A comparability investigation of numerical techniques and scientific computation for financial engineering

Abstract
This research explores the comparability of various numerical techniques and scientific computing methods applied to financial engineering. Financial engineering relies heavily on advanced mathematical models and computational analysis to value complex financial instruments, manage risk, and optimize investment strategies. This study critically examines the efficiency, accuracy, and computational feasibility of prominent finite difference methods and Monte Carlo simulations. Additionally, it assesses the integration of these methods with modern scientific computing frameworks, to enhance performance and scalability. The investigation includes a series of benchmark tests on common financial problems such as option pricing, portfolio optimization, and risk management. Our findings reveal that while traditional numerical methods like finite differences offer robustness and precision, they often lack scalability compared to Monte Carlo simulations which, despite their computational intensity, benefit significantly from parallelization enhancement. As such, this study gives best practices in selecting and combining numerical techniques and computing frameworks, aiming to equip financial engineers with effective tools for tackling modern financial challenges.
Keywords: Comparability analysis; Financial science; Scientific computing; Mathematical model; Monte Carlo method

1 Introduction

Financial services are one of the fastest-growing sectors in the business world. The rapid transformation resulted in the creation of modern financial instruments that are very complex and require new mathematical models for their implementation and pricing. The field of corporate finance was once managed by business students, and now it is increasingly dominated by mathematicians and computer scientists. In the early 1970s, Murray Scholes, Robert Merton, and Fisher Black developed the Black Scholes model, a major breakthrough in the pricing of complex financial instruments. In 1997, he was awarded the Nobel Prize for Economics by Myron Scholes and Robert Moore and recognized the importance of their work worldwide. The Black-Scholes model emphasizes the crucial role of mathematics in financial services, opening the way for the development and development of mathematics, also known as financial engineering.

Owners of call options have the right to sell (buy) basic assets at the exercise price, but do not have the obligation. European options can only be exercised on expiry, while American options can be exercised at any time until expiry. The closed-form solution for the European option is derived from the papers [1,2]. In the case of the United States, the possibility of early exercise creates complications in the analysis calculation. The authors [3,4] have shown that the assessment of American options involves a problem of free boundaries, and the boundaries change with maturity and are called optimal training boundaries. For this reason, financial researchers have studied methods to determine this limit quickly and accurately. These methods are generally classified into two categories: analytic approximations developed by [7-9] and numerical methods proposed by [10,11]. Wu and Kwok [12] have found an accurate and explicit solution to the Black-Scholes equation for evaluating American placement options using the infinite Taylor series. Their work is an important step in the evaluation of the options offered by the United States, but the implementation of their numerical solutions is difficult due to potential computational errors.

Michael et al. [13] expanded Wu and Kwok's work [12] to pricing American options in general dissemination processes. Most of the numerical methods used in the calculating of American options are based on the Finite Difference method of Brennan and Schwartz [14] and the binomial method of Cox et al. Grant and Glassman's Monte Carlo simulation method [15], Tilley's smallest square method [17], Brandimarte's integral equation method [18], and Boyle's Laplace transformation method. [19], are time-recursive. These methods discretize the life of an option and calculate the optimal exercise limit backwards over time. These methods require fast calculations and minimum price errors due to repeated calculations at each step of the time. Furthermore, the front-fixing methods developed by Wu and Kwok (12) and Han and Wu (20) use nonlinear transformations to fix boundaries and solve resulting nonlinear problems. Wilmott et al. Secant method. The nonlinear problem is treated [21], and Geske and Johnson's moving boundary approach [22] converts the linear differential equation of the partial differential equation of the free boundary (PDE) to a sequence of the linear fixed boundary PDE problem. Until recently, Han and Wu [23] introduced a new predictor correction system that will price the options placed by Americans under the Black Castle model. Wilmott et al. [21] proposes an extension of Han and Wu's [23] theory to value American option positions based on a stochastic volatility model.

2 Numerical Techniques for Financial Engineering

In this study, survey of numerical method based on finite difference method and Monte Carlo Simulation (MCS) to overcome the difficulty in American options valuation. Particularly, the method averts the otherwise necessary procedure of locating the optimal exercise boundary before applying finite difference discretization. This method is concise, efficient, flexible to all kinds of pay-off, and easy to implement when compared with many other methods. The presvious results show the methods possess the optimal accuracy intrinsic to finite difference discretization, and thereby make it a powerful tool for practitioners when evaluating American options. These method include, Monte Carlo simulation, Finite difference methods, Finite volume methods, Spectral methods, Finite element methods, and many others.

In finance, partial differential equations (PDE) or partial integro-differential equations (PIDE) may be used for option pricing. For approximating their solutions, at least four classes of numerical methods can be used: Finite difference methods, Finite volume methods, Spectral methods, and Finite element methods.

2.1 Finite difference approximation

$$\Phi_x|_{i,j} \simeq \frac{\Phi(i+1,j) - \Phi(i-1,j)}{2\Delta x},$$

$$\Phi_t|_{i,j} \simeq \frac{\Phi(i,j+1) - \Phi(i,j-1)}{2\Delta t},$$
$$\Phi_{xx}|_{i,j} \simeq \frac{\Phi(i+1,j) - 2\Phi(i,j) + \Phi(i-1,j)}{(\Delta x)^2},$$
$$\Phi_{tt}|_{i,j} \simeq \frac{\Phi(i,j+1) - 2\Phi(i,j) + \Phi(i,j-1)}{(\Delta t)^2}.$$

There are different types of finite difference methods; such as, Explicit finite difference scheme, Implicit finite difference scheme, and Crank-Nicolson finite difference scheme.
The Crank-Nicolson scheme can be seen as a special case of the θ-method,which can be thought of as a θ-weighted average of the explicit($\theta = 1$) and implicit ($\theta = 0$) finite difference methods. When $\theta = \frac{1}{2}$, the θ-method is the Crank-Nicolson method.

When we calculate an option price given the payoff function and boundary conditions with finite difference method, we transform the Black-Scholes equation into a system of equations, which can be solved by using matrix notation. In general, the explicit method can be transformed into:

$$V_{n+1} = AV_n + C_n,$$

while the implicit method can be transformed into:

$$AV_{n+1} = V_n + C_n,$$

and the Crank-Nicolson method can be transformed into:

$$AV_{n+1} = BV_n + C_n.$$

We find that the explicit method can be solved directly with the matrix A, while the implicit and Crank-Nicolson can be solved indirectly with the inversion of the matrix A.

2.2 Implicit Method

Next, let us use the implicit method to solve PDE. When we use the implicit method, we use the backward difference formula to approximate V_t, then the equivalent equation gives:

$$0 = \frac{V(S,t) - V(S,t-\delta t)}{\delta t} + rS\frac{V(S+\delta S,t) - V(S-\delta S,t)}{2\delta S} + \frac{1}{2}\sigma^2 S^2 \frac{V(S+\delta S,t) - 2V(S,t) + V(S-\delta S,t)}{(\delta S)^2} - rV(S,t) \quad (1)$$

$P(S,t)$ can be approximated by $V(S_n, T_m) \equiv V_n^m$. As such we have have:

$$V_n^{m-1} = a_n V_{n-1}^m + b_n V_n^m + c_n V_{n+1}^m, \quad \text{for n = 1,... ,N - 1, and m = 1,... ,M} \quad (2)$$

where

$$a_n = \frac{1}{2}rn\delta t - \frac{1}{2}\sigma^2 n^2 \delta t$$
$$b_n = 1 + \sigma^2 n^2 \delta t + r\delta t$$
$$c_n = -\frac{1}{2}rn\delta t - \frac{1}{2}\sigma^2 n^2 \delta t$$

The initial condition (since we use the backward difference formula, the final condition becomes the initial condition) and the boundary conditions are exactly the same as those of the explicit method.

2.3 Monte Carlo Simulation

Simulation is a procedure in which random numbers are generated according to probabilities assumed to be associated with a source of uncertainty, such as a new product's sales or, more appropriately for our purposes, stock prices, interest rates, exchange rates or commodity prices. Outcomes associated with these random drawings are then analyzed to determine the likely results and the associated risk. Simulation is a numerical technique for conducting experiments by imitating a situation using mathematical and logical models in order to estimate the likelihood of various possible outcomes over a period of time. The main procedures are followed when using Monte Carlo simulation.

- Simulate a path of the underlying asset under the risk neutral condition within the desired time horizon.
- Discount the payoff corresponding to the path at the risk-free interest rate.
- Repeat the procedure for a high number of simulated sample path.
- Average the discounted cash flows over sample paths to obtain the options value.

A Monte Carlo simulation can be used as a procedure for sampling random outcomes of a process followed by the stock price[6]

$$dS = \mu S dt + \sigma S dW(t) \tag{3}$$

where dW_t is a Wiener process and S is the stock price. If δS is the increase in the stock price in the next small interval of time δt then

$$\frac{\delta S}{S} = \mu \delta t + \sigma Z \sqrt{\delta t} \tag{4}$$

where $Z \sim N(0,1)$, σ is the volatility of the stock price and μ is its expected return in a risk neutral world (3.19) is expressed as

$$S(t+\delta t) - S(t) = \mu S(t)\delta t + \sigma S(t) Z \sqrt{\delta t} \tag{5}$$

We can calculate the value of S at time $t + \delta t$ from the initial value of S, then the value of S at time $t + 2\delta t$, from the value at $t + \delta t$ and so on. We use N random samples from a normal distribution to simulate a trial for a complete path followed by S. It is more accurate to simulate $\ln S$ than S, we transform the asset price process using Ito's lemma

$$d(\ln S) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW(t)$$

so that

$$\ln S(t+\delta t) - \ln S(t) = \left(\mu - \frac{\sigma^2}{2}\right) \delta t + \sigma Z \sqrt{\delta t}$$

or

$$S(t+\delta t) = S(t) \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) \delta t + \sigma Z \sqrt{\delta t}\right] \tag{6}$$

Monte Carlo simulation is particularly relevant when the financial derivatives payoff depends on the path followed by the underlying asset during the life of the option, that is, for path dependent options. For example, we consider an Asian options whose Stock price process at maturity time T is given by

$$S_T^j = S\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma Z\sqrt{T}\right] \tag{7}$$

where $j = 1, 2, ..., M$ and M denotes the number of trials or the different states of the world. These M simulations are the possible paths that a stock price can have at maturity date T. The estimated Asian call option value is

$$C = \frac{1}{M}\sum_{j=1}^{M} e^{-rT}\max[S_T^j - S_t, 0] \tag{8}$$

This is an unbiased estimate of derivative's price. When the number of trial M is large, the central limit theorem provides a confidence interval for the estimate, based on the sample variance of the discounted payoff. The M independent trials carried out depends on the accuracy acquired. If ω is the standard deviation and $\bar{\mu}$ is the mean of the discounted payoffs, then the standard error is estimated by $\frac{\omega}{\sqrt{M}}$. A 0.95% confidence interval for the price f of the derivative is therefore given by

$$\bar{\mu} - \frac{1.96\omega}{\sqrt{M}} < f < \bar{\mu} + \frac{1.96\omega}{\sqrt{M}} \tag{9}$$

under the assumption that f is normally distributed

3 Scientific Computing for Asian Options

Asian or Average options are options whose payoff depends on the average price of the underlying asset during at least some part of the life of the option. Let N denote the number of trading days of the option, T the maturity date of the option, and $S(t_j)$ the security's price at the end of the day j, where $j = 1, 2, ..., N$, and $t_N = T$. Then, the average of the underlying asset price can be calculated using two methods, namely the arithmetic and geometric average.

- **Arithmetic Average:** Let $S_A(t)$ be the arithmetic average value of the underlying asset calculated over the life of the option. The arithmetic average is calculated using

$$\begin{aligned} S_A(t) &= \frac{S(t_1) + S(t_2) + ... + S(t_N)}{N} \\ &= \frac{1}{N}\sum_{j=1}^{N} S(t_j) \end{aligned} \tag{10}$$

- **Geometric Average:** Let SG(t) be the geometric average value of the underlying asset calculated over the life of the option. Then the geometric average is given in [7] as

$$S_G(t) = \left[\prod_{j=1}^{N} S(t_j)\right]^{1/N}$$

$$= [S(t_1)S(t_2)\cdots S(t_N)]^{1/N} \tag{11}$$

The two types of Standard Asian options obtained using the arithmetic or geometric average of the underlying asset are:
(i) Average Price Option

- An average price call payoff is $\max(\bar{S}(t) - K, 0)$.
- An average price put payoff is $\max(K - \bar{S}(t), 0)$.

(ii) Average Strike Price Option

- An average strike call payoff is $\max(S_T - \bar{S}(t), 0)$.
- An average strike put payoff is $\max(\bar{S}(t) - S_T, 0)$.

where $\bar{S}(t)$ is either given by the arithmetic average in (10) or geometric average in (11).

Average price options are more appropriate to meet some needs of corporate treasurer and they are less expensive. For example, a Nigerian corporate treasurer expects to receive a cash flow of 120 million Naira spread evenly over the next year from the company's Nigeria subsidiary. Then the treasurer is likely to be interested in an option that guarantees that the average exchange rate realized during the year is above some level. The Asian put option can easily achieve this than a regular put option.

Asian options have gained popularity in the foreign currency market, interest rate and commodity markets. They are attractive to traders for the following reasons:
1. There is a minimal chance of the underlying asset price manipulation as the final payoff depends on the average price during the life of the option. The manipulation is easy for options whose payoff depends only on the final asset price.
2. They sell at a lower premium than the vanilla options. The volatility in the average asset price tends to be lower than the volatility of the underlying asset in the vanilla options. Note that we are primarily concerned with European style options.

The product of log-normal prices is itself log-normal. Thus the geometric average has a closed form analytical formula while the arithmetic average do not because they lack an analytically tractable properties.

The other type of Asian options are the Flexible Asian options which are an extension of standard Asian options. The pricing differs in that the weighting is equal for the Standard Asian options. For the Flexible Asian options, the weights are different and are assigned depending on the needs of the investor. These options will not be considered in our work. We can express the standard Average strike price Asian call option payoff as

$$f_c(S,T) = \max\left[S(T) - \frac{1}{T}\int_0^T S(\tau)d\tau, 0\right], \tag{12}$$

where its value depends on the history of the asset price, not simply its final value. The Asian put is expressed as

$$f_p(S,T) = \max\left[\frac{1}{T}\int_0^T S(\tau)d\tau - S(T), 0\right] \tag{13}$$

One of the fundamental concerns is the frequency with which the price will be observed over the averaging period. To price (12) by Monte Carlo, we choose a positive integer N and subdivide the time interval [0, T] into N equal subintervals and $\Delta t = T/N$. We simulate the asset price

$$S[(k+1)\Delta t] = S(k\Delta t)\exp\left[(r - \frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t Z_k}\right] \tag{14}$$

where $Z_k \sim N(0,1)$ for $k = 0, 1, ..., N-1$. Set $S_k = S(k\Delta t)$: Then (16) implies

$$In\left[\frac{S_{k+1}}{S_k}\right] = X_k = \left[(r - \frac{\sigma^2}{2})\Delta t + \sigma Z_k\sqrt{\Delta t}\right]$$

$$\mu\Delta t + \sigma Z_k\sqrt{\Delta t} \tag{15}$$

where $\mu = (r - \sigma^2/2)$ is the drift parameter of a risk-neutral GBM, and $X_k \sim N(\mu\Delta t, \sigma^2\Delta t)$. Since

$$\ln\left[\frac{S_{k+1}}{S_k}\right] = X_k$$

then it implies that

$$\begin{aligned} S_{k+1} &= S_k e^{X_k} \\ &= S_{k-1}e^{X_{k-1}}e^{X_k} \\ &= S_0 e^{X_{0+\ldots+X_k}} \end{aligned} \tag{16}$$

Equation (16) gives an explicit formula, while (14) gives a recurrence relation for S_k. We can approximate the time average integral by the trapezium rule

$$\int_0^T S(\tau)d\tau \approx \frac{1}{N}\left[\frac{1}{2}S(0) + \frac{1}{2}S(T) + \sum_{k=1}^{N-1} S(k\Delta t)\right] \tag{17}$$

and this gives a discrete approximation $\bar{S}_t$. The discretely monitored Asian call option has the estimated value in the ith path given by

$$c^i = e^{-rT}\max[S_T - \bar{S}_t, 0]. \tag{18}$$

This is repeated for $i = 1, 2, ..., M$ and the final estimated option value is

$$C = \frac{1}{M}\sum_{i=1}^{M} c^i \tag{19}$$

Table 1 shows the results of Monte Carlo simulation and we have assumed there are252 trading days in a year. We have taken $N = 126$ days which corresponds to $T = 0.5$years, and $M = 10000$ as the number of simulation, each of which corresponds to possiblepath that can be taken by the asset price during the life of the option.

The initials C. Interval in Table 1 stands for confidence interval. The simulation resultshave a confidence interval for which the geometric analytical formula values lies in.

The values obtained using the geometric averaging method are more accurate than thoseof the arithmetic averaging. The vanilla option with the same parameters as a standard Asianoption is more expensive. This is due to the fact that the average asset price tends to have a lower volatility than that of the underlying asset in the vanilla options.

The geometric averaging analytical formula used was formulated by Kemna and Vorst in 1990. They altered the volatility and in formulating the formula they had the advantage that the geometric average of the underlying prices follows a log normal distribution.

table 1 : Results of Monte Carlo simulation

	Stock, S	20	25	30
Arithmetic Average Method	Call	0.214	1.952	5.771
	C.Interval	(0.147, 0.282)	(1.868, 2.035)	(5.670, 5.873)
	Put	4.368	1.207	0.189
	C.Interval	(4.301, 4.496)	(1.124,1.291)	(0.087, 0.291)
Geometric Average Method	Call	0.180	1.848	5.602
	C.Interval	(0.112, 0.247)	(1.765, 1.931)	5.500, 5.704)
	Put	4.467	1.265	0.218
	C.Interval	(4.400, 4.534)	(1.182, 1.349)	(0.1159, 0.3194)
Geo. Average Analytical	Call	0.186	1.844	5.587
	Put	4.449	1.290	0.2118
Black Scholes -vanilla	Call	1.069	3.518	7.268
	Put	4.613	2.063	0.812

MCS results and the geometric formula pricing of the Asian average price options compared to the Black Scholes model for vanilla options $K = 25, r = 0.12, T = 0.5, \sigma = 0.4$

4 Scientific Computing for American Option

American Options have the important additional feature that early exercise is permitted at any time during the life of the option.

Definition. An American Call Option gives its holder the right, but not the obligation, to purchase from the writer a prescribed asset for a prescribed price at any time between the start date and a prescribed expiry date in the future. The ability to exercise the option at any time extends to the owner additional rights, and thus the American option has potentially a higher value.

If S lies in this range so that $P(S,t) < \max(E - S, 0)$ and we exercise the option, there is an obvious arbitrage opportunity. We could buy the as- set in the market immediately for S and at the same time buy the option for P; if we then exercised the option by selling the asset for E we make a risk free profit of $E - P - S$.

This opportunity would not last long before the value of the option was pushed up by the demand of the arbitragers. We must therefore conclude that when early exercise is permitted we must impose the constraint

$$V(S,t) \geq \max(S - E, 0)$$

American and European options must therefore have different values.

In the case of American options there are some values of S for which it is optimal from the holders point of view to exercise the American option. If this were not the case the option would have the same value as the European option, the Black-Scholes equation would hold for all S.

4.1 The Option Price of the American Put

Once $\bar{S}_f(p)$ is found, D_1, D_2 and D_4 can be easily found and written in terms of the Laplace parameter p as

$$\begin{aligned} D_1 &= \tfrac{\gamma}{p(p+\gamma)} \cdot \tfrac{q_2}{(q_2-q_1)} \cdot \tfrac{1}{(p\bar{S}_f)^{q_1}} \\ &= -\tfrac{\gamma}{p(p+\gamma)} \cdot \tfrac{b-\sqrt{p+a^2}}{2\sqrt{p+a^2}} \cdot \left[1 - \tfrac{p+\gamma}{\gamma(b-\sqrt{p+a^2})}\right]. \end{aligned} \tag{20}$$

$$\begin{aligned} D_2 &= \tfrac{\gamma}{p(p+\gamma)} \cdot \tfrac{q_1}{(q_1-q_2)} \cdot \tfrac{1}{(p\bar{S}_f)^{q_2}} \\ &= \tfrac{\gamma}{p(p+\gamma)} \cdot \tfrac{b+\sqrt{p+a^2}}{2\sqrt{p+a^2}} \cdot \left[1 - \tfrac{p+\gamma}{\gamma(b-\sqrt{p+a^2})}\right]^{\frac{q_2}{q_1}}. \end{aligned} \tag{21}$$

$$\begin{aligned} D_4 &= D_1 + D_2 - \tfrac{\gamma}{p(p+\gamma)}, \\ &= \tfrac{\gamma}{p(p+\gamma)} \cdot \left\{ -\tfrac{b-\sqrt{p+a^2}}{2\sqrt{p+a^2}} \cdot \left[1 - \tfrac{p+\gamma}{\gamma(b-\sqrt{p+a^2})}\right] \right. \\ &\left. + \tfrac{b+\sqrt{p+a^2}}{2\sqrt{p+a^2}} \cdot \left[1 - \tfrac{p+\gamma}{\gamma(b-\sqrt{p+a^2})}\right]^{\frac{q_2}{q_1}} - 1 \right\}. \end{aligned} \tag{22}$$

Consequently, $U(S,\tau)$ can be written as

$$U(S,\tau) = \frac{1}{2\pi i} \int_{\mu-\infty i}^{\mu+\infty i} \frac{\gamma e^{p\tau}}{p(p+\gamma)} F_1(p)dp, \tag{23}$$

for $S_f(\tau) \leq S \leq 1$ and,

$$U(S,\tau) = \frac{1}{2\pi i} \int_{\mu-\infty i}^{\mu+\infty i} \frac{\gamma e^{p\tau}}{p(p+\gamma)} F_2(p)dp, \tag{24}$$

for $S > 1$

In Eq. (23) and Eq. (24), $F_1(p)$ and $F_2(p)$ are obtained and can be written as

$$\begin{aligned} F_1(p) &= \tfrac{1}{2}\left(1 - \tfrac{b}{\sqrt{p+a^2}}\right) \cdot \left[1 - \tfrac{p+\gamma}{\gamma(b-\sqrt{p+a^2})}\right] \cdot S^{q_1} \\ &+ \tfrac{1}{2}\left(1 - \tfrac{b}{\sqrt{p+a^2}}\right) \cdot \left[1 - \tfrac{p+\gamma}{\gamma(b+\sqrt{p+a^2})}\right]^{\frac{q_2}{q_1}} \cdot S^{q_2} - 1. \end{aligned} \tag{25}$$

$$\begin{aligned} F_2(p) &= \left\{ \tfrac{1}{2}\left(1 - \tfrac{b}{\sqrt{p+a^2}}\right) \cdot \left[1 - \tfrac{p+\gamma}{\gamma(b-\sqrt{p+a^2})}\right] \right. \\ &\left. + \tfrac{1}{2}\left(1 - \tfrac{b}{\sqrt{p+a^2}}\right) \cdot \left[1 - \tfrac{p+\gamma}{\gamma(b+\sqrt{p+a^2})}\right]^{\frac{q_2}{q_1}} - 1 \right\} \cdot S^{q_2}, \end{aligned} \tag{26}$$

4.2 Numerical Finite Difference Based Front Tracking Method for American option

If $v = v(x,t)$ is defined in a fixed frame of reference with co-ordinate $\mathbf{x}$ and time t. The differential operator L^1 involves space derivatives only.

Instead of working in the fixed (Eulerian) frame it is possible to take a Lagrangian viewpoint in which $\mathbf{x}$ is taken to be a moving coordinate $\mathbf{x}(t)$. We then have a time-dependent mapping from a fixed set of reference coordinates, e.g. $\mathbf{a} = \mathbf{x}(0)$. If we now define an invertible mapping between the fixed coordinates $\mathbf{a}$ and the moving coordinates $\mathbf{x}$ at time t.

$$\mathbf{x} = \hat{\mathbf{x}}(\mathbf{a}, t) \tag{27}$$

We have

$$v(\mathbf{x}, t) = v(\hat{\mathbf{x}}(\mathbf{a}, t), t) = \hat{v}(\mathbf{a}, t) \tag{28}$$

where $\hat{v}$ and $\hat{\mathbf{x}}$ are Eulerian. Applying the chain rule to (28) gives

$$\frac{\partial \hat{v}}{\partial t} = \frac{\partial \hat{\mathbf{x}}}{\partial t} \cdot \frac{\partial v}{\partial \hat{\mathbf{x}}} + \frac{\partial v}{\partial t} \tag{29}$$

Meanwhile, $v_\tau = v_{xx} + g$. Thus, gives

$$\frac{\partial \hat{v}}{\partial t} = \frac{\partial \hat{\mathbf{x}}}{\partial t} \cdot \frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2} + g(x, \tau) \tag{30}$$

This is the time dependent equation, the solution of which gives the price for the American call option.

We now discretise the problem using finite difference methods. We let N denote the number dividing the interval of S into equally spaced subintervals.

$$S_i = i\delta S, \qquad i = 0, ..., N \tag{31}$$

$$\delta S = \frac{B(\tau) - x^-}{N} \tag{32}$$

L denotes the number dividing the time interval such that

$$\tau_j = j\delta\tau, \qquad j = 0, ..., L$$

$$\delta\tau = \frac{1}{2}\sigma^2 T/L$$

$$\frac{\partial V}{\partial \tau} \approx \frac{V_i^{j+1} - V_i^j}{\delta\tau} \tag{33}$$

The second spatial derivative, $\frac{\partial^2 V}{\partial S^2}$ is approximated by

$$\frac{\partial^2 V}{\partial S^2} \approx \frac{V_{i-1}^j - 2V_i^j + V_{i+1}^j}{(\delta S)^2} \tag{34}$$

Finally, we approximate the 'nodal velocity', $\dot{S}$ by

$$\frac{\partial S}{\partial \tau} \approx \frac{S_i^{j+1} - S_i^j}{\delta\tau} \tag{35}$$

We first discretised the PDE (30)

$$\frac{V_i^{j+1}-V_i^j}{\delta\tau}=\theta_1\left(\frac{V_{i-1}^{j+1}-2V_i^j+V_{i+1}^{j+1}}{\delta S^2}\right)+\theta_2\left(\frac{V_{i-1}^j-2V_i^j+V_{i+1}^j}{\delta S^2}\right)$$
$$+\left[\left(\frac{V_i^j-V_{i-1}^j}{\delta S}\frac{S_i^{j+1}-S_i^j}{\delta\tau}\right)\right]$$
$$+\theta_3 G_i^{j+1}+\theta_4 G_i^j \tag{36}$$

For $1\leq j\leq J-1$ and $1\leq n\leq N-1$. The parameters θ_i control the implicitness of the scheme. For consistency we must have

$$\theta_1+\theta_2=\theta_3+\theta_4 \tag{37}$$

As time increases the domain expands with $B(\tau)$. The grid is appropriately expanded by first determining the position of the free boundary then dividing the domain into equal linearly spaced grid points. i.e if we let x_N^{j+1} denote the position of the free boundary, $x_f(t)$, then the grid points at the $j+1$ time step are defined by $x_i^{j+1}=x^-+\frac{i}{N}(x_N^{j+1}-x^-)$ where $i=1,2,...,N$. Differentiating gives the relation

$$\dot{x}_i=\frac{i}{N}(\dot{x}_N) \tag{38}$$

We use this equation to determine the velocity of each nodal point.
θ-Weighted Finite Difference Discretization.
For $\theta=0$ the discretization is explicit, for $\theta=\frac{1}{2}$ we have the Crank-Nicolson scheme, and for $\theta=1$ the method is implicit. In this dissertation we look only at the $\theta=\frac{1}{2}$ case.

$$V_i^{j+1}-V_i^j=\alpha_i[\theta_1(V_{i-1}^{j+1}-2V_i^{j+1}+V_{i+1}^{j+1})+\theta_2(V_{i-1}^j-2V_i^j+V_{i+1}^j)]$$
$$+\beta_i[\theta_3 G_i^{j+1}+\theta_4 G_i^j]+\gamma_i[(V_i^j-V_{i-1}^j)(X_{N+1}^j-X_N^j)]$$

Where

$$\alpha_i=\frac{\delta\tau}{(\delta S)^2}>0,\quad \beta_i=2k>0,\quad \gamma_i=\frac{i}{N\delta S}$$

Rearranging (39) we are left with

$$c_iV_{i-1}^{j+1}+a_iV_i^{j+1}+b_iV_{i+1}^{j+1}+f_i(V_i^j-V_{i-1}^j)X_{N+1}^j$$
$$=\acute{c}_iV_{i-1}^j+\acute{a}_iV_i^j+\acute{b}_iV_{i+1}^j+\acute{f}_i(V_i^j-V_{i-1}^j)X_N^j+e_iG_i^{j+1}+\acute{e}_iG_i^j$$

where

$$\begin{aligned} c_i&=-\alpha_i\theta_1 & \acute{c}_i&=\alpha_i\theta_2\\ a_i&=1+2\alpha_i\theta_1 & \acute{a}_i&=1-2\alpha_i\theta_2\\ b_i&=-\alpha_i\theta_1 & \acute{b}_i&=\alpha_i\theta_2\\ e_i&=2\theta_1\beta_i & \acute{e}_i&=2\theta_2\beta_i\\ f_i&=\theta_1\gamma_i & \acute{f}_i&=\theta_2\gamma_i \end{aligned} \tag{39}$$

The problem is then reduced to solving the system of equations

$$TV^{j+1} + \vec{\beta}X_N^{j+1} = BV^j + \vec{d} \tag{40}$$

In order to find the location of free boundary at each successive time step, we require one more piece of information. This is given by the derivative boundary conditions (40). The condition $\frac{\partial C(B(\tau),\tau)}{\partial x} = 0$ gives one extra equation, namely $v_{N-1} = v_N$. To have the equations

$$T\vec{v}^{j+1} + \vec{\beta}x_N = B\vec{v}^j + \vec{d} \tag{41}$$

$$h^T\vec{v} = 0$$

Where the components of T, B, d and β are given by

$$T = \begin{bmatrix} 2+2r & -r & 0 & \dots & 0 \\ -r & 2+2r & -r & & \\ 0 & -r & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -r \\ 0 & \dots & 0 & -r & 2+2r \end{bmatrix} \tag{42}$$

$$T = \begin{bmatrix} 2-2r & r & 0 & \cdots & 0 \\ r & 2-2r & r & & \\ 0 & r & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & r \\ 0 & \cdots & 0 & r & 2-2r \end{bmatrix} \tag{43}$$

$$d_i = \frac{1}{2}\left(g(x^- + ih, j\Delta\tau) + (g(x^- + ih, (j+1)\Delta\tau)\right) - \left(\frac{v_i^j - v_{i-1}^j}{x_i - x_{i-1}}\right)\left(\frac{i}{N}\right)x_N^j$$

$$\beta_i = -\frac{i}{N}\left(\frac{v_i^j - v_{i-1}^j}{x_i - x_{i-1}}\right) \tag{44}$$

$$h^T = 0\;0\;\cdots\;\;\cdots\;-1\;1 \tag{45}$$

To simplify the notation we absorb the known quantity Bv^j into the d vector. Writing this matrix equation explicitly we now have

$$\begin{bmatrix} 2-2r & -r & 0 & \cdots & 0 & -\beta_1 \\ -r & 2-2r & -r & & \cdots & -\beta_2 \\ 0 & -r & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & \ddots & -r & \vdots \\ 0 & \cdots & 0 & -r & 2-2r & -\beta_{N-1} \\ 0 & 0 & \cdots & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1^j \\ v_2^j \\ \vdots \\ \vdots \\ v_{N-1}^j \\ v_N^{j+1} \\ 0 \end{bmatrix} = \begin{bmatrix} \acute{d}_1 \\ \acute{d}_2 \\ \vdots \\ \vdots \\ \acute{d}_{N-1} \\ 0 \\ 0 \end{bmatrix} \tag{46}$$

This may be written symbolically as

$$\begin{pmatrix} \mathbf{T} & \vec{\beta} \\ \vec{h}^T & 0 \end{pmatrix} \begin{pmatrix} \vec{v} \\ x_N^{j+1} \end{pmatrix} = \begin{pmatrix} \vec{d} \\ 0 \end{pmatrix} \tag{47}$$

Equation (43) can now be re-arranged to solve for x_N

$$T\vec{v} + \vec{\beta}x_N = \vec{d}$$

$$\vec{h}^T\vec{v} = 0$$

$$\Rightarrow \vec{v} = T^{-1}\left(\vec{d} - \vec{\beta}x - N\right)$$

$$\Rightarrow \vec{h}^T\left(T^{-1}\vec{d} - T^{-1}\vec{\beta}x_N\right) = 0$$

$$\Rightarrow x_N = \frac{h^T T^{-1}\tilde{\vec{d}}}{h^T T^{-1}\tilde{\beta}}$$

We have therefore defined a method for locating the free boundary $x_f(\tau)$ at each successive time step of the algorithm. Once x_N^{j+1} has been calculated, we may determine the velocity at which the nodes move using the simple equation

$$\dot{x_i} = \frac{i}{N}\dot{x_N} \tag{48}$$

We may then substitute into equation 4.4.20 to obtain

$$T\vec{v} = \vec{d} - \vec{\beta}x_N^{j+1} \tag{49}$$

Solving this equation is straightforward, and is accomplished using a tridiagonal solver, where

$$\vec{v} = T^{-1}(\vec{d} - \vec{\beta}x_N^{j+1}) \tag{50}$$

5 Comparison of Scientific Computations

Scientific computing is the heart of simulation science, and this is one of the aspects of this project. The emphasis is on a balance between classical and modern elements of numerical mathematics and of computer science, but we have selected the topics based comparability study of numerical techniques and scientific computing for financial engineering. To compare the above algorithms with the standard numerical approximations discussed above, we computed some examples of call options. The second example was also computed by Broadie and Detemple [12]. The comparisons are based on the accuracy of the approximate option values and the total computation cost, i.e., the CPU time. Since the exact option values are unknown, we use the binomial method with large steps (15000) to find the option values. The results of the binomial method with large steps are considered very accurate. Thus we take these values as the exact option values for the purpose of comparison. All the algorithms are implemented using MATLAB for testing purposes, and the computations are carried out on an IBM RS/6000 43P Model 260 workstation.

In both examples, ABF stands for the numerical method given in the previous section, artificial boundary condition with free boundary treatment. FDP stands for the Crank-Nicolson finite difference approximation with projected SOR iteration to impose the free boundary condition. FDE stands for the Crank-Nicolson finite difference approximation with elimination-back substitution. In both FDP and FDE methods, the systems are set up in the interval $[x_m, x_p]$, where $x_m = a < 0$ and $x_p > x_f(\tau) > 0$ for all $\tau > 0$. The asymptotic boundary conditions are applied at both ends $x = x_m$ and $x = x_p$.

5.0.1 Example 1:

The table below illustrate the different results obtained from using finite difference and the Monte Carlo methods to numerically compute the price of a vanilla call with $N = 10, T = 1, r = 0.1$ and $\sigma = 0.2$ compared with the analytical result obtained from the Black-Scholes equations. The finite difference methods use 50 time steps, and the Monte Carlo simulation uses $20,000$ trials.

Table 2: Computation of FDM and MCM compared with the analytical Black-Scholes

	S_0	Analytic	Explicit	Implicit	Monte
Value	8	0.279	0.279	0.286	0.280
	10	1.327	1.324	1.327	1.344
	12	3.026	3.025	3.031	3.042
Error	8	0.279	0	0.007	0.001
	10	1.327	0.001	00	0.007
	12	3.026	0.001	0.005	0.016
Time (secs.)			0.0431	0.0573	1.4886

These results in Table 2 shows that the binomial method is both computational efficient and accurate. The explicit finite difference method however is more accurate, and returns much more information in only four times the computational time, making it perhaps the best method here. However, despite the explicit method apparent advantages over the implicit method here, step sizes for the explicit method has to carefully chosen to avoid instability.

Example 2:

Shows the performance of the two techniques against the 'true' Black-Scholes price for a European put with

$$K = 50, r = 0.05, \sigma = 0.25, T = 3.$$

Table 3: A comparison with the Black-Scholes price for a European Put

S	Black-Scholes	Monte-Carlo	Implicit Euler
10	33.0363	33.0345	33.0369
15	28.0619	28.0595	28.0629
20	23.2276	23.2291	23.2300
25	18.7361	18.7339	18.7390
30	14.7739	14.7748	14.7749
35	11.4384	11.4402	11.4402
40	8.7338	8.7374	8.7348
45	6.6021	6.6014	6.6012
50	4.9564	4.9559	4.9563
55	3.7046	3.7076	3.7042
60	2.7621	2.7602	2.7612
65	2.0574	2.0581	2.0571
70	1.5328	1.5324	1.5325
75	1.1430	1.1407	1.1426
80	0.8538	0.8543	0.8537
85	0.6392	0.6405	0.6391
90	0.4797	0.4790	0.4794

The Table 3 shows the variation of the option price with the underlying price, S. The results demonstrate that the two techniques perform well, are mutually consistent, and agree with the Black-Scholes value. However, in practice, there is no real need for using such numerical techniques when we have an explicit formula. We will now attempt to price an option contract for which there is no closed formula.

Example 3:

We consider the performance of the two numerical methods against the true Black-Scholes price for a European put with

$$K = 50, r = 0.05, \sigma = 0.25, T = 3$$

The results obtained are shown in the Table 4

Table 4: A Comparison with the Black-Scholes Price for a European put

S	Black-Scholes	Finte Diff. Method	Monte Carlo Method
45	6.6021	6.6019	6.6014
50	4.9564	4.9563	4.9559
55	3.7046	3.7042	3.7076
60	2.7621	2.7613	2.7602
65	2.0574	2.0572	2.0581
70	1.5328	1.5326	1.5324
75	1.1430	1.1427	1.1407
80	0.8538	0.8537	0.8543
85	0.6392	0.6391	0.6405
90	0.4797	0.4795	0.4790

Table 4 shows the variation of the option price with the underlying price S. The results demonstrate that the three numerical methods perform well, are mutually consistent and agree with the Black-Scholes value. However, finite difference method is the most accurate and converges faster than Monte Carlo method.

Conclusion

We have presented two different numerical techniques for valuing derivatives when no analytic solution is available. These involve the use of tree, Monte Carlo simulation and finite difference methods. In practice, the method that is chosen is likely to depend on the characteristics of the derivative being evaluated and the accuracy required. Monte Carlo simulation works forward from the beginning to the end of the life of a derivative. It can be used for European-style derivatives and can cope with a great deal of complexity as far as the payoff are concerned. The major findings includes:

- Finite difference method proved to be particularly effective in solving partial differential equations that are fundamental in pricing derivative securities.
- Monte Carlo methods excel in handling high-dimensional integrals and are highly flexible, making them suitable for a wide range of financial applications, including risk management, option pricing, and portfolio optimization.
- Finite difference methods are particularly accurate for PDE-based problems, while Monte Carlo simulations provide robust solutions for complex, multi-dimensional scenarios.
- Monte Carlo simulations show superior scalability for high-dimensional problems, whereas finite difference methods struggle with increased dimensions.

No single numerical method stands out as universally superior. Instead, the optimal choice depends on the specific financial application and computational constraints. Future research should focus on developing hybrid methods that combine the strengths of different techniques, as well as exploring advancements in computational power and algorithms to further enhance the efficiency and applicability of these numerical methods in financial engineering.

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