

## CORRESPONDENCE OF FIXED POINT THEOREM IN $T_2, T_3 - SPACE$

### Abstract

Fixed-point theory (FPT) has lot of applications not only in the field of mathematics but also in various other disciplines. Fixed Point Theorem presents that if  $T: X \rightarrow X$  is a contraction mapping on a complete metric space  $(X, d)$  then there exists a unique fixed point in  $X$ . FPT is also essential in game theory, in this case Brouwer Fixed Point has an application in game theory specifically in non-cooperative games and existence of Equilibrium. In particular, a game is a set of actions done by the participants defined by a set of rules. This is commonly described using mathematical concepts, which offers a concrete model to describe a variety of situations. On the other hand, the separation axioms  $T_i, i = 0,1,2,3,4$  are vital properties that describes the topological spaces  $T_0, T_1, T_2, T_3$  and  $T_4$ . It is noted that a  $T_3 - space$  is a generalized version of  $T_2$ -space and since various results on application of fixed point theory in game theory on an arbitrary locally convex  $T_2 - space$  has been established, in this study we sort to extend this concept to the general  $T_3 - space$ . The utilization of a symmetric property of Hausdorff space established that if two continuous commutative mappings are defined on a  $T_3 - space$ , then the two maps achieves unique fixed points.

**Keywords:** Fixed-point theory, Game theory, Brouwer Fixed Point, Separation axiom

### 1. Introduction

A topological space  $X$  is said to be a Hausdorff space ( $T_2 - space$ ) if  $\forall a, b \in X, a \neq b$ , then  $\exists$  an open sets  $U$  and  $V$  such that  $a \in U, b \in V, and U \cap V = \phi$ , whereas a  $T_3 - space$  is a regular  $T_1 - space$  (Kuratowski, K. (2014). Furthermore, a topological space  $X$  is regular if  $\forall a \in X$  and any closed set  $A$  of  $X$ ,  $\exists$  open sets say  $U$  and  $V$  such that  $a \in U, A \subseteq U$  and  $U \cap V = \emptyset$ , while a topological space  $X$  is said to be  $T_1 - space$  if for any two points  $a, b \in X$ , where  $a \neq b$  there exist open sets  $U$  and  $V$  such that  $a \in U, b \notin U$  and  $b \in V, a \notin V$  (Thron, W. J. (1962). In this study, we consider  $T_i - spaces$  for  $i = 0,1,2,3,4$  which are key

separable topological spaces. In this case,  $T_4 \rightarrow T_3 \rightarrow T_2 \rightarrow T_1 \rightarrow T_0$ . A case of interest in our study is the result that a  $T_3$  – space implies  $T_2$  – space as illustrated by the following lemma.

**Lemma 1.1:** Every topological space  $T_3$  – space is a  $T_2$  – space (Munkres et al., 2000)

We also give a brief discussion of an important property of a symmetric Hausdorff space as presented by ( Gupta, V., Aydi, H., & Mani, N. (2015) whereby, it is illustrated that a Hausdorff space  $X$  with a continuous mapping  $H$  is said to be a symmetric if it satisfies the following axioms;

- I.  $H(x, y) = 0$  iff  $x = y$
- II.  $H(x, y) = H(y, x)$
- III. It is a  $T_1$  – space

And thus  $H$  is said to be a symmetric on a  $T_2$  – space .

Having seen that  $T_3$  – space is a generalization of  $T_2$  – space and suppose that the  $T_3$  – space satisfies a symmetric conditions, then as a consequence of Lemma 1.1 the following result is evident,

**Lemma 1.2:** Let  $X$  be a  $T_3$  – space and  $R: X \rightarrow X$  be a continuous mapping such  $\forall x, y \in X$ , then

- I.  $R(x, y) = 0$  iff  $x = y$ .
- II. It is a regular  $T_1$  – space
- III.  $R(x, y) = R(y, x)$

Thus  $R$  is said to be symmetric on  $T_3$  – space.

From lemma 1.1 and lemma 1.2 we have established  $T_2$  – space and  $T_3$  – space are symmetric .

(Popa, 1983) established a generalized results of Banach Fixed point theorem through Hausdorff topological space by considering the properties below:

1.  $H(x, y) \neq 0$ .....(1)
2.  $H(Fx, Fy) \leq \alpha M(x, y) + \beta H(x, y)$ .....(2)
3.  $H(Fx, Fy) \leq \alpha \left( \frac{H(x, Fx)H(y, Fy)}{H(x, y)} + \beta(H(x, y)) \right)$ ..(triangular inequality).....(3)

$$4. M(x, y) = \max \left\{ H(x, y), \frac{H(x, F(x))H(y, f(y))}{H(x, y)} \right\} \dots \dots \dots (4)$$

$$5. H^2(x, y) \geq H(x, x)H(y, y) \dots \dots \dots (5)$$

Where  $\alpha, \beta > 0$  and  $\alpha + \beta < 1$  for some  $x_0 \in X$  such that  $Fx_0 = x_1$ , defining the sequence  $x_n$  in  $X$  such that  $Fx_n = x_{n+1}$  and  $Fx_{n-1} = x_n$ , then through iteration process the sequence  $x_n = \{F^n x_0\}$  has a convergent subsequence. As a result fixed point of  $F$  is attain.

Based on the results discussed above by (Popa, 1983) and utilizing the concept of symmetric Hausdorff space, we similarly make an effort of derivation of Fixed Point Theorem under  $T_3$  – space in the proceeding proposition.

## 2. Main Results

### Proposition 2.1:

Let  $T: X \rightarrow X$  be a continuous mapping on  $T_3$  – space  $X$  into itself and let  $R: X \rightarrow X$  be a continuous mapping which commutes with  $T$  satisfying the following conditions;

1.  $R(x) \subset T(x)$
2.  $d(Rx, Ry) \leq \alpha(Tx, Ty), \forall x, y \in X,$

Then  $T$  and  $R$  have a fixed point.

### Proof:

Choosing  $x_0 \in X$  such that  $Rx_0 = Tx_1$ . Based on this, we define a sequence  $x_n$  in  $X$  such that

$$Tx_n = R(x_{n-1})$$

### Step 1:

Let  $x_0 \in X$  and  $x_1$  be such that

$$Tx_1 = Rx_0, \text{ in general we choose } x_n \text{ so that}$$

$$Tx_n = R(x_{n-1})$$

$$T(x_{n+1}, x_n) \leq \alpha(Tx_n, Tx_{n-1}) \text{ For all } n$$

It follows from property (2)

$$T(x_n, x_{n+1}) \leq T(Rx_{n-1}, Rx_n) \leq \alpha T(x_{n-1}, x_n) + \beta T(x_{n-1}, x_n) < T(x_{n-1}, x_n) \dots (1)$$

Then from property (4)

$$\begin{aligned} T(x_{n-1}, x_n) &= \max \left\{ T(x_{n-1}, x_n), \frac{T(x_{n-1}, Rx_{n-1})T(x_n, Rx_n)}{T(x_{n-1}, x_n)} \right\} \\ &= \max \{ T(x_{n-1}, x_n), T(x_n, x_{n+1}) \} \end{aligned}$$

Suppose that

$T(x_n, x_{n+1}) > T(x_{n-1}, x_n)$ , it follows from equation (1)

$$T(x_n, x_{n+1}) \leq \alpha T(x_{n-1}, x_n) + \beta T(x_{n-1}, x_n) \dots (2)$$

Also if

$$T(x_n, x_{n+1}) \leq T(x_{n-1}, x_n)$$

It follows again from (I)

$$T(x_n, x_{n+1}) \leq (\alpha + \beta)T(x_{n-1}, x_n) < T(x_{n-1}, x_n) \dots (3)$$

From (2) and (3) we obtain

$$\begin{aligned} \alpha T(x_n, x_{n+1}) + \beta T(x_{n-1}, x_n) &\leq (\alpha + \beta)T(x_{n-1}, x_n) \\ &\leq \alpha T(x_{n-1}, x_n) + \beta T(x_{n-1}, x_n) \\ \alpha T(x_n, x_{n+1}) + \beta T(x_{n-1}, x_n) &\leq \alpha T(x_{n-1}, x_n) + \beta T(x_{n-1}, x_n) \\ \alpha T(x_n, x_{n+1}) &\leq \alpha T(x_{n-1}, x_n) \end{aligned}$$

$$T(x_n, x_{n+1}) \leq T(x_{n-1}, x_n) < T(x_{n-1}, x_n) \dots (4)$$

Using equation (3) and (4), and Repeating the process  $n$  times then,

$$T(x_n, x_{n+1}) < T(x_{n-1}, x_n) < \dots < T(x_1, x_0) < T(x_0, x_1)$$

Since  $\lim_{n \rightarrow \infty} x_n$  is bounded and letting  $x_n \forall n \in N$  be a convergent sequence, denote the limit by  $t$  where  $t \in X$ . Let  $x_{nk} \forall n \in N$  be subsequence such that if  $\varepsilon > 0$  then by definition of

convergence for  $x_n$  as  $n \in \mathbb{N}$ , there exists  $N \in \mathbb{N}$  such that  $|x_n - t| < \varepsilon$  for  $n \geq N$ , but this value  $N$  will also work for  $x_{nk}$ , this is because if  $n \geq N$  then  $x_{nk} = x_m$  for some  $m \geq n \geq N$  and so  $|x_{nk} - t| = |x_m - t| < \varepsilon$ . Thus  $|x_{nk} - t| < \varepsilon$ , as  $n \rightarrow \infty$ ,  $x_{nk} = t$

Hence, we obtain a monotone sequence which converge with all its subsequence to some real number  $t \in X$ .

## Step 2

Next is to show  $t$  is a fixed point for  $T$  and  $R$

Where  $t \in X$  such that  $Tx_n \rightarrow t, Rx_n \rightarrow t$

Since  $T$  is continuous it implies  $R$  is also continuous.

Since  $T$  and  $R$  commutes it follows that

$R(T(x_n)) \rightarrow R(t)$  and  $T(R(x_n)) \rightarrow T(t)$  So that

$$R(T(x_n)) = T(R(x_n))$$

then

$$T(t) = R(t)$$

and

$$T(T(t)) = T(R(t)) = R(R(t)) \text{ (by commutativity)}$$

From contraction mapping

$$d(R(t), R(R(t))) \leq \alpha d(T(t), T(R(t))) = \alpha d(R(t), R(R(t)))$$

$$d(R(t), R(R(t))) \leq \alpha d(T(t), T(R(t)))$$

$$d(R(t), R(R(t))) - \alpha d(R(t), R(R(t))) \leq 0, \text{ Since } \alpha \in (0, 1)$$

$$d(R(t), R(R(t)))(1 - \alpha) \leq 0$$

$R(t) = R(R(t)) = T(R(t))$ , then  $R(t)$  is a fixed point for  $T$  and  $R$

### Step3

To show that  $R$  and  $T$  have unique fixed point, we

Suppose  $R(t) = T(t) = t$  and  $R(t') = T(t') = t'$

Then it follows from contraction principle,

$$d(t, t') = d(R(t), R(t')) \leq \alpha d(T(t), T(t')) = \alpha d(t, t')$$

but  $\alpha < 1$

And thus

$$t = t'$$

Thus  $R$  and  $T$  have unique fixed point ■

### Conclusion:

In this study, it has been established that the generalized result by (Popa, 1983) of Banach Fixed point theorem in a  $T_2$  topological space can be extended to a  $T_3$  space if the considered  $T_3$  – space possesses two continuous maps that commutates with one another.

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