Original Research Article

A scienti_c computing analysis of _nancial Black-Scholes and Monte Carlo di_erential equation: An American option

Abstract
This study presents a systematic computing analysis of _nancial models, precisely focusing on the Black-Scholes and Monte Carlo derivative equations, to evaluate American options. American selections are exercised at any time before expiration, posing unique challenges in _nancial modelling due to their complex early exercise features. The Black-Scholes formulation gives a foundational framework for choice pricing, utilizing partial derivative formulations to estimate the fair value of options under de_nite assumptions. Nevertheless, because of its restriction, Monte Carlo computations are taken to give a better simulation scheme to overcome the posed challenges by computing wider likely underlying price path assets. This study implements a computational approach to compare the e_cacy of the Black-Scholes formulation and Monte Carlo methods in selected American pricing. A numerical scheme for solving the Black-Scholes derivative systems and a variance reduction technique for enhancing the e_ectiveness of Monte Carlo simulations are adopted. Our analysis reveals that while the Black-Scholes model provides a useful approximation, Monte Carlo simulations deliver more accurate and exible results for American options, especially in scenarios with substantial volatility and early exercise potential. The outcomes underscore the importance of sophisticated numerical methods in _nancial engineering and highlight the trade-o_s between analytical tractability and numerical precision.
Keywords: Comparability analysis; Financial science; Scienti_c computing; Mathematical model; Monte Carlo method

1 Introduction

Finance is the fastest-rising and highly speedily varying area in corporate organisation. Following its swift variation, _nancial contemporary apparatuses are now really intricate. As such, innovative scienti_c formulations are indispensable for implementing and pricing these novel

monetary apparatuses. The _nance commercial world, earlier controlled by corporate apprentices, is nowadays governed by computer and mathematician experts. In the early 1970s, Fisher Black, Robert Merton, and Myron Scholes accomplished essential success in the _nancial complex pricing tools by formulating a well-known model called the Black-Scholes technique. In 1997, the strength and essential of the modelled technique was acknowledged globally when Robert Merton and Myron Scholes were given an Economics Nobel Prize. The Black-Scholes formulation demonstrated the importance of mathematics in the _nance _eld. This led to the evolution and victory recorded in _nancial engineering or _nancial mathematical _elds.
The cell (put) option owner has no obligation but the right to buy (sell) a primary bene_t

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at the price exercise. The European option may be practised or executed at the expiration time only, but the American option can be any time executed and practised until the expiration time. A closed-form solution for the European option is developed from the ideas in [1,2].
For the American option, due to the early possibility practice, the problem of pricing gives a complicated analytical calculation. The scholars [3,4] presented that the American valuation option establishes a boundary-free problem for time changes in the boundary to maturity, frequently referred to as boundary optimum exercise. Therefore, _nance scientists have considered approaches to correctly and rapidly _nd the boundary optimum exercise. The techniques are fundamentally di_erent, i.e., approximating analytical schemes as demonstrated by [7-9] and computational techniques as illustrated by [10,11]. Wu and Kwok [12] _nd an explicit and precise solution for the Black-Scholes model in evaluating the Put American option through the Taylor series of inde_nitely several parameters. Their study gives an exceptional outcome for the Put American option valuations; meanwhile, it is complex to carry out the numerical analysis and solution. The in_nite totality yields several computational errors.
Michael et a. [13] studied expanded the Wu and Kwok [12] to American pricing option with wide-ranging distribution procedures. The most common computational scheme for the

American pricing option includes the _nite di_erence technique by Brennan and Schwartz [14], binomial scheme by Cox et al. [15], Monte Carlo computing scheme by Glasserman [16], least squares technique by Tilley [17], integral-equation technique by Brandimarte [18], and the Laplace transform procedure by Boyle et al. [19] are recursive time-depended. The lifetime discretization of options is the basis of their idea in calculating the backward boundary optimal time exercise. Since recursive time-dependent ways produced a repeated time step calculation, it requires quick time computations and lower pricing errors. Moreover, the front-_xing schemes [12, 20] formulated use a quasilinear transformation to solve the resultant quasilinear model and _x the boundary. A Secant technique formulated by Wilmott et al. [21] is required to provide a solution to a quasilinear equation, and a boundary-moving procedure presented by Geske and Johnson [22] changes the resultant boundary-free linear partial derivative model into a _xed-boundary linear sequence PDE model. Recently, Han and Wu [23] developed a novel corrector-predictor method for the Put American pricing option by applying the Black-Scholes formulation. After that, Wilmott et al. [21] presented an expansion of Han and Wu [23] to provide a valuation for the Put American option with volatility stochastic formulation. The investigation adopts a numerical procedure to analyse the e_ectiveness of the Black-Scholes formulation and Monte Carlo schemes for American pricing options. Numerical procedures for solving the Black-Scholes partial derivative models and variance reduction techniques for enhancing the pro_ciency of Monte Carlo simulations are adopted.

2 The Black-Scholes Formulation

Following the derivative of lognormal dynamics, the Merton, Myron Scholes, and Fischer Black assumptions are developed for the European pricing option formulation. Additional assumptions were made as:

_ The stock rate of return probability is lognormally distributed with the same return freerisk and mean,

_ Transaction taxes or costs ignored,

_ There is no arbitrage risk-free occur,

_ During the option life dividends do not exist,
_ The interest risk-free rate in time is constant and known,
_ The return variance life option is _xed,
_ There is continuous trading of asset with continuous varying price.
Given the stock price as
dS = _Sdt + _SdW (1)
where W is the Wiener process, _ dentes volatility, and _ de_nes trend. The call option is f and
other contingent derivative presents S where f is a function t and S. Thus, by Ito's lemma
df = _@F
@S
_S +
@F
@t
+
1
2
@2F
@S2 _2S2_dt +
@F
@S
_SdW (2)
Equations (1) and (2) in discrete for
@S = _S@t + _S@W
@f = _@F
@S
_S +
@F
@t
+
1
2
@2F
@S2 _2S2_@t +
@F
@S
_SdW (3)
The underlying Wiener procedure for S and f are equal and can be removed through right
choice of portfolio for the derivative and stock. A portfolio is chosen as:
+
@f
@S
: shares and □ 1 : derivative

It is a derivative of short holder and share long amount of @f=@S. De_ne _ to be the portfolio
value to have
_ =
@f
@S
S □ f: (4)
The portfolio value is the change in @_ with time variance of @t is describes as:
@_ =
@f
@S
_S □ @f: (5)
Using model (3) in model (5) gives
@_ = _□
@f
@t □
1
2
@$_2$f
@S$_2$ _$_2$S$_2$_@t (6)
By the removal of the term @W, the portfolio less-risk. A return must then earn equivalent to
other free-risk securities. Hence,
__ = r_@t (7)
where the interest free-risk rate is r. Applying (4) and (6) on (7) results to
_1
2
@$_2$f
@S$_2$ _$_2$S$_2$ +
@f
@t _@t = □r _@f
@s
S □ f_@t: (8)
3
To have,
1
2
@$_2$f
@S$_2$ _$_2$S$_2$ + rS
@f
@S
+
@f
@t
= fr (9)
This gives Black-Scholes-Merton derivative model.

Solving model (9) results to an European pricing option analytical, which is exercised at the due time and not applicable for early exercise pricing. Therefore, the European boundary constraints option can be used on model (9).

2.1 The Black-Scholes Model Solution

The condition for payo_ is given as $\max(S - N; 0) = f(S; t = T)$. The boundary upper and upper constraints are consider as $c(S;K;t);C(S;K;t) \geq -Ke^{(t-T)}+S_t$ and $c(S;t;K);C(S;t;K) \leq S_t$. These are the satis_ed PDE constraints

Let $\tau = T - t$; where T denotes time expiration and t is time; thus, mode; (9) becomes

$$\frac{\partial f}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial S} rS - fr \quad (10)$$

Let

$$\frac{\partial f}{\partial S} = \frac{1}{S}\frac{\partial f}{\partial y}$$

and

$$\frac{\partial^2 f}{\partial S^2} = -\frac{1}{S^2}\frac{\partial f}{\partial y} + \frac{1}{S^2}\frac{\partial^2 f}{dy^2} \quad (11)$$

If $y = \ln S$;, new notation is introduced

$$w(y;\tau) = e^{r\tau} f(y;\tau)$$

Using (11), the di_usion PDE Black-Scholes model gives

$$\frac{\partial w}{\partial \tau} =$$

_2
2
@2w
@y2 + _r □
_2
2 _
@w
@y
(12)
with a basic normal solution as:
_(y; _) =
1
_p2__
exp h□
[(r □ _2
2)_]2 + y
2_2_ i (13)
Hence, equation (13) results to:
w(y; z) = Z 1
□1
_(y □ _; _)w(_; 0)d_ (14)
Introducing the payo_ constraint to have the basic solution for equation (13) as:
w(y; _) =
1
_p2__ Z 1
□1
max(e_ □ K; 0)exph[y □ _ + (r □ _2
2)_]2
2_2_ id_
4
=
1
_p2__ Z 1
lnK
(e_ □ K; 0) exp h□
[(r □ _2
2)_ □ _ + y]2
2_2_ id_ (15)
The normal variable distribution function is presented as
K(x) =
1
p2_ Z x
□1
e□
u2
2 du (16)
We can express (15) as
w(y; _) =

1
p2_2__ Z 1
lnK
exp h□
(A □ _)2
2__2 ie_d_
□
N
p2_2__ Z 1
lnK
exp h□
(A □ _)2
2__2 id_ (17)
where A = y + (r □ _2
2)_ = ln S + (r □ _2
2)_: The RHS second term of (17) can be simpli_ed to
have
z =
(A □ _)
p_2_
(18)
then d_ gives
d_ = □
p_2_dz
The limit _ = 1 and z = □1; are taken for (17) using (18),
z =
A □ lnK
p_2_
=
(r □ _2
2)_ + ln S □ lnK
p_2_ _ d2 where _ = lnK (19)
Carry out variable from _ to z changes, the equation (17) second term gives
K
p2_ Z d1
d2
e□z2=2dz = □
K
p2_x Z 1
1
e□z2=2dz
The equation (17) _rst term integral results to
exp h□
(A □ _)2
2__2 ie_
= exp h□
A2 □ 2(__2 + A)_ + _2_2
2__2 i

= exp h□
A2 □ (__2 + A) + (__2 + A)2 □ 2(__2 + A)_ + _2
2__2 i
= exp h□
[□(__2 + A) + _]2
2__2 + A +
1
2
__2i
= eA+1
2 __2
exp h□
[□(__2 + A)]2 + _
2__2 i (20)
By the A description,
eA+1
2 __2
= e_r+y = Se_r (21)
5
Use (20, 21) on the equation (17) _rst term gives
1
p2_2__
Se_r Z 1
lnK
exp h□
[□(__2 + A) + _]2
2__2 id_: (22)
By variable changes,
1
p2_
Ser_ Z d1
□1
e□
z2
2 dz = Ser_K(d1) (23)
The last line of (15) can be written as;
w(y; _) = er_SNhln(S
K) + (r + _2
2)_
_p_ i□ KNhln(S
K) + (r □ _2
2)_
_p_ i (24)
this implies that
C = □KNe□_r(d2) + NS(d1) (25)
where
d1 =
_ (_2

2 + r) + ln(S
K)
_p_
and $d_2 = d_1$ □
p_2_ : (26)
Thus, the time zero pricing for Black-Scholes model of a paying stock non-dividend European
call option .
By using P = □S + Ke□Tr + c:, The analytical put European model is obtained as
P = □NS(□d_1) + KNe□_rN(□d_2): (27)
The analytical put and call European model have become very popular in the _nancial world
since it is easily used to determine European options.

3 The American Option

For the call American option, the holder has the choice and not responsibility to buy a prescribed
asset price from the start time to the expiry time; hence, the American option has higher values
potentially.
Given S in between P(S; t) < max(E □ S; 0) for an option exercise, there exist a arbitrage
privilege with free-risk pro_t of E □ S □ P. Therefore, it can be concluded that constraints
must be imposed when permitting early exercise.
V (S; t) _ max(S □ E; 0)
The American option-valuation is complex than its counterpart European option-valuation
since the option value only needs to be determined. A unique speci_cation of American optionvaluation
is done by the conditions
_ the value for the option is equal or higher than the function of the payo_;
_ an inequality is used to replace Black-Scholes model;
_ a S continuous function for option value exist and;
_ the delta option (@V
@S) is continuous.
Holding a call American option to maturity is optimal since no payment of dividend on the
underlying asset.

3.1 The Put American Optimal price Exercise

Let the Put American option be denoted by V (S; t), asset price is given as S and current time
is t. Black and Scholes [2] gives the put and call value option V in partial derivative model as:
@V
@t

$$+ \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial S} rS - V r = 0; \quad (28)$$

where interest free-risk rate is r and underlying volatility asset is defined as S. Eq. (28) is described Black-Scholes model; thus, σ and r and constant. Equation (28) is the Black-Scholes financial derivative. For put American and European options, there exist far-field boundary constraint

$$\lim_{S \to \infty} V(S; t) = 0 \quad (29)$$

this implies put option is worthless as the underlying price asset is getting increase unlike European option. A asset critical price $S_f(t)$ is given, which is equal or below its optimal exercise of the put American option. With optimal price exercise, according to Wilmott et al. [21], the boundary constraint of the optimal price exercise defined as $S = S_f(t)$ is given as

$$V(S_f(t); t) = -S_f(t) - X \quad \frac{\partial V}{\partial t}(S_f(t); t) + 1 = 0; \text{ or a continuous } \frac{\partial V}{\partial t} \text{ on } S = S_f(t); \quad (30)$$

for which option strice price is X. From mathematical perspective, free value boundary model is constituted, where the location of the boundary formed the solution part of the model. Though, in terms of V , the governing derivative model is linear but made nonlinear due to the unknown boundary constraint. The unknown product function $\frac{1}{S_f}\frac{\partial v}{\partial s}\frac{dS_f}{dt}$, which exist in the partial derivative model gives a better measure for nonlinearity strength. The put value option been equal to the payoff setup functions ends the condition

$$V(S; T) = \max\{X - S; 0\}; \quad (31)$$

where the option expiration time is T. As such, equations (28)-(31) give a derivative system, which the solution resulted in an American option value before expiration of time T at any price S.

To appropriately solve the system, non-dimensionl variables are introduced to have a dimensionless system

$$S' = \frac{S}{X}; \quad V' = \frac{V}{X}; \quad \tau' = \frac{\sigma^2}{2}(T - t) = \frac{\sigma^2}{2}.\tau.$$

The dimensionless system with prime dropped becomes

$$\begin{cases} -V - \dfrac{\partial V}{\partial \tau} + S\dfrac{\partial V}{\partial S} + S^2\dfrac{\partial^2 V}{\partial S^2} = 0; \\ \dfrac{\partial V}{\partial S}(S_r(\tau); \tau) = -1; \\ V(S; 0) = \max\{1 - S; 0\}; \\ \lim_{S \to \infty} V(S; \tau) = 0; \\ V(S_r(\tau); \tau) = -S_r(\tau) + 1; \end{cases} \quad (32)$$

where relative interest rate to the volatility price asset is $\alpha = \frac{2r}{\sigma^2}$. The dimensionless derivative system (32) depicts a family of two-parameter solution. The solutions are the relative interest

rate, and total dimensionless time, $\tau_{exp} = \frac{\sigma^2}{2}.T$, for option initial time t = 0 to the expire time T. Due to the introduced expire time as the di_erence between times T and t, an terminal constraint (31) gives an initial constraint (32). Consider a new function $U(S; \tau)$ described as

$$U = \begin{cases} S - 1 + V; & \text{if } S_r \le S < 1; \\ V; & \text{if } S \ge 1, \end{cases} \quad (33)$$

The derivative model (32) can be presented in two equations and boundary constraints

8>
><
>>:
S2 @2U
@S2 + S @U
@S @U
@_ U = ;
U(Sf (_); _) = 0; if Sf _ S < 1;
@U
@S (Sf (_); _) = 0;
U(S; 0) = 0;
(34)

8<:
S2 @2U
@S2 + S @U
@S @U
@_ U = 0;
limS!1
U(S; _) = 0; if S _ 1;
U(S; 0) = 0;
(35)

As presented, the constraints in (34) and (35) now appeared easier to handle than that of (31).

The moving boundary constraints $S = S_r(t)$ is homogeneous because the derivative model of (34) is non-homogeneous, which facilitates considerable solution step. To establish V as a contentious function V (S; _), its derivatives are needed on the boundary S = 1, which resulted into matching interfacial constraints

lim
S!1+
U = lim
S!1
U; (36)

1 + lim
S!1+
@U
@S
= lim
S!1
@U
@S
; (37)

where 1_+ and $1_{}$ respectively indicate S is approaching 1 from the right and left. Taking Laplace transform of the systems (34)-(37) for the optimal price exercise $S_r(t)$ and option price

U(S; _), the existence conditions for Laplace transform are satis_ed as given by Karatzas [24].
All quantities shall be denoted in the Laplace space as:
LU(S; _) = Z 1
0
e□p_U(S; _)d_ = U(S; p); LSr(_) = Z 1
0
e□p_Sr(_)d_ = _ Sf (p):
The Laplace transform of the systems (34)-(37) give the respective ordinary derivative equations
in terms of p with the substitution of the initial constraints
8<:
□[p_U □ 0] + S2 d2_U
dS2 + S d_U
dS □ _U =
p ;
_U
(pSf ; p) = 0;
d_U
dS (p _ Sf ; p) = 0;
(38)
_ □[p_U □ 0] + S2 d2_U
dS2 + S d_U
dS □ _U = 0;
limS!1
_U
(S; p) = 0;
(39)
_ _U (1+; p) = _U (1□; p);
1
p + d_U
dS (1+; p) = d_U
dS (1□; p)
(40)
8
It is observed that under the Laplace transform derivation model of (38) and (39), the matching
interfacial constraints in (40) and the boundary far-_eld constraint of (39) are direct. Meanwhile,
a Laplace transformation is carried out on the boundary constraints of (34), which is demonstrated on the moving boundary Sr(_); S in
LU(S; _) = Z 1
0
U(S; _)e□_pd_;
According to an approximate steady pseudo-state, if a slow movement optimal boundary exercise
assumed and compared to option price di_usion, S will remain constant in the Laplace transform

and be replaced interfacial constraint $S = S_r(\_)$ (i.e., $LS_r(\_) = LS$) s
$_p = \_ S_r$). Also, the same
argument is established for the second moving boundary constraint of (38). From the Stefan
classical model, the steady pseudo-state approximate solution gives inde_nite speed at $t = 0$,
like the Stefan model exact solution (see Kemna and Vorst [25]). The system derivative solutions
of (38)-(40) is done as:
_U
$= \_ D_2S^{q_2} + D_1S^{q_1}$ □
$_{p(p+)}$; if $S_f \_ S < 1$;
$D_3S^{q_1} + D_4S^{q_2}$; if $S \_ 1$;
(41)
where the characteristic roots are given as q_1 and q_2 respectively for the corresponding homogeneous
equation
$q_{1;2} =$
1 □
2 _s_1 □
2 $\__2$
+ (p +); (42)
and the unknown complex arbitrary constants are presented as $D_1;D_2;D_3$ and D_4, which are
to be evaluated for all conditions to be satis_ed. Hence, equation (42) can be expressed as
$q_{1;2} = m \_pm^2 + (+ p) = m \_$
pa^{2+p}; (43)
where m = □(□1)
$_2$ and a = $_{+1}$
$_2$. If varied in (0;1), a is positive while m is either negative
or positive. In fact, m takes value between ($_1$
$_2$;□1). Thus, m and a are associated as
$a + m = 1$; a □ m = ; $a^2 = + m^2$:
Taking an appropriate contour for the Laplace inverse transform, the q_1 is the positive real
part and the q_2 is the negative real part. As such, for the boundary far-_eld constraint (29)
to be satis_ed, D_3 must be of zeroth order. The satisfying other boundary constraints and the
interface constraints of (38)-(40) results in algebraic equations
8>
><
>>:
$D_1(p \_ S_f)^{q_1} + D_2(p \_ S_f)^{q_2} =$
$_{p(p+)}$;

$D_1 q_1 (p \_ S_f)_{q_1 □ 1} + D_2 q_2 (p \_ S_f)_{q_2 □ 1} = 0;$
$D_1 (1)_{q_1} + D_2 (1)_{q_2}$ □
$_{p(p+)} = D_4 (1)_{q_2};$
$D_1 q_1 (1)_{q_1} + D_2 q_2 (1)_{q_2} = D_4 q_2 (1)_{q_2} + {}_1$
$_p$;
(44)
the solution yields a elegant and simple optimal price exercise formula in the Laplace space
$\_ S_f =$
1
$p \_ q_2$
q_2 □ (p +)_
$^{1}_{q1}$
; (45)
9
therefore, three unknown constants from U(S; _) is to be evaluated for the option price. The
optimal price exercise S_f (_) is obtained from the inversion of (45). Hence, the Laplace inverse
transform gives
$\_ S_f (\_) =$
1
2_i Z $_{\_+1i}$
$_{\_□1i}$
$e^{p\_}$
p _ _ (p +)
q_2 □ (p +)_
$^{1}_{q1}$
dp;
=
1
2_i Z $_{\_+1i}$
$_{\_□1i}$
$e^{p\_}$
p _ exp8>><
>>:
□log _1 □ (p+)
(b□pp+a2)_
b +pp + a_2
9>
>=
>>;
dp; (46)
to ensure that $Re(q_2) < 0$ and $Re(q_1) > 0$, it is su_cient to assume any _, such that
_ > 0 (47)
Following Cauchy's residue theorem de_ned as
6

Xj=1 ZCj
ep_ _ Sf (p)dp = 2_i
n
Xk=1
Resp=pkfep_g _ Sf (p)
= 2_i
n
Xk=1
Resp=pk8>><
>>:
ep_
p
exp8>><
>>:
log _1 (p+)
(b pp+a2)_
b +pp + a2
9>
>=
>>;
9>
>=
>>;
; (48)
One simple isolated pole ep_ exist when p = 0. As given by Broadie and Glasserman [26], an
optimal perpetual price exercise residue ep_ is determined. In relation to the derivation, the
integrand residue of (46) is obtained as:
I3 + I5 = 2ie a2r Z 1
0
e __
_ + a2 _ Im8<
:
exp2
4
log _1 + (b ip_)
_ b ip_ 3
5
9=;
d_; (49)
In (49), Imf_g denotes complex function imaginary part. On dividing equation (49) on both
sides by 2_ and use the outcome on equations (46) and (48), an analytical optimal price exercise
formula is gotten Sr
Sf (_) =

(1 +)
+
e□a2_
_ Z 1
0
e□__
a2 + _
e□f1(_) sin[f2(_)]d_; (50)
where
f1(_) =
1
(_ + b2) "b ln p_ + a2
!+ p_ tan□1 _p_
a _#;
f2(_) =
1
(b2 + _) "p_ ln pa2 + _
!□ b tan□1 _p_
a _#;
10

3.2 The Put American Price Option

The respective D_1;D_2 and D_4 can be determined once _ S_r(p) is known from (17) and expressed
in Laplace term p as
D1= 1
(_ Srp)q1 _ q2
(q2□q1) _
(+p)p
=_1 □ +p
(m□pa2+p) __ m□pa2+p
2pa2+p _ □
(+p)p :
(51)
D2= 1
(_ Srp)q2 _ q1
(q1□q2) _
(+p)p
=h1 □ +p
(m□
pa2+p) i
q2
q1 _
pa2+p+m
2pa2+o _
(+p)p :
(52)
D4=D2 + D1 □

(+p)p ;
=
(+p)p _ (□_1 □ +p
(m□pa2+p) ___ m□pp+a2
2pp+a2
+ _1 □ +p
(b□pa2+p) _
q2
q1
_
pa2+p+m
2pa2+p □ 1):
(53)

As such, U(S; _) is expressed as

U(S; _) =
1
2_i Z 1i+_
_□1i
e_p
(+ p)p
$F_1(p)dp$; (54)

for $S_r(\_) \_ S \_ 1$ and,

U(S; _) =
1
2_i Z 1i+_
_□1i
e_p
(+ p)p
$F_2(p)dp$; (55)

for S > 1

In equation (54) and (55), $F_2(p)$ and $F_1(p)$ are determined by applying equations (51) to (53) on (41) to have

$F_1(p)=_1$
2 _1 □ b pa2+p___ _1 □ +p
(b□pa2+p) ___ S_{q1}
$+_1$
2 _1 □ b pa2+p___ _1 □ +p
(pp+a2+b) _
q2
q1
_ S_{q2} □ 1:
(56)

$F_2(p) = \_{}_1$
2 _1 □ m pa2+p___ _1 □ +p
(m□pa2+p) _
$+_1$
2 _1 □ m pa2+p___ _1 □ +p
(pp+a2+m) _
q2
q1

□ 1)_ S_{q2};
(57)

3.3 The American Options Partial Derivative Equation

For the calls American valuation, a transformation is introduced. The Black-Scholes price
model of a deterministic asset paying dividend V yields D, interest rate r and volatility _ are
illustrated as:

V_t +
1
2
$\_^2 S^2 V_{ss}$ + SV_s(r □ D) □ V r = 0 (58)

Here, the call option underlying asset is denoted as S. The early American option exercise gives
an optimal boundary exercise model, which is considered as a boundary free problem in the

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PDE setting. The domain (0; ~B(t))_[0; T) for the equation (58) and the boundary constraints
are taken as:

V (S; T) = S 2 (0; ~B(T)); max(S □ K; 0); (59)
~B
(T) = max_K;
Kr
D _ (60)
V (0; t) = 0; (61)
V (~B(t); t) = □K + ~B (t); (62)
V_s(~B (t); t) □ 1 = 0; (63)

3.4 American Call with Dividends

From the Black-Scholes equation, the American option value C(S; t) satis_es

@C
@t
+
1
2
$\_^2 S^2$ $@^2$C
$@S^2$ + S(r □ D_0)
@C
@S □ rC = 0; (64)

This remains for non-optimal exercise. The payo_ constraint is

max(S □ E; 0) = C(S; T) (65)

Also, the option exercised can take place at any period; as such,

max(S □ E; 0) _ C(S; t) (66)

provided the boundary optimal exercise S = S_r(t)

C(S_r(t); t) = □E(S_r(t); t)
@C

@S
+ $S_r(t)$ = 1 (67)
If the boundary optimal exercise occur therefore (64) is valid when max(S□E; 0) < C(S; t) due
to the fact that max(S □E; 0) is not the Black-Scholes model solution. (64) can be replaced by
an inequality
S(r □ D_0)S
@C
@S
+
1
2
_^{2}S^2 @^{2}C
@S^2 +
@C
@t □ rC _ 0 (68)
The inequality is valid if max(S □ E; 0) < C(S; t).

3.5 Call Dividends Analysis

Here, the simpli_cation of dividend payments with Black-Scholes model is carried out by taking
that the dividend payments and interest rate satisfy 0 < D_0 < r. Therefore, equations (64) to
(66) are transformed to dimensionless form by introducing the variables
S = Ee^x; t = T □
_
1
2_2
; C(S; t) = Ec(x; _) □ E + S (69)
the result is
@c
@_
=
@^{2}c
@x^2 + (k_0 □ 1)S
@c
@x □ kc + f(x) (70)
12
where k = r
1
2_2 , k_0 = r□D_0
1
2_2 for □1 < x;1 and _ > 0. The function c(x; 0) gives the initial formation
max(1 □ e^x; 0) = c(x; 0) (71)
The function is resulted from f
f(x) = k □ (k □ k_0)e^x (72)
Taking that free boundary occur, x = x_r(t), at the boundary to obtain

c(xr(_); _) = (xr(_); _)
@c
@_
(73)
From equation (73), at expiration time gives
@c
@_
= f(x) (74)
For f(x) > 0, x_0 > x > 0, and c is positive. Give x_0 < x when f(x) < 0, then c is negative; as such, when x > 0, c > 0 therefore constraint is not satis_ed.

4 Monte Carlo Technique

The idea of Monte Carlo computation into _nance was introduced by Boyle. The Monte Carlo
numerical technique is used when closed-form solution is not available. The approach is a better
pricing dependent path option. Monte Carlo computational scheme employes the valuation risk
results; in risk-neutral world, expected payo_ is computed by adopting a sampling technique.
The major Monte Carlo simulation techniques are as follows:
_ Compute an asset underlying path with the risk-neutralization constraint in between the
time horizon.
_ The payo_ discounted is equivalent to the interest free-risk path rate.
_ The approach is repeated for higher simulation numbers of sample path.
_ The cash ow discounted average over the sample value option paths is determined.
A Monte Carlo computation uses approach for random sampling outcome processes according
to the stock price [6]
dS = _Sdt + _SdW(t) (75)
where stock price is S and Wiener process is dW_t. Given that _S is the stock price increment
in the time interval of _t, then
_S
S
= __t + _Zp_t (76)
where Z _ N(0; 1), expected risk-neutral return is _ and stock price volatility is _, which can
be expressed as
S(t + _t) S(t) = _S(t)_t + _S(t)Zp_t (77)

At time t+_t the value of S is computed taking from the S initial value, then at time t+2_t the
value of S calculated and so on. A random N sample from a normal distribution for complete

trial calculations is done in S. Its give better accurate simulation of ln S than S by using Ito's lemma to modify the asset price procedure

d(ln S) = _dW(t) __2
2 __dt

so that

ln(t + _t)S ln S(t) = _Zp_t __2
2 ___t

or

S(t + _t) = S(t) exp __Zp_t __2
2 ___t_ (78)

The Monte Carlo computation is speci_cally useful when there is payo_ _nancial derivatives depending on the path and life asset underlying option. For instance, considering a maturity time T of Asian Stock price options de_ned as

S^j_T = S exp __ZpT __2
2 __T_ (79)

The evaluated Call Asian value option is

C =
1
M
M
$X_{j=1}$
e Tr max[S^j_T S_t; 0] (80)

where trial numbers is M and j = 1; 2; 3; :::;M, this is an unbiased price derivative evaluation.

When M is high, the limit central theorem gives an estimation realistic interval based on the discounted payo_ of sample variance. If discounted payo_s mean is _ and standard deviation is !, then !
pM
is the estimated standard error. A price f con_dence interval of 0.95% for the derivative is taken as:

_
1:96!
pM
< f < _ +
1:96!
pM
(81)

with the normally distributed assumption for f.

5 Comparison of Scienti_c Computations

Scienti_c calculation is the science computing, and it is major aspect of this study. The interest on a relative balance between computer science and the modern and classical computational mathematics elements. To compare the results of the standard numerical approximations discussed above, we computed some examples of call options.

Example 1:

Depicts the two schemes performance against the put European Black-Scholes price with K = 50; r = 0:05; _ = 0:25; T = 3:

Table 1: A veri_cation of the Put European Black-Scholes price

S	Black-Scholes	Monte-Carlo	Implicit Euler
10	33.0363	33.0345	33.0369
20	23.2276	23.2291	23.2300
30	14.7739	14.7748	14.7749
40	8.7338	8.7374	8.7348
50	4.9564	4.9559	4.9563
60	2.7621	2.7602	2.7612
70	1.5328	1.5324	1.5325
80	0.8538	0.8543	0.8537

Table 1 displays the option price variation with the asset price underlying, S. The outcomes illustrate that the two schemes are mutually consistent, e_ective, and agree with the values of Black-Scholes. Meanwhile, it may not be essential is applying such numerical schemes in practice where explicit formula exist.

6 Results Discussion

The study presents scienti_c computing for derivatives valuing where analytical solution do not exist. This adopted the uses of Monte Carlo numerical scheme for an American price option.
The Monte Carlo computation has to do with a simple random variable numbers with diverse paths that a derivative could take in a neutral-risk society. The free-risk-discounted interest rate and payo_ are determined for each paths. The discounted payo_s average arithmetic is the derivative estimated value. The implicit _nite implicit di_erence approach more complex but has good because the user do not need to take any particular caution for convergence.
In practice, the accuracy and derivative characteristics determined the applied evaluation technique to be adopted. The Monte Carlo computation works on the life derivative from the beginning to the end forwardly. It may be employed for the derivative of the Europeanstyle

and can adapt to high complexity deals of the payo_. It is adaptively pro_cient as the underlying variable numbers is raised. Moreover, one major setbacks in utilizing Monte-Carlo computation is that it is complex to implement for early American-style, though it performs better for path-dependent European-style pricing.

Conclusion

This research provides a comprehensive analysis of the application of the Black-Scholes and Monte Carlo derivative model methods in the American options context. The analysis underscores the complexity inherent in the American option modeling, majorly as a result of their early exercise characteristics, which is not completely taking care by the Black-Scholes standard equation. Through an advanced Monte Carlo simulations, this research evaluates the performance and accuracy of this computational approaches in pricing American options. Key

outcomes from the analysis can be summarized: The Black-Scholes model o_ers a foundational and relatively straightforward approach to option pricing, its assumptions and limitations become evident when applied to American options; Monte Carlo simulations, with their ability to handle a wide range of potential price paths and complex boundary conditions, demonstrate superior accuracy and exibility; The comparative analysis highlights that although the Black-Scholes model remains a vital tool in _nancial engineering, Monte Carlo simulations provide a more robust and adaptable framework for American option pricing.

In summary, the research contributes valuable insights into the strengths and limitations of diverse scienti_c computing techniques in option pricing. It underscores the importance of selecting appropriate models and methods based on the speci_c characteristics of the _nancial instruments being analyzed. Future work could explore hybrid approaches that integrate the analytical strengths of the Black-Scholes equation with the computational precision of Monte Carlo simulations, potentially o_ering even more e_ective solutions for pricing American options.

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