X such that

$Tx_n = R(x_{n-1})$

Step 1:

Let $x_0 \in X$ and x_1 be such that

$Tx_1 = Rx_0$, in general we choose x_n so that

$Tx_n = R(x_{n-1})$

$T(x_{n+1}, x_n) \leq \alpha(Tx_n, Tx_{n-1})$ For all n

It follows from property (2)

$T(x_n, x_{n+1}) \leq T(Rx_{n-1,} Rx_n) \leq \alpha T(x_{n-1}, x_n) + \beta T(x_{n-1}, x_n) < T(x_{n-1}, x_n) \ldots.$ (1)

Then from property (4)

$$T(x_{n-1}, x_n) = max\left\{T(x_{n-1}, x_n), \frac{T(x_{n-1}, Rx_{n-1})T(x_n, Rx_n)}{T(x_{n-1}, x_n)}\right\}$$

$= max\{T(x_{n-1}, x_n), T(x_n, x_{n+1})\}$

Suppose that

$T(x_n, x_{n+1}) > T(x_{n-1}, x_n)$, it follows from equation (1)

$T(x_n, x_{n+1}) \leq \alpha T(x_{n-1}, x_n) + \beta T(x_{n-1}, x_n) \ldots.$ (2)

Also if

$T(x_n, x_{n+1}) \leq T(x_{n-1}, x_n)$

It follows again from (I)

$T(x_n, x_{n+1}) \leq (\alpha + \beta)T(x_{n-1}, x_n) < T(x_{n-1}, x_n)$...(3)

From (2) and (3) we obtain

$\alpha T(x_n, x_{n+1}) + \beta T(x_{n-1}, x_n) \leq (\alpha + \beta)T(x_{n-1}, x_n)$

$\leq \alpha T(x_{n-1}, x_n) + \beta T(x_{n-1}, \mathrm{x_n})$

$\alpha T(x_n, x_{n+1}) + \beta T(x_{n-1}, x_n) \leq \alpha T(x_{n-1}, x_n) + \beta T(x_{n-1}, \mathrm{x_n})$

$\alpha T(x_n, x_{n+1}) \leq \quad \alpha T(x_{n-1}, x_n)$

$T(x_n, x_{n+1}) \leq T(x_{n-1}, x_n) < T(x_{n-1}, x_n) \ldots..$ (4)

Using equation (3) and (4), and Repeating the process n times then,

$T(x_n, x_{n+1}) < T(x_{n-1}, x_n) < \cdots < T(x_1, x_0) < T(x_0, x_1)$

Since $\lim_{n \to \to \infty} x_n$ is bounded and letting $x_n \ \forall\, n \in N$ be a convergent sequence, denote the limit by t where $t \in X$. Let $x_{nk} \quad \forall\, n \in N$ be subsequence such that if $\varepsilon > 0$ then by definition of convergence for $x_n \ as \ n \in N$,there exists $N \in \mathbb{N}$ such that $|x_n - t| < \varepsilon$ for $n \geq N$,but this value

N will also work for x_{nk},this is because if $n \geq N$ then $x_{nk} = x_m$ for some $m \geq n \geq N$ and so $|x_{nk} - \text{t}| = |x_m - \text{t}\,| < \varepsilon$.Thus $|x_{nk} - \text{t}| < \varepsilon$,as $n \to \infty, x_{nk} = t$

Hence, we obtain a monotone sequence which converge with all its subsequence to some real number $t \in X$.

Step 2

Next is to show t is a fixed point for T and R

Where $t \in X$ such that $Tx_n \to t, Rx_n \to t$

Since T is continuous it implies R is also continuous.

Since T and R commutes it follows that

$R(T(x_n) \to R(t) and\ \ T(R(x_n) \to T(t)$ So that

$R(T(x_n) = T(R(x_n))$

then

$T(t) = R(t)$

and

$T\big(T(t)\big) = T\big(R(t)\big) = R(R(t)$ (by commutativity)

From contraction mapping

$d(R(t), R(R(t)) \leq \alpha d(T(t), T\big(R(t)\big) = \alpha d(R(t), R(R(t))$

$d(R(t), R(R(t)) \leq \alpha d(T(t), T\big(R(t)\big)$

$d(R(t), R\big(R(t)\big) - \alpha d(R(t), R\big(R(t)\big) \leq 0$,Since $\alpha \epsilon (0,1)$

$d(R(t), R\big(R(t)\big)(1 - \alpha) \leq 0$

$R(t) = R(R(t)) = T(R(t))$, then $R(t)$ is a fixed point for T and R

Step3

To show that R and T have unique fixed point, we

Suppose $R(t) = T(t) = t$ and $R(t') = T(t') = t'$

Then it follows from contraction principle,

$d(t,t') = d(R(t), R(t')) \leq \alpha d(T(t), T(t')) = \alpha d(t,t')$

$but\ \alpha < 1$

And thus

$t = t'$

Thus R and T have unique fixed point ■

Conclusion:

In this study, it has been established that the generalized result by (Popa, 1983) of Banach Fixed point theorem in a T_2 topological space can be extended to a T_3 space if the considered $T_3 - space$ possesses two continuous maps that commutates with one another.

Disclaimer (Artificial intelligence)

Authors hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

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