Original Research Article

# A scientific computing analysis of financial Black-Scholes and Monte Carlo differential equation: An American option

**Abstract**
This study presents a scientific computing analysis of financial models, specifically focusing on the Black-Scholes and Monte Carlo differential equations, to evaluate American options. American options, which can be exercised at any time before expiration, pose unique challenges in financial modeling due to their complex early exercise features. The Black-Scholes model provides a foundational framework for option pricing, utilizing partial differential equations to estimate the fair value of options under specific assumptions. However, due to its limit, Monte Carlo simulations is considered to offer a robust numerical method to address these limitations by simulating a wide range of possible price paths for the underlying asset. This study implements a computational approach to compare the efficacy of the Black-Scholes model and Monte Carlo methods in pricing American options. Computation scheme for solving the Black-Scholes partial differential equations and variance reduction technique for enhancing the efficiency of Monte Carlo simulations are adopted. Our analysis reveals that while the Black-Scholes model provides a useful approximation, Monte Carlo simulations deliver more accurate and flexible results for American options, especially in scenarios with significant volatility and early exercise potential. The findings underscore the importance of sophisticated computational methods in financial engineering and highlight the trade-offs between analytical tractability and numerical precision.
**Keywords:** Comparability analysis; Financial science; Scientific computing; Mathematical model; Monte Carlo method

## 1 Introduction

Finance is one of the most rapidly changing and fastest growing areas in the corporate business world. Because of this rapid change, modern financial instruments have become extremely complex. As such, new mathematical models are essential to implement and price these new financial instruments. The world of corporate finance once managed by business students is now controlled by mathematicians and computer scientists. In the early 1970s, Myron Scholes, Robert Merton, and Fisher Black made an important breakthrough in the pricing of complex financial instruments by developing what has become known as the Black-Scholes model. In 1997, the importance of their model was recognized world wide when Myron Scholes and Robert Merton received the Nobel Prize for Economics. The Black-Scholes model displayed the essence mathematics in the field of finance. It also led to the growth and success of the new field of mathematical finance or financial engineering.

The owner of a put (call) option has the right but no obligation to sell (buy) an underlying asset at the exercise price. European options can be exercised only on the expiry date, while American options can be exercised at any time until the expiry date. Closed-form solutions for the European options are derived in papers by [1,2]. In the case of American options, because of the early exercise possibility, the pricing problem leads to complications for analytic calculation. The authors [3,4] show that the valuation of American options constitutes a free boundary problem looking for a boundary changing in time to maturity, mostly called an optimal exercise boundary. Hence, finance researchers have studied methods to quickly and accurately find the optimal exercise boundary. These methods are basically of two types, i.e, analytical approximations such as those developed by [7-9] and numerical methods such as those of [10,11]. Wu and Kwok [12] find an exact and explicit solution of the Black-Scholes equation for the valuation of American put options using Taylor series with infinitely many terms. His work is an excellent result for the valuation of American put options; however, it seems difficult to perform his solution numerically. The infinite sum is likely to yield many computation errors.

Michael et a. [13] studied an extension of Wu and Kwok [12] to price American options under general diffusion processes. The majority of numerical methods for pricing American options, such as the finite difference method of Brennan and Schwartz [14], the binomial method of Cox et al. [15], the Monte Carlo simulation method of Grant, Glasserman [16], the least squares method of Tilley [17], the integral-equation method of Brandimarte [18] and the Laplace transform method of Boyle et al. [19], are time-recursive ways. Their idea is discretization of the lifetime of an option and calculation the optimal exercise boundary backward in time. Since time-recursive ways yield repeated calculations for every time step, they require fast computation times and small pricing errors. Also, front-fixing methods developed by Wu and Kwok [12] and Han and Wu [20] apply a non-linear transformation to fix the boundary and solve the resulting non-linear problem. A secant method developed by Wilmott et al. [21] needs to solve a nonlinear problem and a moving boundary approach developed by Geske and Johnson [22] converts the arising linear free boundary partial differential equation (PDE) problem into a sequence of linear fixed-boundary PDE problems. More recently, Han and Wu [23] introduced a new predictor-corrector scheme to price American put options under the Black-Scholes model and then Wilmott et al. [21] proposed an extension of Han and Wu [23] to solve for the valuation of American put options with stochastic volatility model. This study implements a computational approach to compare the efficacy of the Black-Scholes model and Monte Carlo methods in pricing American options. Computational scheme for solving the Black-Scholes partial differential equations and variance reduction techniques for enhancing the efficiency of Monte Carlo simulations are adopted.

## 2 Black-Scholes Equation

It was under the assumption of the lognormal dynamics of derivatives that Fischer Black, Myron Scholes and Merton developed their European option pricing model. They further made the following assumptions:

- The probability of the rate of return for a stock is lognormally distributed with the mean same as the risk-free rate of return.
- There are no transaction costs or taxes.

- No risk-free arbitrage opportunities exist.
- There are no dividends during the life of the options.
- The risk-free interest rate r is known and constant over time.
- The variance of the return is constant over the life of the option.
- The underlying asset trading is continuous and the change of its price is continuous.

Let a stock price follow

$$dS = \mu S dt + \sigma S dW \tag{1}$$

where $\mu$ is the trend, $\sigma$ is the volatility and $W$ follows a Wiener process. Now, suppose that $f$ is the price of a call option or other derivative contingent on $S$. The variable $f$ must be some function of $S$ and $t$. Hence, by **Ito's lemma**

$$df = \left[\frac{\partial F}{\partial S}\mu S + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S^2\right] dt + \frac{\partial F}{\partial S}\sigma S dW \tag{2}$$

The discrete versions of (1) and (2) are

$$\partial S = \mu S \partial t + \sigma S \partial W$$

$$\partial f = \left[\frac{\partial F}{\partial S}\mu S + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S^2\right] \partial t + \frac{\partial F}{\partial S}\sigma S dW \tag{3}$$

The Wiener process underlying $f$ and $S$ are the same and can be eliminated by choosing an appropriate portfolio of the stock and the derivative. We choose a portfolio of

$$-1 : derivative$$

$$+\frac{\partial f}{\partial S} : shares$$

The holder is short one derivative and long an amount $\partial f/\partial S$ of shares. We define $\Theta$ as the value of the portfolio and we have

$$\Theta = -f + \frac{\partial f}{\partial S}S. \tag{4}$$

The change $\delta\Theta$ in the value of the portfolio in the time interval $\delta t$ is given by

$$\delta\Theta = -\partial f + \frac{\partial f}{\partial S}\delta S. \tag{5}$$

Substituting (3) into (5), we get

$$\delta\Theta = \left[-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right] \partial t \tag{6}$$

The portfolio is now risk-less due to elimination of the $\delta W$ term. It must then earn a return similar to other short term risk-free securities. Therefore

$$\delta\Theta = r\Theta\delta t \tag{7}$$

where $r$ is the risk-free interest rate. Substituting (4) and (6) into (7), we obtain

$$\left[\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right]\delta t = r\left[f-\frac{\partial f}{\partial s}S\right]\delta t. \tag{9}$$

Thus, we have

$$\frac{\partial f}{\partial t}+rS\frac{\partial f}{\partial S}+\frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf \tag{9}$$

which is the Black-Scholes-Merton differential equation.

Solving the partial differential equation in (9) gives an analytical formula for pricing the European style options. These options can only be exercised at the maturity date. The American style options are exercised any time up to the maturity date. Thus, the analytical formula we will derive is not appropriate for pricing them due to this early exercise privilege. In the next section we examine the upper and lower boundary conditions for the American and European style options. Then, the boundary conditions for the European options will be applied to solve (9).

## 2.1 Solution of the Black-Scholes Equation

The payoff condition is $f(S,t=T)=\max(S-N,0)$. The lower and upper boundary conditions are given by $C(S,t,K), c(S,t,K) \geq S_t - Ke^{-(T-t)}$ and $C(S,t,K), c(S,t,K) \leq S_t$. These are the conditions that should be satisfied by the PDE

Let $\tau = T-t$, where $T$ is the expiration time and $t$ the present time, then (9) can be written as

$$\frac{\partial f}{\partial \tau}=\frac{\sigma^2}{2}S^2\frac{\partial^2 f}{\partial S^2}+rS\frac{\partial f}{\partial S}-rf \tag{10}$$

Let

$$\frac{\partial f}{\partial S}=\frac{1}{S}\frac{\partial f}{\partial y}$$

and

$$\frac{\partial^2 f}{\partial S^2}=-\frac{1}{S^2}\frac{\partial f}{\partial y}+\frac{1}{S^2}\frac{\partial^2 f}{dy^2} \tag{11}$$

Taking $y=\ln S$, we now introduce a new notation

$$w(y,\tau)=e^{r\tau}f(y,\tau)$$

Using (11), the Black-Scholes PDE becomes a diffusion equation

$$\frac{\partial w}{\partial \tau}=\frac{\sigma^2}{2}\frac{\partial^2 w}{\partial y^2}+\left[r-\frac{\sigma^2}{2}\right]\frac{\partial w}{\partial y} \tag{12}$$

and has a fundamental solution as a normal function

$$\phi(y,\tau)=\frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left[-\frac{[y+(r-\frac{\sigma^2}{2})\tau]^2}{2\sigma^2\tau}\right] \tag{13}$$

The solution to (13) is

$$w(y,z)=\int_{-\infty}^{\infty} w(\xi,0)\phi(y-\xi,\tau)d\xi \tag{14}$$

we use the payoff condition and the fundamental solution of (13) to obtain

$$w(y,\tau) = \frac{1}{\sigma\sqrt{2\pi\tau}}\int_{-\infty}^{\infty} \max(e^{\xi} - K, 0) exp\Big[\frac{[y-\xi+(r-\frac{\sigma^2}{2})\tau]^2}{2\sigma^2\tau}\Big]d\xi$$

$$= \frac{1}{\sigma\sqrt{2\pi\tau}}\int_{\ln K}^{\infty} (e^{\xi} - K, 0)\exp\Big[-\frac{[y-\xi+(r-\frac{\sigma^2}{2})\tau]^2}{2\sigma^2\tau}\Big]d\xi \quad (15)$$

We denote the distribution function for a normal variable by

$$K(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{u^2}{2}}du \quad (16)$$

We can express (15) as

$$w(y,\tau) = \frac{1}{\sigma\sqrt{2\pi\tau}}\int_{\ln K}^{\infty} e^{\xi}\exp\Big[-\frac{(-\xi+A)^2}{2\sigma^2\tau}\Big]d\xi$$

$$-\frac{N}{\sigma\sqrt{2\pi\tau}}\int_{\ln K}^{\infty}\exp\Big[-\frac{(-\xi+A)^2}{2\sigma^2\tau}\Big]d\xi \quad (17)$$

where $A = y + (r - \frac{\sigma^2}{2})\tau = \ln S + (r - \frac{\sigma^2}{2})\tau$. We now consider the second term in the right hand side of (17)
Let

$$z = \frac{(-\xi + A)}{\sigma\sqrt{\tau}} \quad (18)$$

the $d\xi$ becomes

$$d\xi = -\sigma\sqrt{\tau}dz$$

and the limit of (17) using (18) are given as $z = -\infty$, when $\xi = \infty$

$$z = \frac{-\ln K + A}{\sigma\sqrt{\tau}} = \frac{-\ln K + \ln S + (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \equiv d_2 \quad where \quad \xi = \ln K \quad (19)$$

changing the variable from $\xi$ to $z$, the second term of (17) becomes

$$\frac{K}{\sqrt{2\pi}}\int_{d_2}^{d_1} e^{-\frac{z^2}{2}}dz = -\frac{K}{\sqrt{2\pi x}}\int_{\infty}^{1} e^{-\frac{z^2}{2}}dz$$

The integral of the first term in (17) is expressed as;

$$e^{\xi}\exp\Big[-\frac{(-\xi+A)^2}{2\sigma^2\tau}\Big]$$

$$= \exp\Big[-\frac{\xi^2 - 2(A+\sigma^2\tau)\xi + A^2}{2\sigma^2\tau}\Big]$$

$$= \exp\Big[-\frac{\xi^2 - 2(A+\sigma^2\tau)\xi + (A+\sigma^2\tau)^2 - (A+\sigma^2\tau) + A^2}{2\sigma^2\tau}\Big]$$

$$= \exp\Big[-\frac{[\xi-(A+\sigma^2\tau)]^2}{2\sigma^2\tau}+\frac{1}{2}\sigma^2\tau + A\Big]$$

$$= e^{\frac{1}{2}\sigma^2\tau + A}\exp\Big[-\frac{[\xi-(A+\sigma^2\tau)]^2}{2\sigma^2\tau}\Big] \tag{20}$$

By definition of $A$

$$e^{\frac{1}{2}\sigma^2\tau + A} = e^{y+r\tau} = Se^{r\tau} \tag{21}$$

Put (20) and (21) into the first term of (17), we have

$$\frac{1}{\sigma\sqrt{2\pi\tau}}Se^{r\tau}\int_{\ln K}^{\infty}\exp\Big[-\frac{[\xi-(A+\sigma^2\tau)]^2}{2\sigma^2\tau}\Big]d\xi. \tag{22}$$

By change of variables, we get

$$\frac{1}{\sqrt{2\pi}}Se^{r\tau}\int_{-\infty}^{d_1}e^{-\frac{z^2}{2}}dz = Se^{r\tau}K(d_1) \tag{23}$$

The last line of (15) can be written as;

$$w(y,\tau) = e^{r\tau}SN\Big[\frac{\ln(\frac{S}{K})+(r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\Big] - KN\Big[\frac{\ln(\frac{S}{K})+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\Big] \tag{24}$$

which means that

$$C = SN(d_1) - Ke^{-r\tau}N(d_2) \tag{25}$$

where

$$d_1 = \frac{\ln(\frac{S}{K})+(r+\frac{\sigma^2}{2})\tau}{\tau\sqrt{\tau}} \quad and \quad d_2 = d_1 - \sigma\sqrt{\tau}. \tag{26}$$

This is the Black-Scholes formula for the price at time zero of a European call option on a non dividend paying stock.
We can derive the corresponding European put option formula for a non dividend paying stock by using the call-put parity which is given by
$P = c + Ke^{-rT} - S$. The European put analytical formula is

$$P = Ke^{-r\tau}N(-d_2) - SN(-d_1). \tag{27}$$

The European call and put analytical formula have become very popular in the financial world since it is easily used to value the European options. When evaluating the value of the option, the other parameters apart from volatility can easily be observed from the market. Hence the need to find appropriate methods of evaluating the volatility is of great importance.

# 3 American Option

American Options have the important additional feature that early exercise is permitted at any time during the life of the option.

**Definition 2**. An American Call Option gives its holder the right, but not the obligation, to purchase from the writer a prescribed asset for a prescribed price at any time between the start date and a prescribed expiry date in the future. The ability to exercise the option at any time extends to the owner additional rights, and thus the American option has potentially a higher value.

If $S$ lies in this range so that $P(S,t) < \max(E-S,0)$ and we exercise the option, there is an obvious arbitrage opportunity. We could buy the as- set in the market immediately for $S$ and at the same time buy the option for $P$; if we then exercised the option by selling the asset for $E$ we make a risk free profit of $E-P-S$.

This opportunity would not last long before the value of the option was pushed up by the demand of the arbitragers. We must therefore conclude that when early exercise is permitted we must impose the constraint

$$V(S,t) \geq \max(S-E,0)$$

American and European options **must** therefore have different values.

In the case of American options there are some values of $S$ for which it is optimal from the holders point of view to exercise the American option. If this were not the case the option would have the same value as the European option, the Black-Scholes equation would hold for all $S$.

The valuation of an American option is therefore more complicated than its European counterpart since we have to determine not only the option value but also, for each value of $S$, whether or not it should be exercised. This is what is known as a *free boundary* problem. At each time $t$ there is a particular value of $S$ which marks the boundary between two regions: to one side one should hold the option and to the other side one should exercise it.

We denote this value, which varies with time, by $S_f(t)$, and refer to it as the *optimal exercise price.* As we have already observed, since we do not know $S_f$ a priori unlike the corresponding European problem, we do not know where to apply the boundary conditions, and for this reason, the problem is called a *free boundary problem.* An American option valuation can be shown to be uniquely specified by a set of constraints

- the option value must be greater than or equal to the payoff function;
- the Black-Scholes equation is replaced by an inequality;
- the option value must be a continuous function of $S$ and;
- the option delta $(\frac{\partial V}{\partial S})$ must be continuous.

The New Analytical-Approximation Solution Since one can easily show that, without dividends, American call options would be equivalent to their European counterparts, i.e, it is always optimal to hold an American call to maturity when there are no dividend payments to the underlying asset, we shall thus concentrate on solving the Black-Scholes equation for an American put option with constant interest rate and volatility but no dividend payments to the underlying asset. This section is subdivided into two subsections, in the first of which an analytical-approximation formula for the optimal exercise price of American put options is presented and discussed, and in the second of which, the corresponding formula for the price of the American put is provided.

# 4 The Optimal Exercise Price of the American Put

Let $V(S,t)$ denote the value of an American put option, with $S$ being the price of the underlying asset and $t$ being the current time. With six main assumptions, Black and Scholes [2] showed that the value of a call or put option $V$ can be modeled by the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{28}$$

where $r$ is the risk-free interest rate and s is the volatility of the underlying asset price. Eq. (28) is widely referred to as the Black-Scholes equation. In this paper, $r$ and $\sigma$ are assumed to be constant.

Eq. (28) is the governing equation for any financial derivative in the Black-Scholes world. To model a specific type of options, Eq. (28) needs to be solved together with a set of appropriate boundary conditions. For both European and American put options, there is a far-field boundary condition

$$\lim_{S\to\infty} V(S,t) = 0 \tag{29}$$

which simply states that a put option becomes worthless when the price of the underlying asset becomes very large. On the other hand, unlike European options, there is a critical asset price, $S_f(t)$, below or equal to which it is optimal to exercise the American put option. The optimal nature can be understood by the arbitrage opportunity to make a risk-free profit if the option is not exercised when the stock price is equal to or less than this critical asset price, which is usually referred to, in the literature, as the optimal exercise price (Wilmott et al. [21]), the critical stock price (Geske and Johnson [22] ), or optimal exercise boundary (Wu and Kwok [12]), among some other less frequently used names. In this project, we shall mainly refer to it as the optimal exercise price, but occasionally use the other two names as well. It is the existence of the optimal exercise price that has made the process of finding an exact formula for the valuation of American options much more difficult than that of finding an exact formula for the valuation of European options.

With the presence of the optimal exercise price, it can be shown (see Wilmott et al. [21]) that the boundary conditions at the optimal exercise price $S = S_f(t)$ are

$$\begin{cases} V(S_f(t),t) = X - S_f(t) \\ \frac{\partial V}{\partial t}(S_f(t),t) = -1, \quad \text{or } \frac{\partial V}{\partial t} \text{ be continuous on } S = S_f(t), \end{cases} \tag{30}$$

in which $X$ is the strike price of the option. The equation (28) simply states that the option price should be nothing but the intrinsic value of the option when the optimal exercise price is reached and the second one states that the option price is smoothly connected to the pay-off function at $S = S_f(t)$. From a mathematical point of view, this constitutes a so-called free-boundary value problem, in which the boundary location itself is part of the solution of the problem. Although the governing differential equation itself is linear in terms of the unknown function $V$ , it is the unknown boundary that has made this type of problem highly nonlinear. The nonlinearity of the problem is clearly manifested once a Landau transform is used to convert the moving boundary problem to a fixed boundary value problem as demonstrated by Wu and Kwok [12]; the product term of the unknown functions $\frac{1}{S_f}\frac{dS_f}{dt}\frac{\partial v}{\partial s}$, which now appears in the partial differential equation, gives a good measure of the strength of the nonlinearity.

The fact that the value of a put option must be equal to its payoff function sets up the terminal condition

$$V(S,T) = \max\{X - S, 0\}, \tag{31}$$

where $T$ is the expiration time of the option. Eqs. (28)-(31) constitute a differential system, the solution of which will give rise to the value of the American option at any time $t$ before the expiration time $T$ and at any price $S$.

To solve this system effectively, we shall first non-dimensionalze all variables by introducing dimensionless variables

$$V' = \frac{V}{X}, \qquad S' = \frac{S}{X}, \qquad \tau' = \tau.\frac{\sigma^2}{2} = (T-t)\frac{\sigma^2}{2}.$$

With all primes dropped from now on, the dimensionless system reads as

$$\begin{cases} -\frac{\partial V}{\partial \tau} + S^2 \frac{\partial^2 V}{\partial S^2} + \gamma S \frac{\partial V}{\partial S} - \gamma V = 0, \\ V(S_f(\tau), \tau) = 1 - S_f(\tau), \\ \frac{\partial V}{\partial S}(S_f(\tau), \tau) = -1, \\ \lim_{S \to \infty} V(S, \tau) = 0, \\ V(S, 0) = \max\{1 - S, 0\}, \end{cases} \tag{32}$$

in which $\gamma \equiv \frac{2r}{\sigma^2}$ can be viewed as an interest rate relative to the volatility of the underlying asset price. The nondimensional differential system (32) shows that the solution will be a two-parameter family. That is, the solution of the system depends only on two parameters; one is the relative interest rate, $\gamma$, and the other one is the dimensionless total time, $\tau_{exp} = T.\frac{\sigma^2}{2}$, from the initial time $t = 0$ to the expiration time $T$ of the option. It should also be noticed that due to the introduction of the time to expiration $\gamma$ as the difference between the expiration time $T$ and the current time $t$, terminal condition (31) has become an initial condition in (32).

If we define a new function $U(S, \tau)$ as

$$U = \begin{cases} V + S - 1, & \text{if } S_f \leq S < 1, \\ V, & \text{if } S \geq 1, \end{cases} \tag{33}$$

differential system (32) can be written as the following two sets of equations and boundary conditions

$$\begin{cases} -\frac{\partial U}{\partial \tau} + S^2 \frac{\partial^2 U}{\partial S^2} + \gamma S \frac{\partial U}{\partial S} - \gamma U = \gamma, \\ U(S_f(\tau), \tau) = 0, \\ \frac{\partial U}{\partial S}(S_f(\tau), \tau) = 0, \\ U(S, 0) = 0, \end{cases} \qquad \text{if } S_f \leq S < 1, \tag{34}$$

$$\begin{cases} -\frac{\partial U}{\partial \tau} + S^2 \frac{\partial^2 U}{\partial S^2} + \gamma S \frac{\partial U}{\partial S} - \gamma U = 0, \\ \lim_{S \to \infty} U(S, \tau) = 0, \\ U(S, 0) = 0, \end{cases} \qquad \text{if } S \geq 1, \tag{35}$$

One should notice that the initial condition in (34) and (35) now becomes a much easier form to deal with than that in (31). The boundary conditions at the moving boundary $S = S_f(t)$ also become homogeneous (at the expenses that the differential equation in (34) has now become non-homogeneous), which will considerably facilitate the solution procedure. To guarantee $V$

being a $C^1$ function of $S$, the continuity of the unknown function $V(S,\tau)$ and its derivative are demanded on the boundary $S = 1$, which results in the following interfacial matching conditions

$$\lim_{S\to 1^-} U = \lim_{S\to 1^+} U, \tag{36}$$

$$\lim_{S\to 1^-} \frac{\partial U}{\partial S} = \lim_{S\to 1^+} \frac{\partial U}{\partial S} + 1, \tag{37}$$

where $1^-$ indicates $S$ approaching 1 from the left and $1^+$ indicates $S$ approaching 1 from the right.

Now, we perform the Laplace transform on systems (34)-(37). For the option price $U(S,\tau)$ and the optimal exercise price $S_f(t)$, we can certainly show that all three conditions for the exitance of the Laplace transform (cf. Karatzas [24]) are satisfied, and we shall denote all variables in the Laplace space with bars. For example

$$\mathcal{L}U(S,\tau) = \int_0^\infty e^{-p\tau} U(S,\tau)d\tau = U(S,p), \qquad \mathcal{L}S_f(\tau) = \int_0^\infty e^{-p\tau} S_f(\tau)d\tau = \bar{S}_f(p).$$

Under the Laplace transform, systems (34)-(37) become the following ordinary differential equation systems, respectively, in terms of parameter $p$ after the initial conditions have been substituted in

$$\begin{cases} -[p\bar{U} - 0] + S^2 \frac{d^2\bar{U}}{dS^2} + \gamma S \frac{d\bar{U}}{dS} - \gamma\bar{U} = \frac{\gamma}{p}, \\ \bar{U}(pS_f, p) = 0, \\ \frac{d\bar{U}}{dS}(p\bar{S}_f, p) = 0, \end{cases} \tag{38}$$

$$\begin{cases} -[p\bar{U} - 0] + S^2 \frac{d^2\bar{U}}{dS^2} + \gamma S \frac{d\bar{U}}{dS} - \gamma\bar{U} = 0, \\ \lim_{S\to\infty} \bar{U}(S,p) = 0, \end{cases} \tag{39}$$

$$\begin{cases} \bar{U}(1^-, p) = \bar{U}(1^+, p), \\ \frac{d\bar{U}}{dS}(1^-, p) = \frac{d\bar{U}}{dS}(1^+, p) + \frac{1}{p} \end{cases} \tag{40}$$

One should notice that the derivation of the differential equations in (38) and (39) under the Laplace transform, the interfacial matching conditions in (40) and the far-field boundary condition in (39) is straightforward. However, treatment of the two nonlinear moving boundary conditions in (34) requires an approximation based on the pseudo-steady-state approximation used in the heat transfer for Stefan problems. When the Laplace transform is performed on the boundary conditions defined on the moving boundary $S_f(\tau), S$ in

$$\mathcal{L}U(S,\tau) = \int_0^\infty e^{-p\tau} U(S,\tau)d\tau,$$

should be replaced by $S_f(\tau)$ and the result is a function of $p$ only. Based on the pseudo-steady-state approximation, if we assume that the optimal exercise boundary moves slowly in comparison with the "diffusion" of the option price, $S$ can still be held as a constant during the Laplace transform and will then be replaced by the Laplace transform performed on the interfacial condition $S = S_f(\tau)$, with $S$ being held as a constant as well (i.e.,$\mathcal{L}S = \mathcal{L}S_f(\tau) \Rightarrow \frac{S}{p} = \bar{S}_f$). That is, we argue that the moving boundary condition $U(S_f(\tau), \tau) = 0$ in the original time space can be approximated by the boundary condition $\bar{U}(S,p) = 0$, with $S = p\bar{S}_f$ in the

Laplace space. Similarly, we have the same argument for the 2nd moving boundary condition in (38). Of course, like the pseudo-steady-state approximation used for the classical Stefan problem, the assumption that $S_f(\tau)$ is nearly a constant function during the Laplace transform does not necessarily result in a boundary that is slowly moving. In the classical Stefan problem, the approximate solution based on the pseudo-steady-state approximation has a infinite speed at $t = 0$, just like that of the exact solution of the Stefan problem (see Kemna and Vorst [25]). For the similar reason, we shall expect that the result of using approximation technique will result in an approximation with a reasonably high accuracy. The verification of the accuracy of this approximation will be performed after the approximate solution is obtained.

The solution of differential systems (38)-(40) is of the form

$$\bar{U} = \begin{cases} D_1 S^{q1} + D_2 S^{q2} - \frac{\gamma}{p(p+\gamma)}, & \text{if } S_f \leq S < 1, \\ D_3 S^{q1} + D_4 S^{q2}, & \text{if } S \geq 1, \end{cases} \tag{41}$$

where $q_1$ and $q_2$ are roots of the characteristic equation of the homogeneous part of the corresponding equation

$$q_{1,2} = \frac{1-\gamma}{2} \pm \sqrt{\left(\frac{1-\gamma}{2}\right)^2 + (p+\gamma)}, \tag{42}$$

and $D_1, D_2, D_3$ and $D_4$ are four arbitrary complex constants to be determined in order to satisfy all boundary conditions. To facilitate the derivation, Eq. (42) can be written in di?erent forms as

$$q_{1,2} = b \pm \sqrt{b^2 + (p+\gamma)} = b \pm \sqrt{p + a^2}, \tag{43}$$

where $a = \frac{1+\gamma}{2}$ and $b = \frac{1-\gamma}{2}$. It should be noticed that when $\gamma$ varies in the domain $(0, \infty), a$ is always positive but $b$ can be either positive or negative. In fact, $b$ varies in the range $(\frac{1}{2}, -\infty)$. Furthermore, $a$ and $b$ are related as

$$a + b = 1, \qquad a - b = \gamma, \qquad a^2 = b^2 + \gamma.$$

As shown in Appendix , if we choose a proper contour for the Laplace inverse transform, it can be shown that the real part of $q_1$ is always positive and the real part of $q_2$ is always negative. Therefore, $D_3$ has to be set to zero in order to satisfy the far-field boundary condition (29).

The satisfaction of the remaining boundary conditions as well as the interface conditions in (38)-(40) leads to a set of algebraic equations

$$\begin{cases} D_1 (p\bar{S}_f)^{q1} + D_2 (p\bar{S}_f)^{q2} = \frac{\gamma}{p(p+\gamma)}, \\ D_1 q_1 (p\bar{S}_f)^{q1-1} + D_2 q_2 (p\bar{S}_f)^{q2-1} = 0, \\ D_1 (1)^{q1} + D_2 (1)^{q2} - \frac{\gamma}{p(p+\gamma)} = D_4 (1)^{q2}, \\ D_1 q_1 (1)^{q1} + D_2 q_2 (1)^{q2} = D_4 q_2 (1)^{q2} + \frac{1}{p}, \end{cases} \tag{44}$$

the solution of which leads to a simple and yet elegant formula for the optimal exercise price in the Laplace space

$$\bar{S}_f = \frac{1}{p} \left[ \frac{\gamma q_2}{\gamma q_2 - (p+\gamma)} \right]^{\frac{1}{q_1}}, \tag{45}$$

as well as three coefficients from which the option price $U(S, \tau)$ will be determined.

By definition, the inversion of Eq. (45) should lead to the optimal exercise price $S_f(\tau)$ in the time domain. However, for the inverse Laplace transform

$$\bar{S}_f(\tau) = \frac{1}{2\pi i}\int_{\mu-\infty i}^{\mu+\infty i} \frac{e^{p\tau}}{p}\cdot\left[\frac{(p+\gamma)}{\gamma q_2 - (p+\gamma)}\right]^{\frac{1}{q_1}} dp,$$

$$= \frac{1}{2\pi i}\int_{\mu-\infty i}^{\mu+\infty i} \frac{e^{p\tau}}{p}\cdot\exp\left\{\frac{-\log\left[1-\frac{(p+\gamma)}{\gamma(b-\sqrt{p+a^2})}\right]}{b+\sqrt{p+a^2}}\right\} dp, \tag{46}$$

to exist, we need to show that $\bar{S}_f(p)$ is analytic to the right of the straight line $Re(p) = \mu$, where $\mu$ is an appropriately chosen positive number. In our problem here, other than a branch cup that ends at $p = -a^2, p = 0$ the only simple pole in $\bar{S}_f(p)$. To ensure that $Re(q_1) > 0$ and $Re(q_2) < 0$, we can show that it is sufficient to choose any $\mu$ such that

$$\mu > 0 \tag{47}$$

According to Cauchy's residue theorem, we have

$$\sum_{j=1}^{6}\int_{C_j} e^{p\tau}\bar{S}_f(p)dp = 2\pi i\sum_{k=1}^{n} Res_{p=p_k}\{e^{p\tau}\}\bar{S}_f(p)$$

$$= 2\pi i\sum_{k=1}^{n} Res_{p=p_k}\left\{\frac{e^{p\tau}}{p}\exp\left\{\frac{-\log\left[1-\frac{(p+\gamma)}{\gamma(b-\sqrt{p+a^2})}\right]}{b+\sqrt{p+a^2}}\right\}\right\}, \tag{48}$$

There is only one isolated simple pole of $e^{p\tau}$ at $p = 0$. The residue of $e^{p\tau}$ at this point can be readily evaluated as, which turns out to be the perpetual optimal exercise price shown by Broadie and Glasserman [26]. In comparison to his derivation, here we have amazingly reached the same conclusion naturally as the residue of the integrand in (46).

$$I_3 + I_5 = 2ie^{-a^2\tau}\int_0^\infty \frac{e^{-\rho\tau}}{\rho+a^2}\cdot Im\left\{\exp\left[\frac{-\log\left(1+\frac{(b-i\sqrt{\rho})}{\gamma}\right)}{b-i\sqrt{\rho}}\right]\right\} d\rho, \tag{49}$$

In (49), $Im\{\cdot\}$ stand for taking the imaginary part of the complex function inside of the curly bracket. After dividing both sides of Eq. (49) by $2\pi$ and substituting the result into Eq. (46) and Eq. (48), an analytical formula is obtained for the optimal exercise price $S_f$

$$S_f(\tau) = \frac{\gamma}{(1+\gamma)} + \frac{e^{-a^2\tau}}{\rho}\int_0^\infty \frac{e^{-\rho\tau}}{a^2+\rho}e^{-f_1(\rho)}\sin[f_2(\rho)]d\rho, \tag{50}$$

where

$$f_1(\rho) = \frac{1}{(b^2+\rho)}\left[b\ln\left(\frac{\sqrt{a^2+\rho}}{\gamma}\right) + \sqrt{\rho}\tan^{-1}\left(\frac{\sqrt{\rho}}{a}\right)\right],$$

$$f_2(\rho) = \frac{1}{(b^2+\rho)}\left[\sqrt{\rho}\ln\left(\frac{\sqrt{a^2+\rho}}{\gamma}\right) - b\tan^{-1}\left(\frac{\sqrt{\rho}}{a}\right)\right],$$

### Example 1

Calculate the level of accuracy in the computation of the optimal exercise price at $\tau = 0$. The parameters used by them are

- Strike price $X = \$100$,
- Risk-free interest rate $r = 0.1$,
- Volatility $\sigma = 0.3$,
- Time to expiration $T = 1$ (year).

In terms of the dimensionless variables, the two parameters involved are $\gamma = 2.2222$ and $\tau_{exp} = 0.045$. Since $\gamma \equiv \frac{2r}{\sigma^2}$ and $\tau_{exp} = T.\frac{\sigma^2}{2}$. For the remaining $\tau$ values, the computation was just as easy as it was for the case when $\tau = 0$. An interesting phenomenon observed in the calculation for nonzero $\tau$ is that the upper limit cannot be as large as one wishes once the exponential factor $e^{-t}$ has made the integrand vanish much faster. Generally speaking, the larger the $\tau$ is, the faster the integrand approaches zero and the smaller the upper limit needs to be in order to achieve a certain degree of accuracy.

**Table 1:** Optimal exercise price at the expiration time

| $R$ | $S_f(0)$ |
|---|---|
| $1 \times 10^6$ | 0.9954771159 |
| $1 \times 10^8$ | 0.9994008655 |
| $1 \times 10^{10}$ | 0.9999254241 |
| $1 \times 10^{13}$ | 0.9999969643 |

Table 1 gives the optimal exercise price at the expiration time for the illustration.

## 4.1 The Option Price of the American Put

Once $\bar{S}_f(p)$ is found, $D_1, D_2$ and $D_4$ can be easily found from (17) and written in terms of the Laplace parameter $p$ as

$$\begin{aligned} D_1 &= \frac{\gamma}{p(p+\gamma)} \cdot \frac{q_2}{(q_2-q_1)} \cdot \frac{1}{(p\bar{S}_f)^{q_1}} \\ &= -\frac{\gamma}{p(p+\gamma)} \cdot \frac{b-\sqrt{p+a^2}}{2\sqrt{p+a^2}} \cdot \left[1 - \frac{p+\gamma}{\gamma(b-\sqrt{p+a^2})}\right]. \end{aligned} \tag{51}$$

$$\begin{aligned} D_2 &= \frac{\gamma}{p(p+\gamma)} \cdot \frac{q_1}{(q_1-q_2)} \cdot \frac{1}{(p\bar{S}_f)^{q_2}} \\ &= \frac{\gamma}{p(p+\gamma)} \cdot \frac{b+\sqrt{p+a^2}}{2\sqrt{p+a^2}} \cdot \left[1 - \frac{p+\gamma}{\gamma(b-\sqrt{p+a^2})}\right]^{\frac{q_2}{q_1}}. \end{aligned} \tag{52}$$

$$\begin{aligned} D_4 &= D_1 + D_2 - \frac{\gamma}{p(p+\gamma)}, \\ &= \frac{\gamma}{p(p+\gamma)} \cdot \left\{ -\frac{b-\sqrt{p+a^2}}{2\sqrt{p+a^2}} \cdot \left[1 - \frac{p+\gamma}{\gamma(b-\sqrt{p+a^2})}\right] \right. \\ &\left. + \frac{b+\sqrt{p+a^2}}{2\sqrt{p+a^2}} \cdot \left[1 - \frac{p+\gamma}{\gamma(b-\sqrt{p+a^2})}\right]^{\frac{q_2}{q_1}} - 1 \right\}. \end{aligned} \tag{53}$$

Consequently, $U(S,\tau)$ can be written as

$$U(S,\tau) = \frac{1}{2\pi i}\int_{\mu-\infty i}^{\mu+\infty i} \frac{\gamma e^{p\tau}}{p(p+\gamma)} F_1(p)dp, \tag{54}$$

for $S_f(\tau) \leq S \leq 1$ and,

$$U(S,\tau) = \frac{1}{2\pi i}\int_{\mu-\infty i}^{\mu+\infty i} \frac{\gamma e^{p\tau}}{p(p+\gamma)} F_2(p)dp, \tag{55}$$

for $S > 1$

In Eq. (54) and Eq. (55), $F_1(p)$ and $F_2(p)$ are obtained by substituting Eq. (51)-Eq. (53) into Eq. (41) and can be written as

$$\begin{aligned} F_1(p) = & \tfrac{1}{2}\left(1 - \tfrac{b}{\sqrt{p+a^2}}\right)\cdot\left[1 - \tfrac{p+\gamma}{\gamma(b-\sqrt{p+a^2})}\right]\cdot S^{q_1} \\ & + \tfrac{1}{2}\left(1 - \tfrac{b}{\sqrt{p+a^2}}\right)\cdot\left[1 - \tfrac{p+\gamma}{\gamma(b+\sqrt{p+a^2})}\right]^{\frac{q_2}{q_1}}\cdot S^{q_2} - 1. \end{aligned} \tag{56}$$

$$\begin{aligned} F_2(p) = & \left\{\tfrac{1}{2}\left(1 - \tfrac{b}{\sqrt{p+a^2}}\right)\cdot\left[1 - \tfrac{p+\gamma}{\gamma(b-\sqrt{p+a^2})}\right]\right. \\ & \left. + \tfrac{1}{2}\left(1 - \tfrac{b}{\sqrt{p+a^2}}\right)\cdot\left[1 - \tfrac{p+\gamma}{\gamma(b+\sqrt{p+a^2})}\right]^{\frac{q_2}{q_1}} - 1\right\}\cdot S^{q_2}, \end{aligned} \tag{57}$$

## 4.2 American Options PDE

In this section we introduce a transformation for the valuation of American calls. The Black-Scholes equation modeling the price of a dividend paying asset $V$ , under deterministic yield $D$, volatility $\sigma$ and interest rate $r$ may be written as

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{ss} + (r-D)SV_s - rV = 0 \tag{58}$$

Here $S$ denotes the underlying asset on which the call option is written. The early exercise feature of the American option results in an optimal exercise boundary problem, which in the PDE setting is treated as a free boundary problem.We denoted the free boundary with $\tilde{B}(t)$. The domain of equation (58) is $(0, \tilde{B}(t)) \times [0,T)$. The boundary conditions are given below

$$V(S,T) = \max(S-K,0), S \in (0, \tilde{B}(T)) \tag{59}$$

$$\tilde{B}(T) = \max\left(K, \frac{rK}{D}\right) \tag{60}$$

$$V(0,t) = 0, \tag{61}$$

$$V(\tilde{B}(t),t) = \tilde{B}(t) - K, \tag{62}$$

$$V_s(\tilde{B}(t),t) = 1, \tag{63}$$

## 4.3 American Call with Dividends

From the Black-Scholes equation the value $C(S,t)$ of an American option satisfies

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D_0) S \frac{\partial C}{\partial S} - rC = 0, \tag{64}$$

This holds as long as exercise is not optimal. The payoff condition is

$$C(S,T) = \max(S - E, 0) \tag{65}$$

Also, since the option may be exercised at any time, we have that

$$C(S,t) \geq \max(S - E, 0) \tag{66}$$

Along the optimal exercise boundary $S = S_f(t)$

$$C(S_f(t), t) = S_f(t) - E \frac{\partial C}{\partial S}(S_f(t), t) = 1 \tag{67}$$

If the optimal exercise boundary exists then (64) is 65valid only while $C(S,t) > \max(S - E, 0)$ since $\max(S - E, 0)$ is not a solution of the Black-Scholes equation. (64) can be replaced by an inequality

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D_0) S \frac{\partial C}{\partial S} - rC \leq 0 \tag{68}$$

The inequality holds only if $C(S,t) > \max(S - E, 0)$. If early exercise if optimal, it is because the option would be less valuable than if it were exercised immediately and the funds deposited in an interest paying bank account.

## 4.4 General Analysis of Call with Dividends

We can simplify the Black-Scholes equation with dividend payments by assuming that the interest rate and the dividend payments satisfy $r > D_0 > 0$. We can then make equations (64), (65) and (66), dimensionless and reduce (64) to a constant coefficient forward equation 1. We now also subtract off the payoff $S - E$ for the call value $C(S,t)$

$$S = Ee^x, \quad t = T - \frac{\tau}{\frac{1}{2}\sigma^2}, \quad C(S,t) = S - E + Ec(x,\tau) \tag{69}$$

the result is

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial x^2} + (k' - 1) S \frac{\partial c}{\partial x} - kc + f(x) \tag{70}$$

where $k = \frac{r}{\frac{1}{2}\sigma^2}$, $k' = \frac{r - D_0}{\frac{1}{2}\sigma^2}$ for $-\infty < x, \infty$ and $\tau > 0$. The function $c(x,0)$, the initial profile, is given by

$$c(x,0) = \max(1 - e^x, 0) \tag{71}$$

The function $f$ is given by

$$f(x) = (k' - k)e^x + k \tag{72}$$

Assuming that the free boundary does exist, $x = x_f(t)$, at this boundary we have

$$c(x_f(\tau), \tau) = \frac{\partial c}{\partial \tau}(x_f(\tau), \tau) \tag{73}$$

From equation (73) at expiry we have

$$\frac{\partial c}{\partial \tau} = f(x) \tag{74}$$

For $0 < x < x_0, f(x) > 0$ and thus $c$ is positive. If $x > x_0$ then $f(x) < 0$ and $c$ will be negative. We have the constraint that $c > 0$ for $x > 0$ thus the latter does not satisfy this constraint.

# 5 Monte Carlo Method

Boyle [6] was the first researcher to introduce Monte Carlo simulation into finance. Monte Carlo method is a numerical method that is useful in many situations when no closed form solution is available. This method is good for pricing path dependent options.

The basis of Monte Carlo simulation is the strong law of large numbers, stating that the arithmetic mean of independent, identically distributed random variables, converges towards their mean almost surely. Monte Carlo simulation method uses the risk valuation result. The expected payoff in a risk-neutral world is calculated using a sampling procedure The main procedures are followed when using Monte Carlo simulation.

- Simulate a path of the underlying asset under the risk neutral condition within the desired time horizon.
- Discount the payoff corresponding to the path at the risk-free interest rate.
- Repeat the procedure for a high number of simulated sample path.
- Average the discounted cash flows over sample paths to obtain the options value.

A Monte Carlo simulation can be used as a procedure for sampling random outcomes of a process followed by the stock price [6]

$$dS = \mu S dt + \sigma S dW(t) \tag{75}$$

where $dW_t$ is a Wiener process and $S$ is the stock price. If $\delta S$ is the increase in the stock price in the next small interval of time $\delta t$ then

$$\frac{\delta S}{S} = \mu \delta t + \sigma Z \sqrt{\delta t} \tag{76}$$

where $Z \sim N(0, 1)$, $\sigma$ is the volatility of the stock price and $\mu$ is its expected return in a risk neutral, which can be expressed as

$$S(t + \delta t) - S(t) = \mu S(t)\delta t + \sigma S(t) Z \sqrt{\delta t} \tag{77}$$

We can calculate the value of $S$ at time $t + \delta t$ from the initial value of $S$, then the value of $S$ at time $t + 2\delta t$, from the value at $t + \delta t$ and so on. We use $N$ random samples from a normal distribution to simulate a trial for a complete path followed by $S$. It is more accurate to simulate $\ln S$ than $S$, we transform the asset price process using Ito's lemma

$$d(\ln S) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW(t)$$

so that

$$\ln S(t+\delta t) - \ln S(t) = \left(\mu - \frac{\sigma^2}{2}\right) \delta t + \sigma Z \sqrt{\delta t}$$

or

$$S(t+\delta t) = S(t) \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) \delta t + \sigma Z \sqrt{\delta t}\right] \tag{78}$$

Monte Carlo simulation is particularly relevant when the financial derivatives payoff depends on the path followed by the underlying asset during the life of the option, that is, for path dependent options. For example, we consider an Asian options whose Stock price process at maturity time $T$ is given by

$$S_T^j = S \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) T + \sigma Z \sqrt{T}\right] \tag{74}$$

where $j = 1, 2, ..., M$ and $M$ denotes the number of trials or the different states of the world. These $M$ simulations are the possible paths that a stock price can have at maturity date $T$. The estimated Asian call option value is

$$C = \frac{1}{M} \sum_{j=1}^{M} e^{-rT} \max[S_T^j - S_t, 0] \tag{79}$$

This is an unbiased estimate of derivative's price. When the number of trial $M$ is large, the central limit theorem provides a confidence interval for the estimate, based on the sample variance of the discounted payoff. The $M$ independent trials carried out depends on the accuracy acquired. If $\omega$ is the standard deviation and $\bar{\mu}$ is the mean of the discounted payoffs given, then the standard error is estimated by $\frac{\omega}{\sqrt{M}}$. A 0.95% confidence interval for the price $f$ of the derivative is therefore given by

$$\bar{\mu} - \frac{1.96\omega}{\sqrt{M}} < f < \bar{\mu} + \frac{1.96\omega}{\sqrt{M}} \tag{80}$$

under the assumption that $f$ is normally distributed [31]

# 6 Comparison of Scientific Computations

Scientific computing is the heart of simulation science, and this is one of the aspects of this project. The emphasis is on a balance between classical and modern elements of numerical mathematics and of computer science. To compare the results of the standard numerical approximations discussed above, we computed some examples of call options.

### Example 2:

Shows the performance of the two techniques against the 'true' Black-Scholes price for a European put with

$$K = 50, r = 0.05, \sigma = 0.25, T = 3.$$

**Table 2:** A comparison with the Black-Scholes price for a European Put

| S | Black-Scholes | Monte-Carlo | Implicit Euler |
|---|---|---|---|
| 10 | 33.0363 | 33.0345 | 33.0369 |
| 15 | 28.0619 | 28.0595 | 28.0629 |
| 20 | 23.2276 | 23.2291 | 23.2300 |
| 25 | 18.7361 | 18.7339 | 18.7390 |
| 30 | 14.7739 | 14.7748 | 14.7749 |
| 35 | 11.4384 | 11.4402 | 11.4402 |
| 40 | 8.7338 | 8.7374 | 8.7348 |
| 45 | 6.6021 | 6.6014 | 6.6012 |
| 50 | 4.9564 | 4.9559 | 4.9563 |
| 55 | 3.7046 | 3.7076 | 3.7042 |
| 60 | 2.7621 | 2.7602 | 2.7612 |
| 65 | 2.0574 | 2.0581 | 2.0571 |
| 70 | 1.5328 | 1.5324 | 1.5325 |
| 75 | 1.1430 | 1.1407 | 1.1426 |
| 80 | 0.8538 | 0.8543 | 0.8537 |

Table 2 shows the variation of the option price with the underlying price, $S$. The results demonstrate that the two techniques perform well, are mutually consistent, and agree with the Black-Scholes value. However, in practice, there is no real need for using such numerical techniques when we have an explicit formula. We will now attempt to price an option contract for which there is no closed formula.

# 7 Results Discussion

The study presents scientific computing for valuing derivatives when no analytic solution is available. This involve the use of Monte Carlo simulation and finite difference methods. For an American option, the value at a node is the greater of the value if it is exercised immediately and discounted expected value if it is held for a further period of time $\delta t$. The Monte Carlo simulation involves using random numbers to sample many different paths that the variables underlying the derivative could follow in a risk-neutral world. For each paths, the payoff is calculated and discounted at the risk-free interest rate. The arithmetic average of the discounted payoffs is the estimated value of the derivative. Finite Difference Method solve the underlying differential equation by converting it to a difference equation. They are similar to tree approaches in that the computations work back from the end of the life of the derivative to the beginning. The implicit finite difference method is more complicated but has the advantage that the user does not have to take any special precaution to ensure convergence.

In practice, the method that is chosen is likely to depend on the characteristics of the derivative being evaluated and the accuracy required. Monte Carlo simulation works forward from the beginning to the end of the life of a derivative. It can be used for European-style derivatives and can cope with a great deal of complexity as far as the payoff are concerned. It become relatively more efficient as the number of underlying variables increases. Finite Difference Method work from the end of the life of a security to the beginning and can accommodate American-style as well as European-style derivatives. However, they are difficult to apply when the payoffs depends on the past history of the state variables as well as on their currents values. In addition, they are liable to become computationally very time-consuming when three or more variable are involved. Lastly, one of the main drawbacks to using Monte-Carlo simulation is that though it works very well for pricing European-style path-dependent options, it is difficult to implement for early exercise (American-style) options. Finite Difference methods are, in general, far better suited to pricing these types of contract.

## Conclusion

This research provides a comprehensive analysis of the application of the Black-Scholes and Monte Carlo differential equation methods in the context of American options. The study underscores the inherent complexities in modeling American options, primarily due to their early exercise feature, which is not adequately addressed by the standard Black-Scholes model. Through a combination of finite difference methods and advanced Monte Carlo simulations, this research evaluates the effectiveness and accuracy of these computational approaches in pricing American options. Key findings from the analysis can be surmized: The Black-Scholes model offers a foundational and relatively straightforward approach to option pricing, its assumptions and limitations become evident when applied to American options; Monte Carlo simulations, with their ability to handle a wide range of potential price paths and complex boundary conditions, demonstrate superior accuracy and flexibility; The comparative analysis highlights that although the Black-Scholes model remains a vital tool in financial engineering, Monte Carlo simulations provide a more robust and adaptable framework for American option pricing.

In summary, the research contributes valuable insights into the strengths and limitations of different scientific computing techniques in option pricing. It underscores the importance of selecting appropriate models and methods based on the specific characteristics of the financial instruments being analyzed. Future work could explore hybrid approaches that integrate the analytical strengths of the Black-Scholes model with the numerical precision of Monte Carlo simulations, potentially offering even more effective solutions for pricing American options.

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