

## COMPLETION OF WEAKLY SIGN SYMMETRIC $P_0$ -MATRIX PROBLEM FOR $5 \times 5$ MATRICES SPECIFYING DIGRAPHS OF ORDER 5 WITH UP TO 5 ARCS.

### ABSTRACT

An  $n \times n$  matrix is a weakly sign symmetric matrix if the off-diagonal elements have the property that if the entry in row  $i$  and column  $j$  is non-zero, then the entry in row  $j$  and column  $i$  must have same sign or zero. A digraph  $D$  has a Wss  $P_0$ -matrix completion if every partial weakly sign symmetric  $P_0$ -matrix that describes  $D$  can be extended to a complete weakly sign symmetric  $P_0$ -matrix. This paper investigates the problem of completing weakly sign symmetric  $P_0$ -matrices. It demonstrates that partial matrices representing all directed graphs of order 5 with edge strengths from 0 to 5 can indeed be completed to a weakly sign symmetric  $P_0$ -matrix. Moreover, we established digraph characteristics that the partial Wss  $P_0$ -matrices specifying digraphs of order 5 with up to 5 arcs which have a clique and are cyclic or acyclic have zero completion into a Wss  $P_0$ -matrix.

**Keywords:** cyclic digraphs; acyclic digraphs; clique digraphs; Matrix completion; Digraph; weakly sign symmetric  $P_0$ -matrix.

### 1. INTRODUCTION

Completion of Wss  $P_0$ -matrix for  $4 \times 4$  matrices has been explored by [1] using digraphs of order with 4 arcs. However, the case of digraphs of order  $5 \times 5$  has not been investigated. In this study we determine Wss  $P_0$ -matrix completion of digraphs of order 5 with up to 5 arcs. A  $P_0$ -matrix  $A$  is classified as a weakly sign symmetric  $P_0$ -matrix if  $a_{ij} a_{ji} \geq 0$  for all  $i$  and  $j$  [3]. A sub matrix that has no unspecified entry is said to be fully specified according to [6]. Let  $\alpha$  be a subset of  $N = 1, 2, \dots, n$ . the principal submatrix obtained by deleting all columns and rows not in  $\alpha$  from  $A$  is denoted as  $A(\alpha)$ . Similarly,  $P(\alpha)$  represents the principal sub pattern obtained from the pattern  $P$  by removing all positions where the first and second coordinates are not in  $\alpha$ . A principal minor of  $A$  is the determinant of a principal sub-matrix [5]. A square matrix  $A$  is  $n \times n$ , with equal rows and columns. A partial matrix is an array of numbers with some specified entries while others unspecified. A partial  $P_0$ -matrix has non-negative principal submatrices. A fully specified submatrix of  $A$  has all entries defined [11].

### 2. PRELIMINARIES

Basic concepts in linear algebra, group theory and graph theory that are commonly used in Wss  $P_0$ -matrix completion problems are defined in the section below.

**Definition: 2.1** Graphs and digraphs is used in matrix completion of various classes of matrices. A graph, denoted as  $G = (V_G, E_G)$  comprises a finite non-empty set of positive integers  $V_G$ , where vertices are the members, along with a set of unordered pairs  $\{u, v\}$  of vertices called edges. A null graph is a graph devoid of edges [7].

**Definition: 2.2** A digraph  $D = (V_D, E_D)$  is like a graph  $G$  but includes directed edges  $(u, v)$  where  $u$  is the start vertex and  $v$  is the end vertex. A digraph with cycles is termed a cycle digraph; without cycles, it's acyclic. The order of a digraph is its vertex count, and its size is the number of arcs. A clique in a digraph contains all possible arcs between vertices [5].

**Definition: 2.3** Matrix completion involves identifying feasible positions in matrices using patterns. Patterns like symmetric pairs  $(i, j)$  and  $(j, i)$  are crucial for  $n \times n$  submatrices, focusing on diagonal elements. Symmetric properties in weakly sign symmetric  $P_o$ -matrices facilitate problem-solving in completing matrices, utilizing specified entries corresponding to outlined patterns [4].

**Definition: 2.4** A pattern  $D$  is considered a permutation similar to a pattern  $B$  if there exists a permutation  $\phi$  of  $\{1, 2, \dots, n\}$  such that  $B$  is formed by mapping the pairs  $(i, j)$  in  $D$  to  $(\phi(i), \phi(j))$  [6].

**Lemma: 2.5** Weakly sign symmetric  $P_o$ -matrices exhibit closure under similarity transformations by permutations.

Weakly sign symmetric  $P_o$ -matrices, due to their closure under permutation similarity, allow digraph relabeling. If a  $P_o$ -matrix  $A$  has non-negative elements  $a_{ij}$  and there exists a permutation matrix  $P$  such that  $PAP^T$  is sign symmetric, then  $A$  is considered a weakly sign symmetric  $P_o$ -matrix.

**Definition: 2.6** Let  $A$  be a Wss  $P_o$ - matrix. Then

(i) If  $P$  is a permutation matrix, then  $PAP^T$  is a Wss  $P_o$ - matrix

(ii) Any principal sub-matrix of  $A$  is Wss  $P_o$ - matrix. The set of Wss  $P_o$ - matrices is closed under permutation similarity and left and right diagonal multiplication.

**Proposition: 2.7** Every  $n \times n$  partial Wss  $P_o$ - matrix with all unspecified off diagonal entries has Wss  $P_o$ -matrix completion.

**Theorem: 2.8** Let  $A = [a_{ij}]$  be a partial Wss  $P_o$ -matrix and  $\alpha = \{i: a_{ij}\}$  is specified. If the principal partial submatrix  $A(\alpha)$  of  $A$  has a Wss  $P_o$ -matrix completion, then  $A$  has Wss  $P_o$ -completion.

### 3. ANALYSIS OF $5 \times 5$ MATRICES SPECIFYING DIGRAPHS WITH 5 VERTICES

In the process of constructing a partial Wss  $P_o$ - matrix, we use a digraph where vertices correspond to diagonal entries  $d_{ii}$ . This class of matrices allows 0 as an entry, and by definition, diagonal entries are non-negative ( $\geq 0$ ). Digraphs are utilized to create partial Wss  $P_o$ - matrices, where specific entries were known denoted by  $a_{ij}$  corresponding to existing arcs in the digraph, and other entries are unspecified denoted by  $X_{ij}$  representing missing arcs. digraphs with 5 vertices and 0 to 5 arcs are considered. In this paper we take our  $d_{ii}$  and  $a_{ij}$  to be 1 respectively.

**Digraph D of order 5 without an arc;** consider the digraph  $D = \{(1,1) (2,2), (3,3), (4,4), (5,5)\}$  with 5 vertices and no arc given by:

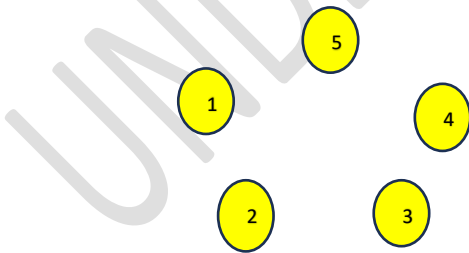


Figure 1 **Digraph D of order 5**

The partial matrix that specifies the digraph  $D$  is

$$A = \begin{pmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$$

By definition of partial Wss P<sub>o</sub>- matrix  $d_1 \geq 0, d_2 \geq 0, d_3 \geq 0, d_4 \geq 0, d_5 \geq 0$ .

For  $d_{ii} = 1, a_{ij} = 1$ , then the partial matrix becomes  $A = \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & 1 & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & 1 & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & 1 & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & 1 \end{pmatrix}$ . Next we

compute the principal minors.

Det (1,2) =  $d_{11}d_{22} - x_{12}x_{21}$ . Setting the unspecified entry to zero, and  $d_{ii} = 1$ , i.e., Det (1,2) =  $1 - 0 = 1 > 0$ . Similarly,

Det (1,3) =  $d_{11}d_{33} - x_{13}x_{31}$ , Det (1,4) =  $d_{11}d_{44} - x_{14}x_{41}$

Det (1,5) =  $d_{11}d_{55} - x_{15}x_{51}$ , Det (2,3) =  $d_{22}d_{33} - x_{23}x_{32}$ , Det (2,4) =  $d_{22}d_{44} - x_{24}x_{42}$

Det (2,5) =  $d_{22}d_{55} - x_{25}x_{52}$ , Det (3,4) =  $d_{33}d_{44} - x_{34}x_{43}$ , Det (3,5) =  $d_{33}d_{55} - x_{35}x_{53}$

Det (4,5) =  $d_{44}d_{55} - x_{45}x_{54}$ .

**TABLE 1: Determinants of 2×2 sub-matrices**

Principal submatrix	Principal minor
A (1,2)	Det A (1,2) = $d_{11}d_{22} \geq 0$ .
A (1,3)	Det A (1,3) = $d_{11}d_{33} \geq 0$ .
A (1,4)	Det A (1,4) = $d_{11}d_{44} \geq 0$ .
A (1,5)	Det A (1,5) = $d_{11}d_{55} \geq 0$ .
A (2,3)	Det A (2,3) = $d_{22}d_{33} \geq 0$ .
A (2,4)	Det A (2,4) = $d_{22}d_{44} \geq 0$ .
A (2,5)	Det A (2,5) = $d_{22}d_{55} \geq 0$ .
A (3,4)	Det A (3,4) = $d_{33}d_{44} \geq 0$ .
A (3,5)	Det A (3,5) = $d_{33}d_{55} \geq 0$ .
A (4,5)	Det A (4,5) = $d_{44}d_{55} \geq 0$ .

By definition of completion, determinants of  $2 \times 2$  submatrices are obtained. i.e.,  $\det A (1,3) = d_{11}d_{33} - x_{13}x_{31} \geq 0$ , since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero then;  $\det A (1,3) = 1 - 0 = 1 > 0$ . Therefore, all the determinants of  $2 \times 2$  submatrices are non-negative then the partial matrix can be completed into Wss P<sub>o</sub>-matrix. Hence it has zero completion into a Wss P<sub>o</sub>-matrix.

**TABLE 2: Determinants of 3×3 sub-matrices**

Principal submatrix	Principal minor
A (1,2,3)	Det A (1,2,3) = $d_{11}d_{22}d_{33} \geq 0$ .
A (1,2,4)	Det A (1,2,4) = $d_{11}d_{22}d_{44} \geq 0$ .
A (1,2,5)	Det A (1,2,5) = $d_{11}d_{22}d_{55} \geq 0$ .
A (1,3,4)	Det A (1,3,4) = $d_{11}d_{33}d_{44} \geq 0$ .
A (1,3,5)	Det A (1,3,5) = $d_{11}d_{33}d_{55} \geq 0$ .

A (1,4,5)	Det A (1,4,5) = $d_{11}d_{44}d_{55} \geq 0$ .
A (2,3,4)	Det A = (2,3,4) $d_{22}d_{33}d_{44} \geq 0$ .
A (2,3,5)	Det A (2,3,5) = $d_{22}d_{33}d_{55} \geq 0$ .
A (2,4,5)	Det A (2,4,5) = $d_{22}d_{44}d_{55} \geq 0$ .
A (3,4,5)	Det A (3,4,5) = $d_{33}d_{44}d_{55} \geq 0$ .

Determinants of  $3 \times 3$  submatrices are then obtained. i.e.,  $\det A (1,2,3) = d_{11} (d_{22}d_{33} - x_{23} x_{32}) - x_{12} (x_{21} d_{33} - x_{23} x_{31}) + x_{13} (x_{21} x_{32} - d_{22}x_{31}) \geq 0$ ., since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero then;  $\det A (1,2,3) = 1 - 0 = 1 > 0$ . Therefore, all the determinants of  $3 \times 3$  submatrices are non-negative then the partial matrix can be completed into Wss  $P_0$ -matrix. Hence it has zero completion into a Wss  $P_0$ -matrix.

**TABLE 3: Determinants of  $4 \times 4$  sub-matrices**

Principal sub-matrix	Principal minor
A (1,2,3,4)	A (1,2,3,4) = $d_{11}d_{22}d_{33}d_{44} \geq 0$ .
A (1,2,3,5)	A (1,2,3,5) = $d_{11}d_{22}d_{33}d_{55} \geq 0$ .
A (1,2,4,5)	A (1,2,4,5) = $d_{11}d_{22}d_{44}d_{55} \geq 0$ .
A (1,3,4,5)	A (1,3,4,5) = $d_{11}d_{33}d_{44}d_{55} \geq 0$ .
A (2,3,4,5)	A (2,3,4,5) = $d_{22}d_{33}d_{44}d_{55} \geq 0$ .

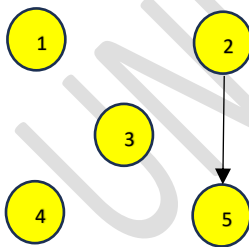
Det (A) =  $d_{11}d_{22}d_{33}d_{44}d_{55} \geq 0$ . Therefore  $\det (A) = 1 > 0$ .

Determinants of  $4 \times 4$  submatrices are obtained. i.e.,  $\det A (1,2,3,4) = \text{Det} (1,2,3,4) = d_{11}[d_{22} (d_{33} d_{44} - x_{34} x_{43}) - x_{23} (d_{44} x_{32} - x_{34} x_{42}) + x_{24} (x_{32} x_{43} - d_{33} x_{42})]$

$$\begin{aligned}
& - x_{12} [x_{21} (d_{33} d_{44} - x_{34} x_{43}) - x_{23} (d_{44} x_{31} - x_{34} x_{41}) + x_{24} (x_{31} x_{43} - d_{33} x_{41})] \\
& + x_{13} [x_{21} (x_{32} d_{44} - x_{34} x_{42}) - d_{22} (d_{44} x_{31} - x_{34} x_{41}) + x_{24} (x_{31} x_{42} - x_{32} x_{41})] \\
& - x_{14} [x_{21} (x_{32} x_{43} - d_{33} x_{42}) - d_{22}(x_{43} x_{31} - d_{33} x_{41}) + x_{23} (x_{31} x_{42} - x_{32} x_{41})] \geq 0.
\end{aligned}$$

since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero then;  $\det A (1,2,3,4) = 1 - 0 = 1 > 0$ . Therefore, all the determinants of  $4 \times 4$  submatrices are non-negative then the partial matrix can be completed into Wss  $P_0$ -matrix. Hence it has zero completion into a Wss  $P_0$ -matrix.

**Digraph D of order 5 and 1 arc:** Consider the digraph  $D = \{(1,1) (2,2), (2,5), (3,3), (4,4), (5,5)\}$  with 5 vertices and one arc given by:



**Figure 2 Digraph D of order 5 and 1 arc**

The Partial matrix that specifies the digraph D is

$$A = \begin{pmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & a_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$$

By definition of partial Wss  $P_0$ - matrix  $d_1 \geq 0, d_2 \geq 0, d_3 \geq 0, d_4 \geq 0, d_5 \geq 0$ .

For  $d_{ii}=1, a_{ij}=1$ , then the partial matrix becomes  $A = \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & 1 & x_{23} & x_{24} & 1 \\ x_{31} & x_{32} & 1 & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & 1 & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & 1 \end{pmatrix}$ . Next, we

compute the principal minors.

$\text{Det}(1,2) = d_{11}d_{22} - x_{12}x_{21}$ . Setting the unspecified entry to zero, and  $d_{ii}=1, a_{ij}=1$ , i.e.,  $\text{Det}(1,2) = 1 - 0 = 1 > 0$ .  $\text{Det}(2,5) = d_{22}d_{55} - a_{25}x_{52}$ , i.e.,  $a_{25}x_{52} = 1 \cdot 0 = 0$ . Similarly,

$\text{Det}(1,3) = d_{11}d_{33} - x_{13}x_{31}$ ,  $\text{Det}(1,4) = d_{11}d_{44} - x_{14}x_{41}$

$\text{Det}(1,5) = d_{11}d_{55} - x_{15}x_{51}$ ,  $\text{Det}(2,3) = d_{22}d_{33} - x_{23}x_{32}$ ,  $\text{Det}(2,4) = d_{22}d_{44} - x_{24}x_{42}$

$\text{Det}(2,5) = d_{22}d_{55} - a_{25}x_{52}$ ,  $\text{Det}(3,4) = d_{33}d_{44} - x_{34}x_{43}$ ,  $\text{Det}(3,5) = d_{33}d_{55} - x_{35}x_{53}$

$\text{Det}(4,5) = d_{44}d_{55} - x_{45}x_{54}$ .

**TABLE 4: Determinants of 2x2 sub-matrices**

Principal submatrix	Principal minor
A (1,2)	$\text{Det A}(1,2) = d_{11}d_{22} \geq 0$ .
A (1,3)	$\text{Det A}(1,3) = d_{11}d_{33} \geq 0$ .
A (1,4)	$\text{Det A}(1,4) = d_{11}d_{44} \geq 0$ .
A (1,5)	$\text{Det A}(1,5) = d_{11}d_{55} \geq 0$ .
A (2,3)	$\text{Det A}(2,3) = d_{22}d_{33} \geq 0$ .
A (2,4)	$\text{Det A}(2,4) = d_{22}d_{44} \geq 0$ .
A (2,5)	$\text{Det A}(2,5) = d_{22}d_{55} \geq 0$ .
A (3,4)	$\text{Det A}(3,4) = d_{33}d_{44} \geq 0$ .
A (3,5)	$\text{Det A}(3,5) = d_{33}d_{55} \geq 0$ .
A (4,5)	$\text{Det A}(4,5) = d_{44}d_{55} \geq 0$ .

By definition of completion, determinants of  $2 \times 2$  submatrices are obtained. i.e.  $\text{det}(2,5) = d_{22}d_{55} - a_{25}x_{52} \geq 0$ , since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and  $a_{ij}=1$  then;  $\text{det A}(1,3) = 1 - 0 = 1 > 0$ . Therefore, all the determinants of  $2 \times 2$  submatrices are non-negative then the partial matrix can be completed into Wss  $P_0$ -matrix. Hence it has zero completion into a Wss  $P_0$ -matrix.

**TABLE 5: Determinants of 3x3 sub-matrices**

Principal submatrix	Principal minor
A (1,2,3)	$\text{Det A}(1,2,3) = d_{11}d_{22}d_{33} \geq 0$ .
A (1,2,4)	$\text{Det A}(1,2,4) = d_{11}d_{22}d_{44} \geq 0$ .
A (1,2,5)	$\text{Det A}(1,2,5) = d_{11}d_{22}d_{55} \geq 0$ .
A (1,3,4)	$\text{Det A}(1,3,4) = d_{11}d_{33}d_{44} \geq 0$ .

A (1,3,5)	Det A (1,3,5) = $d_{11}d_{33}d_{55} \geq 0$ .
A (1,4,5)	Det A (1,4,5) = $d_{11}d_{44}d_{55} \geq 0$ .
A (2,3,4)	Det A = (2,3,4) $d_{22}d_{33}d_{44} \geq 0$ .
A (2,3,5)	Det A (2,3,5) = $d_{22}d_{33}d_{55} \geq 0$ .
A (2,4,5)	Det A (2,4,5) = $d_{22}d_{44}d_{55} \geq 0$ .
A (3,4,5)	Det A (3,4,5) = $d_{33}d_{44}d_{55} \geq 0$ .

Determinants of  $3 \times 3$  submatrices are obtained. i.e.,  $\det A (1,2,3) = d_{11} (d_{22}d_{33} - x_{23} x_{32}) - x_{12} (x_{21} d_{33} - x_{23} x_{31}) + x_{13} (x_{21} x_{32} - d_{22}x_{31}) \geq 0$ ., since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and  $a_{ij}=1$  then;  $\det A (1,2,3) = 1 - 0 = 1 > 0$ . Therefore, all the determinants of  $3 \times 3$  submatrices are non-negative then the partial matrix can be completed into Wss  $P_o$ -matrix. Hence it has zero completion into a Wss  $P_o$ -matrix.

**TABLE 6: Determinants of 4x4 sub-matrices**

Principal sub-matrix	Principal minor
A (1,2,3,4)	A (1,2,3,4) = $d_{11}d_{22}d_{33}d_{44} \geq 0$ .
A (1,2,3,5)	A (1,2,3,5) = $d_{11}d_{22}d_{33}d_{55} \geq 0$ .
A (1,2,4,5)	A (1,2,4,5) = $d_{11}d_{22}d_{44}d_{55} \geq 0$ .
A (1,3,4,5)	A (1,3,4,5) = $d_{11}d_{33}d_{44}d_{55} \geq 0$ .
A (2,3,4,5)	A (2,3,4,5) = $d_{22}d_{33}d_{44}d_{55} \geq 0$ .

Det (A) =  $d_{11}d_{22}d_{33}d_{44}d_{55} \geq 0$ . Therefore  $\det (A) = 1 > 0$ .

Determinants of  $4 \times 4$  submatrices are obtained. i.e.,  $\det A (1,2,3,4) = \text{Det} (1,2,3,4) = d_{11}[d_{22} (d_{33} d_{44} - x_{34} x_{43}) - x_{23} (d_{44} x_{32} - x_{34} x_{42}) + x_{24} (x_{32} x_{43} - d_{33} x_{42})]$

$$- x_{12} [x_{21} (d_{33} d_{44} - x_{34} x_{43}) - x_{23} (d_{44} x_{31} - x_{34} x_{41}) + x_{24} (x_{31} x_{43} - d_{33} x_{41})]$$

$$+ x_{13} [x_{21} (x_{32} d_{44} - x_{34} x_{42}) - d_{22} (d_{44} x_{31} - x_{34} x_{41}) + x_{24} (x_{31} x_{42} - x_{32} x_{41})]$$

$$- x_{14} [x_{21} (x_{32} x_{43} - d_{33} x_{42}) - d_{22}(x_{43} x_{31} - d_{33} x_{41}) + x_{23} (x_{31} x_{42} - x_{32} x_{41})] \geq 0$$

since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and  $a_{ij}=1$  then;  $\det A (1,2,3,4) = 1 - 0 + 0 - 0 = 1 > 0$ . Therefore, all the determinants of  $4 \times 4$  submatrices are non-negative then the partial matrix can be completed into Wss  $P_o$ -matrix. Hence it has zero completion into a Wss  $P_o$ -matrix.

**Digraph D of order 5 and 2 arcs:** consider a digraph  $D = \{(1,1), (2,2), (3,3), (3,4), (4,3), (4,4), (5,5)\}$  with 5 vertices and 2 arcs given by:

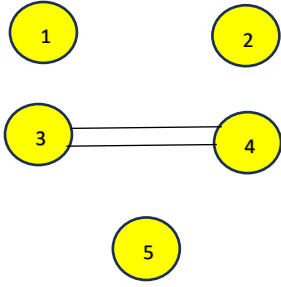


Figure 3. Digraph D of order 5 and 2 arcs

The Partial matrix that specifies the digraph D is

$$A = \begin{pmatrix} d_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & a_{34} & x_{35} \\ x_{41} & x_{42} & a_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}.$$

By definition of partial WSS P<sub>o</sub>- matrix  $d_1 \geq 0, d_2 \geq 0, d_3 \geq 0, d_4 \geq 0, d_5 \geq 0$ .

For  $d_{ii} = 1, a_{ij} = 1$ , then the partial matrix becomes  $A = \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & 1 & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & 1 & 1 & x_{35} \\ x_{41} & x_{42} & 1 & 1 & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & 1 \end{pmatrix}$ . Next, we

compute the principal minors.

Det (1,2) =  $d_{11}d_{22} - x_{12}x_{21}$ , Setting the unspecified entry to zero, i.e.,  $x_{12}x_{21} = 0$ , Det (3,4) =  $d_{33}d_{44} - a_{34}a_{43} = 1-1 \geq 0$ . Similarly;

Det (1,3) =  $d_{11}d_{33} - x_{13}x_{31}$ , Det (1,4) =  $d_{11}d_{44} - x_{14}x_{41}$

Det (1,5) =  $d_{11}d_{55} - x_{15}x_{51}$ , Det (2,3) =  $d_{22}d_{33} - x_{23}x_{32}$ ,

Det (2,4) =  $d_{22}d_{44} - x_{24}x_{42}$ , Det (2,5) =  $d_{22}d_{55} - x_{25}x_{52}$ ,

Det (3,5) =  $d_{33}d_{55} - x_{35}x_{53}$ , Det (4,5) =  $d_{44}d_{55} - x_{45}x_{54}$

**TABLE 7: Determinants of 2×2 sub-matrices**

Principal submatrix	Principal minor
A (1,2)	Det A (1,2) = $d_{11}d_{22} \geq 0$ .
A (1,3)	Det A (1,3) = $d_{11}d_{33} \geq 0$ .
A (1,4)	Det A (1,4) = $d_{11}d_{44} \geq 0$ .
A (1,5)	Det A (1,5) = $d_{11}d_{55} \geq 0$ .
A (2,3)	Det A (2,3) = $d_{22}d_{33} \geq 0$ .
A (2,4)	Det A (2,4) = $d_{22}d_{44} \geq 0$ .

A (2,5)	Det A (2,5) = $d_{22}d_{55} \geq 0$ .
A (3,4)	Det A (3,4) = $(d_{33}d_{44}) - (a_{34}a_{43}) \geq 0$
A (3,5)	Det A (3,5) = $d_{33}d_{55} \geq 0$ .
A (4,5)	Det A (4,5) = $d_{44}d_{55} \geq 0$ .

By definition of completion, determinants of  $2 \times 2$  submatrices are obtained. i.e.  $\text{Det} (1,2) = d_{11}d_{22} - x_{12}x_{21} \geq 0$ ,  $\text{Det} (3,4) = d_{33} d_{44} - a_{34} a_{43} \geq 0$ . Since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and  $a_{ij}=1$  then;  $\det A (1,2) = 1 - 0 = 1 > 0$ ,  $\text{Det} (3,4) = 1-1 \geq 0$ . Therefore, all the determinants of  $2 \times 2$  submatrices are non-negative then the partial matrix can be completed into Wss  $P_0$ -matrix. Hence it has zero completion into a Wss  $P_0$ -matrix.

**TABLE 8: Determinants of  $3 \times 3$  sub-matrices**

Principal submatrix	Principal minor
A (1,2,3)	Det A (1,2,3) = $d_{11}d_{22}d_{33} \geq 0$ .
A (1,2,4)	Det A (1,2,4) = $d_{11}d_{22}d_{44} \geq 0$ .
A (1,2,5)	Det A (1,2,5) = $d_{11}d_{22}d_{55} \geq 0$ .
A (1,3,4)	Det A (1,3,4) = $d_{11}d_{33}d_{44} \geq 0$ .
A (1,3,5)	Det A (1,3,5) = $d_{11}d_{33}d_{55} \geq 0$ .
A (1,4,5)	Det A (1,4,5) = $d_{11}d_{44}d_{55} \geq 0$ .
A (2,3,4)	Det A = (2,3,4) $d_{22}d_{33}d_{44} \geq 0$ .
A (2,3,5)	Det A (2,3,5) = $d_{22}d_{33}d_{55} \geq 0$ .
A (2,4,5)	Det A (2,4,5) = $d_{22}d_{44}d_{55} \geq 0$ .
A (3,4,5)	Det A (3,4,5) = $d_{55}(d_{33}d_{44}) - (a_{34}a_{43}) \geq 0$

Determinants of  $3 \times 3$  submatrices are obtained. i.e.,  $\text{Det} (3,4,5) = d_{33} (d_{44} d_{55} - x_{45} x_{54}) - a_{34} (a_{43} d_{55} - x_{45} x_{53}) + x_{35} (a_{43} x_{54} - d_{44} x_{53}) \geq 0$ . Since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and  $a_{ij}=1$  then;  $\det A (3,4,5) = 1 - 1 = 0$ . Therefore, all the determinants of  $3 \times 3$  submatrices are non-negative then the partial matrix can be completed into Wss  $P_0$ -matrix. Hence it has zero completion into a Wss  $P_0$ -matrix.

**TABLE 9: Determinants of  $4 \times 4$  sub-matrices**

Principal sub-matrix	Principal minor
A (1,2,3,4)	Det (1,2,3,4) = $d_{11}d_{22}d_{33}d_{44} \geq 0$ .
A (1,2,3,5)	Det (1,2,3,5) = $d_{11}d_{22}d_{33}d_{55} \geq 0$ .
A (1,2,4,5)	Det (1,2,4,5) = $d_{11}d_{22}d_{44}d_{55} \geq 0$ .
A (1,3,4,5)	Det (1,3,4,5) = $d_{11}d_{55}(d_{33}d_{44}) - (a_{34}a_{43}) \geq 0$ .
A (2,3,4,5)	Det (2,3,4,5) = $d_{22}d_{33}d_{44}d_{55} \geq 0$ .

$\text{Det} (A) = d_{11}d_{22}d_{33}d_{44}d_{55} - (a_{34}a_{43}) \geq 0$ . therefore  $\text{Det} (A) = 1 - 1 = 0$ .

Determinants of  $4 \times 4$  submatrices are obtained. i.e.,  $\text{Det} (1,3,4,5) = d_{11} [d_{33} (d_{44} d_{55} - x_{45} x_{54}) - a_{34} (d_5 a_{43} - x_{45} x_{53}) + x_{35} (a_{43} x_{54} - d_{44} x_{53})]$

$$- x_{13} [x_{31} (d_{44} d_{55} - x_{45} x_{54}) - a_{34} (d_5 x_{41} - x_{45} x_{51}) + x_{35} (x_{41} x_{54} - d_{44} x_{51})]$$

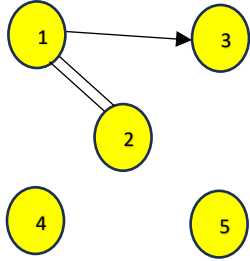
$$+ x_{14} [x_{31} (d_{55} a_{43} - x_{45} x_{53}) - d_{33} (d_5 x_{41} - x_{45} x_{51}) + x_{35} (x_{41} x_{53} - a_{43} x_{51})]$$

$$- x_{15} [x_{31} (a_{43} x_{54} - d_{44} x_{53}) - d_{33} (x_{41} x_{54} - d_{44} x_{51}) + a_{34} (x_{41} x_{53} - x_{51} a_{43})] \geq 0$$

Since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to

zero and  $a_{ij} = 1$  then;  $\det A(1,3,4,5) = 0 - 0 + 0 - 0 = 0$ . Therefore, all the determinants of  $4 \times 4$  submatrices are non-negative then the partial matrix can be completed into Wss  $P_0$ -matrix. Hence it has zero completion into a Wss  $P_0$ -matrix.

**Digraph D of order 5 and 3 arcs;** consider a digraph  $D = \{(1,1), (1,2), (1,3), (2,1), (2,2), (3,3), (4,4), (5,5)\}$  with 5 vertices and 3 arcs given by:



**Figure 4. Digraph D of order 5 and 3 arcs**

The partial matrix that specifies the digraph D is  $A = \begin{pmatrix} d_{11} & a_{12} & a_{13} & x_{14} & x_{15} \\ a_{21} & d_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$

By definition of partial Wss  $P_0$ - matrix  $d_1 \geq 0, d_2 \geq 0, d_3 \geq 0, d_4 \geq 0, d_5 \geq 0$ .

For  $d_{ii} = 1, a_{ij} = 1$ , then the partial matrix becomes  $A = \begin{pmatrix} 1 & 1 & 1 & x_{14} & x_{15} \\ 1 & 1 & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & 1 & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & 1 & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & 1 \end{pmatrix}$ . Next, we

compute the principal minors.

$\text{Det}(1,2) = d_{11}d_{22} - a_{12}a_{21}$ . Setting unspecified entries to zero,  $d_{ii} = 1, a_{ij} = 1$ , then we have;  $1 - 1 \geq 0$ .

$\text{Det}(1,3) = d_{11}d_{33} - a_{13}a_{31} = 1 - x_{31} = 1 - 0 = 1 \geq 0$ . Similarly

$\text{Det}(1,2) = d_{11}d_{22} - a_{12}a_{21}$ ,  $\text{Det}(1,4) = d_{11}d_{44} - x_{14}x_{41}$ ,

$\text{Det}(1,5) = d_{11}d_{55} - x_{15}x_{51}$ ,  $\text{Det}(2,3) = d_{22}d_{33} - x_{23}x_{32}$ ,  $\text{Det}(2,4) = d_{22}d_{44} - x_{24}x_{42}$

$\text{Det}(2,5) = d_{22}d_{55} - x_{25}x_{52}$ ,  $\text{Det}(3,4) = d_{33}d_{44} - x_{34}x_{43}$ ,  $\text{Det}(3,5) = d_{33}d_{55} - x_{35}x_{53}$

$\text{Det}(4,5) = d_{44}d_{55} - x_{45}x_{54}$ .

**TABLE 10: Determinants of 2×2 sub-matrices**

Principal submatrix	Principal minor
A (1,2)	Det A (1,2) = $(d_{11}d_{22}) - (a_{12}a_{21}) \geq 0$
A (1,3)	Det A (1,3) = $d_{11}d_{33} \geq 0$ .
A (1,4)	Det A (1,4) = $d_{11}d_{44} \geq 0$ .
A (1,5)	Det A (1,5) = $d_{11}d_{55} \geq 0$ .
A (2,3)	Det A (2,3) = $d_{22}d_{33} \geq 0$ .
A (2,4)	Det A (2,4) = $d_{22}d_{44} \geq 0$ .
A (2,5)	Det A (2,5) = $d_{22}d_{55} \geq 0$ .
A (3,4)	Det A (3,4) = $d_{33}d_{44} \geq 0$ .
A (3,5)	Det A (3,5) = $d_{33}d_{55} \geq 0$ .
A (4,5)	Det A (4,5) = $d_{44}d_{55} \geq 0$ .

By definition of completion, determinants of  $2 \times 2$  submatrices are obtained. i.e.  $\text{Det}(1,2) = d_{11}d_{22} - a_{12}a_{21} \geq 0$ .  $\text{Det}(1,3) = d_{11}d_{33} - a_{13}a_{31} \geq 0$ . Since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and  $a_{ij}=1$  then;  $\text{Det}(1,2) = 1-1 \geq 0$ ,  $\text{det} A(1,3) = 1-0 = 1 > 0$ . Therefore, all the determinants of  $2 \times 2$  submatrices are non-negative then the partial matrix can be completed into Wss  $P_0$ -matrix. Hence it has zero completion into a Wss  $P_0$ -matrix.

**TABLE 11: Determinants of 3×3 sub-matrices**

Principal submatrix	Principal minor
A (1,2,3)	Det A (1,2,3) = $d_{33}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$ .
A (1,2,4)	Det A (1,2,4) = $d_{44}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$ .
A (1,2,5)	Det A (1,2,5) = $d_{55}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$ .
A (1,3,4)	Det A (1,3,4) = $d_{11}d_{33}d_{44} \geq 0$ .
A (1,3,5)	Det A (1,3,5) = $d_{11}d_{33}d_{55} \geq 0$ .
A (1,4,5)	Det A (1,4,5) = $d_{11}d_{44}d_{55} \geq 0$ .
A (2,3,4)	Det A = (2,3,4) $d_{22}d_{33}d_{44} \geq 0$ .
A (2,3,5)	Det A (2,3,5) = $d_{22}d_{33}d_{55} \geq 0$ .
A (2,4,5)	Det A (2,4,5) = $d_{22}d_{44}d_{55} \geq 0$ .
A (3,4,5)	Det A (3,4,5) = $d_{33}d_{44}d_{55} \geq 0$ .

Determinants of  $3 \times 3$  submatrices are obtained. i.e.,  $\text{Det}(3,4,5) = d_{11}(d_{22}d_{33} - x_{23}x_{32}) - a_{12}(a_{21}d_{33} - x_{23}x_{31}) + a_{13}(a_{21}x_{32} - d_{22}x_{31}) \geq 0$ . Since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and  $a_{ij}=1$  then;  $\text{det} A(3,4,5) = 1-1+0 = 1 > 0$ . Therefore, all the determinants of  $3 \times 3$  submatrices are non-negative then the partial matrix can be completed into Wss  $P_0$ -matrix. Hence it has zero completion into a Wss  $P_0$ -matrix.

**TABLE 12: Determinants of 4×4 sub-matrices**

Principal sub-matrix	Principal minor
A (1,2,3,4)	Det A (1,2,3,4) = $d_{33}d_{44}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$ .
A (1,2,3,5)	Det A (1,2,3,5) = $d_{33}d_{55}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$ .
A (1,2,4,5)	Det A (1,2,4,5) = $d_{33}d_{55}(d_{11}d_{22} - a_{12}a_{21}) \geq 0$ .
A (1,3,4,5)	Det A (1,3,4,5) = $d_{11}d_{33}d_{44}d_{55} \geq 0$ .

$A(2,3,4,5)$	$\text{Det } A(2,3,4,5) = d_{22}d_{33}d_{44}d_{55} \geq 0.$
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$\text{Det } (A) = d_{11}d_{22}d_{33}d_{44}d_{55} - (a_{12}a_{21}d_{33}d_{44}d_{55}) \geq 0.$  Therefore,  $\text{Det } (A) = 1 - 1 = 0.$

Determinants of  $4 \times 4$  submatrices are obtained. i.e.,  $\text{Det } (1,2,3,4) = d_{11}[d_{22}(d_{33}d_{44} - x_{34}x_{43}) - x_{23}(d_{44}x_{32} - x_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})]$

$$\begin{aligned}
 & - a_{12}[a_{21}(d_{33}d_{44} - x_{34}x_{43}) - x_{23}(d_{44}x_{31} - x_{34}x_{41}) + x_{24}(x_{31}x_{43} - d_{33}x_{41})] \\
 & + a_{13}[a_{21}(x_{32}d_{44} - x_{34}x_{42}) - d_{22}(d_{44}x_{31} - x_{34}x_{41}) + x_{24}(x_{31}x_{42} - x_{32}x_{41})] \\
 & - x_{14}[a_{21}(x_{32}x_{43} - d_{33}x_{42}) - d_{22}(x_{43}x_{31} - d_{33}x_{41}) + x_{23}(x_{31}x_{42} - x_{32}x_{41})] \geq 0.
 \end{aligned}$$

Since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and  $a_{ij}=1$  then;  $\text{det } A(1,2,3,4) = 1 - 1 + 0 - 0 = 0.$  Therefore, all the determinants of  $4 \times 4$  submatrices are non-negative then the partial matrix can be completed into Wss  $P_o$ -matrix. Hence it has zero completion into a Wss  $P_o$ -matrix.

**Digraph D of order 5 and 4 arcs;** consider a digraph  $D = \{(1,1), (1,2), (2,2), (2,3), (3,3), (3,4), (3,5), (4,4), (5,5)\}$  with 5 vertices and 4 arcs given by:

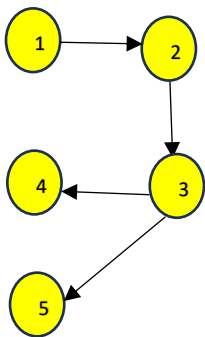


Figure 5. Digraph D of order 5 and 4 arcs

The Partial matrix that specifies the digraph D is  $A = \begin{pmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & a_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & a_{34} & a_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}$ . By

definition of partial Wss  $P_o$ - matrix  $d_1 \geq 0, d_2 \geq 0, d_3 \geq 0, d_4 \geq 0, d_5 \geq 0.$

For  $d_{ii} = 1, a_{ij} = 1,$  then the partial matrix becomes  $A = \begin{pmatrix} 1 & 1 & x_{13} & x_{14} & x_{15} \\ x_{21} & 1 & 1 & x_{24} & x_{25} \\ x_{31} & x_{32} & 1 & 1 & 1 \\ x_{41} & x_{42} & x_{43} & 1 & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & 1 \end{pmatrix}$  Next, we

compute the principal minors.

$\text{Det } (1,2) = d_{11}d_{22} - a_{12}x_{21}$  Setting unspecified entries to zero,  $d_{ii}=1, a_{ij}=1,$  then we have;  $1 - 0 = 0.$  Similarly;  $\text{Det } (1,3) = d_{11}d_{33} - x_{13}x_{31}, \text{Det } (1,4) = d_{11}d_{44} - x_{14}x_{41}, \text{Det } (1,5) = d_{11}d_{55} - x_{15}x_{51}, \text{Det } (2,3) = d_{22}d_{33} - a_{23}x_{32}, \text{Det } (2,4) = d_{22}d_{44} - x_{24}x_{42}, \text{Det } (2,5) = d_{22}d_{55} - x_{25}x_{52}, \text{Det } (3,4) = d_{33}d_{44} - a_{34}x_{43}, \text{Det } (3,5) = d_{33}d_{55} - a_{35}x_{53}$

$$\text{Det } (4 \ 5) = d_{44} d_{55} - x_{45} x_{54}.$$

**TABLE 13: Determinants of 2×2 sub-matrices**

Principal submatrix	Principal minor
A (1,2)	Det A (1,2) = $d_{11}d_{22} \geq 0$ .
A (1,3)	Det A (1,3) = $d_{11}d_{33} \geq 0$ .
A (1,4)	Det A (1,4) = $d_{11}d_{44} \geq 0$ .
A (1,5)	Det A (1,5) = $d_{11}d_{55} \geq 0$ .
A (2,3)	Det A (2,3) = $d_{22}d_{33} \geq 0$ .
A (2,4)	Det A (2,4) = $d_{22}d_{44} \geq 0$ .
A (2,5)	Det A (2,5) = $d_{22}d_{55} \geq 0$ .
A (3,4)	Det A (3,4) = $d_{33}d_{44} \geq 0$ .
A (3,5)	Det A (3,5) = $d_{33}d_{55} \geq 0$ .
A (4,5)	Det A (4,5) = $d_{44}d_{55} \geq 0$ .

By definition of completion, determinants of  $2 \times 2$  submatrices are obtained. i.e.  $\text{Det } (1,2) = d_{11}d_{22} - a_{12}x_{21} \geq 0$ . Since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and  $a_{ij}=1$  then;  $\text{det } A (1,2) = 1 - 0 = 1 > 0$ . Therefore, all the determinants of  $2 \times 2$  submatrices are non-negative then the partial matrix can be completed into Wss  $P_0$ -matrix. Hence it has zero completion into a Wss  $P_0$ -matrix.

**TABLE 14: Determinants of 3×3 sub-matrices**

Principal submatrix	Principal minor
A (1,2,3)	Det A (1,2,3) = $d_{11}d_{22}d_{33} \geq 0$ .
A (1,2,4)	Det A (1,2,4) = $d_{11}d_{22}d_{44} \geq 0$ .
A (1,2,5)	Det A (1,2,5) = $d_{11}d_{22}d_{55} \geq 0$ .
A (1,3,4)	Det A (1,3,4) = $d_{11}d_{33}d_{44} \geq 0$ .
A (1,3,5)	Det A (1,3,5) = $d_{11}d_{33}d_{55} \geq 0$ .
A (1,4,5)	Det A (1,4,5) = $d_{11}d_{44}d_{55} \geq 0$ .
A (2,3,4)	Det A = (2,3,4) $d_{22}d_{33}d_{44} \geq 0$ .
A (2,3,5)	Det A (2,3,5) = $d_{22}d_{33}d_{55} \geq 0$ .
A (2,4,5)	Det A (2,4,5) = $d_{22}d_{44}d_{55} \geq 0$ .
A (3,4,5)	Det A (3,4,5) = $d_{33}d_{44}d_{55} \geq 0$ .

Determinants of  $3 \times 3$  submatrices are obtained. i.e.,  $\text{Det } (1,2,3) = d_{11} (d_{22}d_{33} - a_{23} x_{32}) - a_{12} (a_{21} d_{33} - a_{23} x_{31}) + x_{13} (x_{21} x_{32} - d_{22} x_{31}) \geq 0$ . Since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and  $a_{ij}=1$  then;  $\text{det } A (1,2,3) = 1 - 1 + 0 = 1 > 0$ . Therefore, all the determinants of  $3 \times 3$  submatrices are non-negative then the partial matrix can be completed into Wss  $P_0$ -matrix. Hence it has zero completion into a Wss  $P_0$ -matrix.

**TABLE 15: Determinants of 4×4 sub-matrices**

Principal sub-matrix	Principal minor
A (1,2,3,4)	Det A (1,2,3,4) = $d_{11}d_{22}d_{33}d_{44} \geq 0$ .
A (1,2,3,5)	Det A (1,2,3,5) = $d_{11}d_{22}d_{33}d_{55} \geq 0$ .
A (1,2,4,5)	Det A (1,2,4,5) = $d_{11}d_{22}d_{44}d_{55} \geq 0$ .
A (1,3,4,5)	Det A (1,3,4,5) = $d_{11}d_{33}d_{44}d_{55} \geq 0$ .
A (2,3,4,5)	Det A (2,3,4,5) = $d_{22}d_{33}d_{44}d_{55} \geq 0$ .

$\text{Det}(A) = d_{11}d_{22}d_{33}d_{44}d_{55} \geq 0$ . Therefore  $\text{Det}(A) = 1 > 0$ .

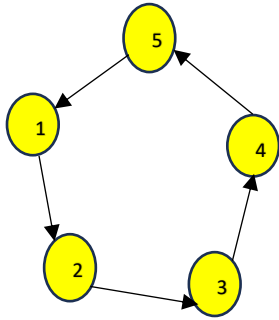
Determinants of  $4 \times 4$  submatrices are obtained. i.e.,  $\text{Det}(1,2,3,4) = d_{11}[d_{22}(d_{33}d_{44} - a_{34}x_{43}) - a_{23}(d_{44}x_{32} - a_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})]$

$$- a_{12}[x_{21}(d_{33}d_{44} - a_{34}x_{43}) - a_{23}(d_{44}x_{31} - a_{34}x_{41}) + x_{24}(x_{31}x_{43} - d_{33}x_{41})]$$

$$+ x_{13}[x_{21}(x_{32}d_{44} - a_{34}x_{42}) - d_{22}(d_{44}x_{31} - a_{34}x_{41}) + x_{24}(x_{31}x_{42} - x_{32}x_{41})]$$

$- x_{14}[x_{21}(x_{32}x_{43} - d_{33}x_{42}) - d_{22}(x_{43}x_{31} - d_{33}x_{41}) + a_{23}(x_{31}x_{42} - x_{32}x_{41})] \geq 0$ . Since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and  $a_{ij}=1$  then;  $\det A(1,2,3,4) = 1 - 0 + 0 - 0 = 1 > 0$ . Therefore, all the determinants of  $4 \times 4$  submatrices are non-negative then the partial matrix can be completed into Wss  $P_o$ -matrix. Hence it has zero completion into a Wss  $P_o$ -matrix.

**Digraph D of order 5 and 5 arcs:** consider a digraph  $D = \{(1,1), (1,2), (2,2), (2,3), (3,3), (3,4), (4,4), (4,5), (5,1), (5,5)\}$  with 5 vertices and 5 arcs given by:



**Figure 6. Digraph D of order 5 and 5 arcs**

The partial matrix that specifies the digraph D is

$$A = \begin{pmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & d_{22} & a_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & d_{33} & a_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & d_{44} & a_{45} \\ a_{51} & x_{52} & x_{53} & x_{54} & d_{55} \end{pmatrix}.$$

By definition of partial Wss  $P_o$ - matrix  $d_1 \geq 0, d_2 \geq 0, d_3 \geq 0, d_4 \geq 0, d_5 \geq 0$ . By definition of partial Wss  $P_o$ - matrix  $d_1 \geq 0, d_2 \geq 0, d_3 \geq 0, d_4 \geq 0, d_5 \geq 0$ .

For  $d_{ii} = 1$ ,  $a_{ij} = 1$ , then the partial matrix becomes  $A = \begin{pmatrix} 1 & 1 & x_{13} & x_{14} & x_{15} \\ x_{21} & 1 & 1 & x_{24} & x_{25} \\ x_{31} & x_{32} & 1 & 1 & x_{35} \\ x_{41} & x_{42} & x_{43} & 1 & 1 \\ 1 & x_{52} & x_{53} & x_{54} & 1 \end{pmatrix}$ . Next, we

compute the principal minors.

Det (1,2) =  $d_{11} d_{22} - a_{12} x_{21}$ . Det (1,2) =  $d_{11} d_{22} - a_{12} x_{21}$  Setting unspecified entries to zero,  $d_{ii} = 1$ ,  $a_{ij} = 1$ , then we have;  $1 - 0 = 1 > 0$ . similarly; Det (1,3) =  $d_{11} d_{33} - x_{13} x_{31}$ , Det (1,4) =  $d_{11} d_{44} - x_{14} x_{41}$

Det (1,5) =  $d_{11} d_{55} - x_{15} a_{51}$ , Det (2,3) =  $d_{22} d_{33} - a_{23} x_{32}$ , Det (2,4) =  $d_{22} d_{44} - x_{24} x_{42}$

Det (2,5) =  $d_{22} d_{55} - x_{25} x_{52}$ , Det (3,4) =  $d_{33} d_{44} - a_{34} x_{43}$ , Det (3,5) =  $d_{33} d_{55} - x_{35} x_{53}$

Det (4,5) =  $d_{44} d_{55} - a_{45} x_{54}$ .

**TABLE 16: Determinants of 2×2 sub-matrices**

Principal submatrix	Principal minor
A (1,2)	Det A (1,2) = $d_{11}d_{22} \geq 0$ .
A (1,3)	Det A (1,3) = $d_{11}d_{33} \geq 0$ .
A (1,4)	Det A (1,4) = $d_{11}d_{44} \geq 0$ .
A (1,5)	Det A (1,5) = $d_{11}d_{55} \geq 0$ .
A (2,3)	Det A (2,3) = $d_{22}d_{33} \geq 0$ .
A (2,4)	Det A (2,4) = $d_{22}d_{44} \geq 0$ .
A (2,5)	Det A (2,5) = $d_{22}d_{55} \geq 0$ .
A (3,4)	Det A (3,4) = $d_{33}d_{44} \geq 0$ .
A (3,5)	Det A (3,5) = $d_{33}d_{55} \geq 0$ .
A (4,5)	Det A (4,5) = $d_{44}d_{55} \geq 0$ .

Determinants of  $2 \times 2$  submatrices are obtained. i.e. Det (1,2) =  $d_{11}d_{22} - a_{12}x_{21} \geq 0$ . Since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and  $a_{ij} = 1$  then;  $\det A (1,2) = 1 - 0 = 1 > 0$ . Therefore, all the determinants of  $2 \times 2$  submatrices are non-negative then the partial matrix can be completed into Wss  $P_0$ -matrix. Hence it has zero completion into a Wss  $P_0$ -matrix.

**TABLE 17: Determinants of 3×3 sub-matrices**

Principal submatrix	Principal minor
A (1,2,3)	Det A (1,2,3) = $d_{11}d_{22}d_{33} \geq 0$ .
A (1,2,4)	Det A (1,2,4) = $d_{11}d_{22}d_{44} \geq 0$ .
A (1,2,5)	Det A (1,2,5) = $d_{11}d_{22}d_{55} \geq 0$ .
A (1,3,4)	Det A (1,3,4) = $d_{11}d_{33}d_{44} \geq 0$ .
A (1,3,5)	Det A (1,3,5) = $d_{11}d_{33}d_{55} \geq 0$ .
A (1,4,5)	Det A (1,4,5) = $d_{11}d_{44}d_{55} \geq 0$ .
A (2,3,4)	Det A (2,3,4) = $d_{22}d_{33}d_{44} \geq 0$ .
A (2,3,5)	Det A (2,3,5) = $d_{22}d_{33}d_{55} \geq 0$ .
A (2,4,5)	Det A (2,4,5) = $d_{22}d_{44}d_{55} \geq 0$ .
A (3,4,5)	Det A (3,4,5) = $d_{33}d_{44}d_{55} \geq 0$ .

Determinants of  $3 \times 3$  submatrices are obtained. i.e., Det (1,2,3) =  $d_{11} (d_{22}d_{33} - a_{23} x_{32}) - a_{12} (x_{21} d_{33} - a_{23} x_{31}) + x_{13} (x_{21} x_{32} - d_{22} x_{31}) \geq 0$ . Since diagonal entries is given to be 1 or larger than other

entries while setting unspecified entries to zero and  $a_{ij}=1$  then;  $\det A(1,2,3) = 1 - 0 + 0 = 1 > 0$ . Therefore, all the determinants of  $3 \times 3$  submatrices are non-negative then the partial matrix can be completed into Wss  $P_o$ -matrix. Hence it has zero completion into a Wss  $P_o$ -matrix.

**TABLE 18: Determinants of  $4 \times 4$  sub-matrices**

Principal sub-matrix	Principal minor
A (1,2,3,4)	Det A (1,2,3,4) = $d_{11}d_{22}d_{33}d_{44} \geq 0$ .
A (1,2,3,5)	Det A (1,2,3,5) = $d_{11}d_{22}d_{33}d_{55} \geq 0$ .
A (1,2,4,5)	Det A (1,2,4,5) = $d_{11}d_{22}d_{44}d_{55} \geq 0$ .
A (1,3,4,5)	Det A (1,3,4,5) = $d_{11}d_{33}d_{44}d_{55} \geq 0$ .
A (2,3,4,5)	Det A (2,3,4,5) = $d_{22}d_{33}d_{44}d_{55} \geq 0$ .

$\text{Det}(A) = (d_{11}d_{22}d_{33}d_{44}d_{55}) - (a_{12}a_{23}a_{34}a_{45}a_{51}) \geq 0$ . Therefore  $\text{Det}(A) = 1 - 1 = 0$ . By definition of completion, determinants of  $4 \times 4$  submatrices are obtained. i.e.,  $\text{Det}(1,2,3,4) = d_{11}[d_{22}(d_{33}d_{44} - a_{34}x_{43}) - a_{23}(d_{44}x_{32} - a_{34}x_{42}) + x_{24}(x_{32}x_{43} - d_{33}x_{42})]$

$$\begin{aligned}
 & - a_{12}[x_{21}(d_{33}d_{44} - a_{34}x_{43}) - a_{23}(d_{44}x_{31} - a_{34}x_{41}) + x_{24}(x_{31}x_{43} - d_{33}x_{41})] \\
 & + x_{13}[x_{21}(x_{32}d_{44} - a_{34}x_{42}) - d_{22}(d_{44}x_{31} - a_{34}x_{41}) + x_{24}(x_{31}x_{42} - x_{32}x_{41})] \\
 & - x_{14}[x_{21}(x_{32}x_{43} - d_{33}x_{42}) - d_{22}(x_{43}x_{31} - d_{33}x_{41}) + a_{23}(x_{31}x_{42} - x_{32}x_{41})] \geq 0.
 \end{aligned}$$

Since diagonal entries is given to be 1 or larger than other entries while setting unspecified entries to zero and  $a_{ij}=1$  then;  $\det A(1,2,3,4) = 1 - 0 + 0 - 0 = 1 > 0$ . Therefore, all the determinants of  $4 \times 4$  submatrices are non-negative then the partial matrix can be completed into Wss  $P_o$ -matrix. Hence it has zero completion into a Wss  $P_o$ -matrix.

All the digraphs mentioned above are not isomorphic to each other.

#### 4.0 CONCLUSION

Upon comprehensive examination and analysis of various digraph patterns, the research arrived at significant conclusions. We used the zero-completion method where unspecified entries in partial Wss  $P_o$ - matrices are set to zero to evaluate principal minors. We concluded that a null graph of order 5 has Wss  $P_o$ - completion. Additionally, we demonstrated that digraphs of order 5 with one to five arcs whether cyclic, acyclic, or with cliques, can be completed into a Wss  $P_o$ - matrix. Therefore, we found that all partial Wss  $P_o$ - matrices have completion into a Wss  $P_o$ - matrix. Conversely, no partial Wss  $P_o$ - matrix formed by digraphs of order 5 with up to 5 arcs lacks the ability to be completed into a Wss  $P_o$ - matrix. Insights gained from these studies could be applied to fill gaps in data from retail surveys, aiding in the prediction of market trends.

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