Common Fixed-Point of Dass-Gupta Rational Contractions and E-Contractions

ABSTRACT: In this paper, we establish some common fixed-point theorems in super-metric space for E-contraction and Dass-Gupta Rational Contraction. Additionally, these theorems expand and generalize several intriguing findings from metric fixed-point theory to the super metric setting. Furthermore, we offer an instance to clarify our theorems.

Keywords: fixed point; E-contraction; rational contraction; super-metric space.

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1. Introduction

A fixed point of a function is a point that doesn't move when the function is applied to it. In many branches of mathematics and its applications, including numerical analysis, optimization, and the study of dynamical systems, fixed points play a crucial role. They frequently depict equilibrium states of systems or solutions to equations. Finding a fixed point in an iterative process, for instance, can be comparable to finding a solution to the equation that is being iterated in the context of numerical equation-solving techniques.

A key finding in the theory of metric spaces is the Banach Contraction Principle, sometimes referred to as the Contraction Mapping Theorem. It gives the circumstances in which there is a unique fixed point for a mapping from a metric space to itself. This idea is fundamental to many branches of mathematics and its applications, such as functional analysis, numerical techniques, analysis, and optimization. It offers a strong tool for proving convergence in iterative algorithms and ensures the existence and uniqueness of solutions to certain equations and problems. The literature then extensively generalized the Banach contraction principle (see [1-15]). It is widely used in applied and pure mathematics alike.

In 1968, Kannan [11] developed a modified version of this theory and removed the continuity requirement. The first important variation of Banach's remarkable finding on the metric fixed-point theory is Kannan's fixed-point theorem. There are various ways to generalize Kannan's theorem.

Dass and Gupta [5] presented the Rational Contraction, which is a generalization of the Banach Contraction Mapping Principle. By using rational functions as the contraction condition rather than constants, it expands the concept of contraction maps to a more generic context. The traditional contraction mapping principle is made broader by the Dass-Gupta Rational Contraction condition, which permits the contraction factor to change based on the points being mapped. In certain applications, this enables a more flexible foundation. Similar to mappings satisfying the Banach Contraction Mapping Principle, the existence and uniqueness of fixed points for mappings satisfying the Dass-Gupta Rational Contraction condition can be determined by taking advantage of the rational function's properties as well as the underlying metric space's completeness. The notion of E-contraction was introduced by Fulga and Proca [7]. Later, this concept has been improved by several authors, e.g., [1, 6, 13].

Super-metric space was introduced by Fulga and Karapinar [12]. In this framework, we were able to derive various fixed-point theorems, and we think this approach could help relieve the congestion and squeeze issues previously mentioned.

In super metric space, we establish some common fixed-point theorems for rational contractions and E-contractions. these theorems expand and generalize several intriguing findings from metric fixed-point theory to the super metric setting. Furthermore, we present an example to illustrate our theorems.

2. Preliminaries

First, we recall the basic results and definitions.

***Definition* 2.1** (see [12]) Consider $\mathfrak{D}$ to be a non-empty set. A function $\mathfrak{d}: \mathfrak{D} \times \mathfrak{D} \to [0, +\infty)$ is considered a super metric if it fulfills the subsequent axioms:

(s1). $\forall\, \theta, \vartheta \in \mathfrak{D}$, if $\mathfrak{d}(\theta, \vartheta) = 0 \Longrightarrow\ \theta = \vartheta$.

(s2). $\forall\, \theta, \vartheta \in \mathfrak{D}, \mathfrak{d}(\theta, \vartheta) = \mathfrak{d}(\vartheta, \theta)$.

(s3). There exists $\mathfrak{s} \geq 1$ such that for every $\vartheta \in \mathfrak{D}$, there exist distinct sequences $\{\theta_i\}, \{\vartheta_i\} \subset \mathfrak{D}$, with $\mathfrak{d}(\theta_i, \vartheta_i) \to 0$ when $i \to \infty$, such that

$$\limsup_{i\to\infty} \mathfrak{d}(\vartheta_i, \vartheta) \leq \mathfrak{s} \limsup_{i\to\infty} \mathfrak{d}(\theta_i, \vartheta)$$

The tripled $(\mathfrak{D}, \mathfrak{d}, \mathfrak{s})$ is called a super metric space.

***Definition* 2.2** (see [12]) A sequence $\{\theta_i\}$ on a super metric space $(\mathfrak{D}, \mathfrak{d}, \mathfrak{s})$:

1) converges to $\theta \in \mathfrak{D} \Leftrightarrow \lim\limits_{i\to\infty} \mathfrak{d}(\theta_i, \theta) = 0$.
2) is a Cauchy sequence in $\mathfrak{D} \Leftrightarrow \lim\limits_{i\to\infty} \sup\{\mathfrak{d}(\theta_i, \theta_j): j > i\} = 0$.

***Proposition* 2.3** (see [12]) The limit of a convergent sequence is unique on a super metric space.

***Definition* 2.4** (see [12]) A super-metric space $(\mathfrak{D}, \mathfrak{d}, \mathfrak{s})$ is called complete iff each Cauchy sequence is convergent in $\mathfrak{D}$.

***Theorem* 2.5** (see [12]) Let $(\mathfrak{D}, \mathfrak{d}, \mathfrak{s})$ be a complete super-metric space and let $\Delta: \mathfrak{D} \to \mathfrak{D}$ be a mapping. Suppose that $0 < c < 1$ such that

$$\mathfrak{d}(\Delta\theta, \Delta\vartheta) \leq c\, \mathfrak{d}(\theta, \vartheta)$$

for all $(\theta, \vartheta) \in \mathfrak{D}$. Then, Δ has a unique fixed point in $\mathfrak{D}$.

***Theorem* 2.6** (see [12]) Let $(\mathfrak{D}, \mathfrak{d}, \mathfrak{s})$ be a complete super metric space and $\Delta: \mathfrak{D} \to \mathfrak{D}$ be a mapping, such that there exist $c \in [0,1)$ and that

$$\mathfrak{d}(\Delta\theta, \Delta\vartheta) \leq c \max\left\{\mathfrak{d}(\theta, \vartheta), \frac{\mathfrak{d}(\theta,\Delta\theta)\eta(\vartheta,\Delta\vartheta)}{\eta(\theta,\vartheta)+1}\right\}$$

Then, Δ has a unique fixed point.

3. Main Results

Our first main result as follows.

Theorem 3.1 Let $(\mathfrak{D}, \mathfrak{d}, \mathfrak{s})$ be a complete super-metric space and let Υ, Δ be self-mappings of $\mathfrak{X}$. If there exist real numbers $\mathfrak{r}_1, \mathfrak{r}_2 \geq 0$ with $\mathfrak{r}_1 + \mathfrak{r}_2 < 1$ such that

$$\mathfrak{d}(\Upsilon\theta, \Delta\vartheta) \leq \mathfrak{r}_1 \frac{\mathfrak{d}(\theta,\Upsilon\theta)\mathfrak{d}(\vartheta,\Delta\vartheta)}{1+\mathfrak{d}(\theta,\vartheta)} + \mathfrak{r}_2\mathfrak{d}(\theta,\vartheta) \tag{1}$$

for all $\theta, \vartheta \in \mathfrak{D}$. Then, Υ and Δ have a unique common fixed point in $\mathfrak{X}$.

Proof. Let $\theta_0 \in \mathfrak{X}$ and we define the class of iterative sequences $\{\theta_i\}$ such that $\theta_{i+1} = \Upsilon\theta_i$, $\theta_{i+2} = \Delta\theta_{i+1}$ for all $i \in \mathbb{N}$. Without loss of generality, we assume that $\theta_{i+2} \neq \Delta\theta_{i+1}$ for each nonnegative integer i. Indeed, if there exist a nonnegative integer i_0 such that $\theta_{i_0+2} = \Delta\theta_{i_0+1}$, then our proof of the Theorem proceeds as follows. Thus, from (1), we have

$$\begin{aligned}
0 < \mathfrak{d}(\theta_{i+1}, \theta_{i+2}) &= \mathfrak{d}(\Upsilon\theta_i, \Delta\theta_{i+1}) \\
&\leq \mathfrak{r}_1 \frac{\mathfrak{d}(\theta_i,\Upsilon\theta_i)\mathfrak{d}(\theta_{i+1},\Delta\theta_{i+1})}{1+\mathfrak{d}(\theta_i,\theta_{i+1})} + \mathfrak{r}_2\mathfrak{d}(\theta_i, \theta_{i+1}) \\
&= \mathfrak{r}_1 \frac{\mathfrak{d}(\theta_i,\theta_{i+1})\mathfrak{d}(\theta_{i+1},\theta_{i+2})}{1+\mathfrak{d}(\theta_i,\theta_{i+1})} + \mathfrak{r}_2\mathfrak{d}(\theta_i, \theta_{i+1}) \\
&\leq \mathfrak{r}_1\mathfrak{d}(\theta_{i+1}, \theta_{i+2}) + \mathfrak{r}_2\mathfrak{d}(\theta_i, \theta_{i+1})
\end{aligned}$$

The last inequality gives

$$0 < \mathfrak{d}(\theta_{i+1}, \theta_{i+2}) \leq \frac{\mathfrak{r}_2}{1-\mathfrak{r}_1}\mathfrak{d}(\theta_i, \theta_{i+1}) = c\,\mathfrak{d}(\theta_i, \theta_{i+1})$$

where $c = \frac{\mathfrak{r}_2}{1-\mathfrak{r}_1}$. From this, we can write

$$0 < \mathfrak{d}(\theta_{i+1}, \theta_{i+2}) \leq c\, \mathfrak{d}(\theta_i, \theta_{i+1}) \leq c^2\, \mathfrak{d}(\theta_{i-1}, \theta_i) \leq \cdots \leq c^{i+1}\, \mathfrak{d}(\theta_0, \theta_1) \tag{2}$$

On the other hand, one writes

$$\begin{aligned} 0 < \mathfrak{d}(\theta_{i+1}, \theta_i) &= \mathfrak{d}(\Upsilon\theta_i, \Delta\theta_{i-1}) \\ &\leq \mathfrak{r}_1 \frac{\mathfrak{d}(\theta_i, \Upsilon\theta_i)\mathfrak{d}(\theta_{i-1}, \Delta\theta_{i-1})}{1+\mathfrak{d}(\theta_i, \theta_{i-1})} + \mathfrak{r}_2 \mathfrak{d}(\theta_i, \theta_{i-1}) \\ &= \mathfrak{r}_1 \frac{\mathfrak{d}(\theta_i, \theta_{i+1})\mathfrak{d}(\theta_{i-1}, \theta_i)}{1+\mathfrak{d}(\theta_i, \theta_{i-1})} + \mathfrak{r}_2 \mathfrak{d}(\theta_i, \theta_{i-1}) \\ &\leq \mathfrak{r}_1 \mathfrak{d}(\theta_i, \theta_{i+1}) + \mathfrak{r}_2 \mathfrak{d}(\theta_i, \theta_{i-1}) \end{aligned}$$

which yields that,

$$0 < \mathfrak{d}(\theta_{i+1}, \theta_i) \leq \frac{\mathfrak{r}_2}{1-\mathfrak{r}_1} \mathfrak{d}(\theta_i, \theta_{i-1}) = c\, \mathfrak{d}(\theta_i, \theta_{i-1})$$

And then, we can write

$$0 < \mathfrak{d}(\theta_i, \theta_{i+1}) \leq c\, \mathfrak{d}(\theta_i, \theta_{i-1}) \leq c^2\, \mathfrak{d}(\theta_{i-1}, \theta_{i-2}) \leq \cdots \leq c^i\, \mathfrak{d}(\theta_0, \theta_1) \tag{3}$$

By appealing to (2) and (3), we find that

$$0 < \mathfrak{d}(\theta_i, \theta_{i+1}) \leq c^i\, \mathfrak{d}(\theta_0, \theta_1) \tag{4}$$

Taking the limit i tends to infinity in inequality (4), we get

$$\lim_{i\to\infty} \mathfrak{d}(\theta_i, \theta_{i+1}) = 0. \tag{5}$$

In what follows, we want to show that the sequence $\{\theta_i\}$ is a Cauchy sequence. Now suppose that, $i, j \in \mathbb{N}$ with $i > j$.Then from inequality (5) and using (s3), we get

$$\lim_{i\to\infty} \sup \mathfrak{d}(\theta_i, \theta_{i+2}) \leq \mathfrak{s} \lim_{i\to\infty} \sup \mathfrak{d}(\theta_{i+1}, \theta_{i+2}) \leq \mathfrak{s} \lim_{i\to\infty} \sup\{c^{i+1}\, \mathfrak{d}(\theta_0, \theta_1)\}. \tag{6}$$

Hence, $\lim_{i\to\infty} \sup \mathfrak{d}(\theta_i, \theta_{i+2}) = 0$. Similarly, we have

$$\lim_{i\to\infty}\sup \mathfrak{d}(\theta_i,\theta_{i+3}) \le \mathfrak{s}\lim_{i\to\infty}\sup \mathfrak{d}(\theta_{i+2},\theta_{i+3}) \le \mathfrak{s}\lim_{i\to\infty}\sup\{c^{i+2}\,\mathfrak{d}(\theta_0,\theta_1)\}. \tag{7}$$

Inductively, one can conclude that $\lim_{i\to\infty}\sup\{\mathfrak{d}(\theta_i,\theta_j): i > j\} = 0$. Thus, $\{\theta_i\}$ is a Cauchy sequence in a complete super-metric space $(\mathfrak{D},\mathfrak{d},\mathfrak{s})$, the sequence $\{\theta_i\}$ converges to $\theta^* \in \mathfrak{D}$ and then $\lim_{i\to\infty}\mathfrak{d}(\theta_i,\theta^*) = 0$. Further, we show that θ^* is the fixed point of Υ and Δ. If not, $\theta^* \ne \Upsilon\theta^* \ne \Delta\theta^*$, and then $\mathfrak{d}(\theta^*,\Upsilon\theta^*) > 0$ and $\mathfrak{d}(\theta^*,\Delta\theta^*) > 0$. Note that

$$\begin{aligned} 0 < \mathfrak{d}(\theta_{i+2},\Upsilon\theta^*) &= \mathfrak{d}(\Upsilon\theta^*,\theta_{i+2}) = \mathfrak{d}(\Upsilon\theta^*,\Delta\theta_{i+1}) \\ &\le \mathfrak{r}_1\frac{\mathfrak{d}(\theta^*,\Upsilon\theta^*)\mathfrak{d}(\theta_{i+1},\Delta\theta_{i+1})}{1+\mathfrak{d}(\theta^*,\theta_{i+1})} + \mathfrak{r}_2\mathfrak{d}(\theta^*,\theta_{i+1}) \\ &= \mathfrak{r}_1\frac{\mathfrak{d}(\theta^*,\Upsilon\theta^*)\mathfrak{d}(\theta_{i+1},\theta_{i+2})}{1+\mathfrak{d}(\theta^*,\theta_{i+1})} + \mathfrak{r}_2\mathfrak{d}(\theta^*,\theta_{i+1}) \end{aligned}$$

Taking $i \to \infty$, we derive $\lim_{i\to\infty}\sup \mathfrak{d}(\theta_{i+2},\Upsilon\theta^*) \le 0$. Thus, we have,

$$0 < \mathfrak{d}(\theta^*,\Upsilon\theta^*) \le \lim_{i\to\infty}\sup \mathfrak{d}(\theta_{i+2},\Upsilon\theta^*) \le 0 \tag{8}$$

and one can conclude that $\mathfrak{d}(\theta^*,\Upsilon\theta^*) = 0$, which implies that $\Upsilon\theta^* = \theta^*$.On the other hand,

$$\begin{aligned} 0 < \mathfrak{d}(\theta_{i+2},\Delta\theta^*) &= \mathfrak{d}(\Upsilon\theta_{i+1},\Delta\theta^*) \\ &\le \mathfrak{r}_1\frac{\mathfrak{d}(\theta_{i+1},\Upsilon\theta_{i+1})\mathfrak{d}(\theta^*,\Delta\theta^*)}{1+\mathfrak{d}(\theta_{i+1},\theta^*)} + \mathfrak{r}_2\mathfrak{d}(\theta_{i+1},\theta^*) \\ &= \mathfrak{r}_1\frac{\mathfrak{d}(\theta_{i+1},\theta_{i+2})\mathfrak{d}(\theta^*,\Delta\theta^*)}{1+\mathfrak{d}(\theta_{i+1},\theta^*)} + \mathfrak{r}_2\mathfrak{d}(\theta_{i+1},\theta^*) \end{aligned}$$

Taking $i \to \infty$, we derive $\lim_{i\to\infty}\sup\ \mathfrak{d}(\theta_{i+2},\Delta\theta^*) \le 0$. Thus, we have,

$$0 < \mathfrak{d}(\theta^*,\Delta\theta^*) \le \lim_{i\to\infty}\sup\ \mathfrak{d}(\theta_{i+2},\Delta\theta^*) \le 0 \tag{9}$$

and one can conclude that $\mathfrak{d}(\theta^*,\Upsilon\theta^*) = 0$, which implies that $\Delta\theta^* = \theta^*$. Hence, θ^* is a common fixed point of Υ and Δ. We shall now prove the uniqueness of θ^*. Suppose there exists another point $\vartheta^* \in \mathfrak{D}$ such that $\Upsilon\vartheta^* = \Delta\vartheta^* = \vartheta^*$. Then, by inequality (1), we have

$$\mathfrak{d}(\Upsilon\theta^*, \Delta\vartheta^*) \leq \mathfrak{r}_1 \frac{\mathfrak{d}(\theta^*, \Upsilon\theta^*)\mathfrak{d}(\vartheta^*, \Delta\vartheta^*)}{1+\mathfrak{d}(\theta^*, \vartheta^*)} + \mathfrak{r}_2\mathfrak{d}(\theta^*, \vartheta^*)$$

$$\leq \mathfrak{r}_2\mathfrak{d}(\theta^*, \vartheta^*) < \mathfrak{d}(\theta^*, \vartheta^*) \tag{10}$$

which is a contradiction.

If we take $\Upsilon = \Delta$ in condition (1), then we obtain the following corollary.

Corollary 3.2 Let $(\mathfrak{D}, \mathfrak{d}, \mathfrak{s})$ be a complete super-metric space and let Υ be a self-mapping of $\mathfrak{X}$. If there exist real numbers $\mathfrak{r}_1, \mathfrak{r}_2 \geq 0$ with $\mathfrak{r}_1 + \mathfrak{r}_2 < 1$ such that

$$\mathfrak{d}(\Upsilon\theta, \Upsilon\vartheta) \leq \mathfrak{r}_1 \frac{\mathfrak{d}(\theta, \Upsilon\theta)\mathfrak{d}(\vartheta, \Upsilon\vartheta)}{1+\mathfrak{d}(\theta, \vartheta)} + \mathfrak{r}_2\mathfrak{d}(\theta, \vartheta) \tag{11}$$

for all $\theta, \vartheta \in \mathfrak{D}$. Then, Υ has a unique fixed point in $\mathfrak{X}$.

If we take $\mathfrak{r}_1 = 0$ in Theorem 3.1 and Corollary 3.2, respectively, then we obtain the following corollaries.

Corollary 3.3 Let $(\mathfrak{D}, \mathfrak{d}, \mathfrak{s})$ be a complete super-metric space and let Υ, Δ be self-mappings of $\mathfrak{X}$. If there exists real number $0 \leq \mathfrak{r}_2 < 1$ such that

$$\mathfrak{d}(\Upsilon\theta, \Delta\vartheta) \leq \mathfrak{r}_2\mathfrak{d}(\theta, \vartheta) \tag{12}$$

for all $\theta, \vartheta \in \mathfrak{D}$. Then, Υ and Δ have a unique common fixed point in $\mathfrak{X}$.

Corollary 3.4 Let $(\mathfrak{D}, \mathfrak{d}, \mathfrak{s})$ be a complete super-metric space and let Υ be a self-mapping of $\mathfrak{X}$. If there exists real number $0 \leq \mathfrak{r}_1 < 1$ such that

$$\mathfrak{d}(\Upsilon\theta, \Upsilon\vartheta) \leq \mathfrak{r}_1\mathfrak{d}(\theta, \vartheta) \tag{13}$$

for all $\theta, \vartheta \in \mathfrak{D}$. Then, Υ has a unique fixed point in $\mathfrak{X}$.

We give an example which satisfy the conditions of Theorem 2.1.

Example 3.5 Let $\mathfrak{s} = 1$, and the function $\mathfrak{d}: [0, 1] \times [0, 1] \to [0, +\infty)$ be defined as follows:

$\mathfrak{d}(\theta,\vartheta) = \theta\vartheta$ for all $\theta \neq \vartheta$, and $\theta,\vartheta \in (0,1)$;

$\mathfrak{d}(\theta,\vartheta) = 0$ for all $\theta = \vartheta$, and $\theta,\vartheta \in [0,1]$;

$\mathfrak{d}(0,\vartheta) = \mathfrak{d}(\vartheta,0) = \vartheta$ for all $\vartheta \in (0,1]$;

$\mathfrak{d}(1,\vartheta) = \mathfrak{d}(\vartheta,1) = 1 - \frac{\vartheta}{2}$ for all $\vartheta \in [0,1)$;

First, we claim that $\mathfrak{d}$ is super-metric on $[0,1]$. We will concentrate on (s3) because (s1) and (s2) are simple to confirm. For any $\vartheta \in (0,1)$, we can choose the sequences $\{\theta_i\},\{\vartheta_i\} \subset [0,1]$, where

$$\theta_i = \frac{i^2+1}{i^2+2}, \text{ and } \vartheta_i = \frac{i+1}{i^2+2}, \text{ for any } n \in \mathbb{N}.$$

Since

$$\lim_{i\to\infty} \theta_i = \lim_{i\to\infty} \frac{i^2+1}{i^2+2} = \lim_{i\to\infty} \frac{1+\frac{1}{i^2}}{1+\frac{2}{i^2}} = 1$$

and
$$\lim_{i\to\infty} \vartheta_i = \lim_{i\to\infty} \frac{i+1}{i^2+2} = \lim_{i\to\infty} \frac{1+\frac{1}{i}}{i\left(1+\frac{2}{i^2}\right)} = 0.$$

Then, we have

$$\lim_{i\to\infty} \mathfrak{d}(\theta_i,\vartheta_i) = \lim_{i\to\infty} \theta_i\vartheta_i = \lim_{i\to\infty} \frac{i^2+1}{i^2+2}\frac{i+1}{i^2+2} = \lim_{i\to\infty} \frac{1+\frac{1}{i^2}}{1+\frac{2}{i^2}} \lim_{i\to\infty} \frac{1+\frac{1}{i}}{i\left(1+\frac{2}{i^2}\right)} = 0.$$

Thus,

$$\lim_{i\to\infty}\sup\ \mathfrak{d}(\theta_i,\vartheta) = \lim_{i\to\infty}\sup\ \theta_i\vartheta = \lim_{i\to\infty}\sup\left\{\left(\frac{i^2+1}{i^2+2}\right)\vartheta\right\} = \vartheta \lim_{i\to\infty}\sup\left(\frac{i^2+1}{i^2+2}\right) = \vartheta,$$

$$\lim_{i\to\infty}\sup\ \mathfrak{d}(\vartheta_i,\vartheta) = \lim_{i\to\infty}\sup\ \vartheta_i\vartheta = \lim_{i\to\infty}\sup\left\{\left(\frac{i+1}{i^2+2}\right)\vartheta\right\} = \vartheta \lim_{i\to\infty}\sup\left(\frac{i+1}{i^2+2}\right) = 0,$$

Therefore,

$$\lim_{i\to\infty}\sup\ \mathfrak{d}(\vartheta_i,\vartheta) = 0 < \vartheta = \mathfrak{s} \lim_{i\to\infty}\sup\ \mathfrak{d}(\theta_i,\vartheta),$$

and (s3) holds.

If $\vartheta = 0$, using the same sequences, we get

$$\limsup_{i\to\infty} \mathfrak{d}(\theta_i, \vartheta) = \limsup_{i\to\infty} \theta_i = \limsup_{i\to\infty} \frac{i^2+1}{i^2+2} = 1,$$

$$\limsup_{i\to\infty} \mathfrak{d}(\vartheta_i, \vartheta) = \limsup_{i\to\infty} \vartheta_i = \limsup_{i\to\infty} \frac{i+1}{i^2+2} = 0,$$

Therefore,

$$\limsup_{i\to\infty} \mathfrak{d}(\vartheta_i, \vartheta) = 0 < 1 = \mathfrak{s} \limsup_{i\to\infty} \mathfrak{d}(\theta_i, \vartheta),$$

and again (s3) holds.

If $\vartheta = 1$, using choosing $\theta_i = \frac{i+1}{i^2+2}$, and $\vartheta_i = \frac{i+2}{i+3}$, for any $n \in \mathbb{N}$. Then

$$\lim_{i\to\infty} \theta_i = \lim_{i\to\infty} \frac{i+1}{i^2+2} = 0 \text{ and } \lim_{i\to\infty} \vartheta_i = \lim_{i\to\infty} \frac{i+2}{i+3} = 1.$$

Then, we have

$$\lim_{i\to\infty} \mathfrak{d}(\theta_i, \vartheta_i) = \lim_{i\to\infty} \theta_i \vartheta_i = \lim_{i\to\infty} \frac{i+1}{i^2+2}\frac{i+2}{i+3} = 0.$$

Thus,

$$\limsup_{i\to\infty} \mathfrak{d}(\theta_i, \vartheta) = \limsup_{i\to\infty} \left(1 - \frac{\theta_i}{2}\right) = \limsup_{i\to\infty} \left(1 - \frac{i+1}{2(i^2+2)}\right) = \limsup_{i\to\infty} \frac{2i^2 - i + 3}{2(i^2+2)} = 1,$$

$$\limsup_{i\to\infty} \mathfrak{d}(\vartheta_i, \vartheta) = \limsup_{i\to\infty} \left(1 - \frac{\vartheta_i}{2}\right) = \limsup_{i\to\infty} \left(1 - \frac{i+2}{2(i+3)}\right) = \limsup_{i\to\infty} \frac{i+4}{2(i+3)} = \frac{1}{2},$$

Therefore,

$$\limsup_{i\to\infty} \mathfrak{d}(\vartheta_i, \vartheta) = \frac{1}{2} < 1 = \mathfrak{s} \limsup_{i\to\infty} \mathfrak{d}(\theta_i, \vartheta),$$

and again (s3) holds. Hence, $\mathfrak{d}$ defines a super-metric on $[0,1]$. Define mappings $\Upsilon, \Delta: [0,1] \to [0,1]$ as

$$\Upsilon\theta = \frac{\theta}{4}, \text{ if } \theta \in [0,1) \text{ and } \Upsilon\theta = \frac{1}{16}, \text{ if } \theta = 1,$$

$$\Delta\theta = \frac{\theta}{2}, \text{ if } \theta \in [0,1) \text{ and } \Delta\theta = \frac{1}{8}, \text{ if } \theta = 1.$$

Taking $\mathfrak{r}_1 = \frac{1}{9}$, $\mathfrak{r}_2 = \frac{1}{2}$

We consider the following cases:

1. If $\theta, \vartheta \in (0,1)$, we have

$$\mathfrak{d}(\Upsilon\theta, \Delta\vartheta) = \mathfrak{d}\left(\frac{\theta}{4}, \frac{\vartheta}{2}\right) = \frac{\theta\vartheta}{8} \le \frac{1}{9}\frac{\theta^2\vartheta^2}{(8+\theta\vartheta)} + \frac{1}{2}\theta\vartheta$$
$$\le \mathfrak{r}_1 \frac{\mathfrak{d}(\theta,\Upsilon\theta)\mathfrak{d}(\vartheta,\Delta\vartheta)}{1+\mathfrak{d}(\theta,\vartheta)} + \mathfrak{r}_2\mathfrak{d}(\theta,\vartheta)$$

2. If $\theta = 0, \vartheta \in (0,1)$, we have

$$\mathfrak{d}(\Upsilon\theta, \Delta\vartheta) = \mathfrak{d}(\Upsilon 0, \Delta\vartheta) = \mathfrak{d}\left(0, \frac{\vartheta}{2}\right) = \frac{\vartheta}{2} \le +\frac{1}{9}(0) + \frac{1}{2}\vartheta$$
$$\le \mathfrak{r}_1 \frac{\mathfrak{d}(\theta,\Upsilon\theta)\mathfrak{d}(\vartheta,\Delta\vartheta)}{1+\mathfrak{d}(\theta,\vartheta)} + \mathfrak{r}_2\mathfrak{d}(\theta,\vartheta)$$

3. If $\theta = 0, \vartheta = 0$, or $\theta = 1, \vartheta = 1$, we have

$$\mathfrak{d}(\Upsilon\theta, \Delta\vartheta) = 0 \le \frac{1}{9}\frac{\mathfrak{d}(\theta,\Upsilon\theta)\mathfrak{d}(\vartheta,\Delta\vartheta)}{1+\mathfrak{d}(\theta,\vartheta)} + \frac{1}{2}\mathfrak{d}(\theta,\vartheta)$$
$$\le \mathfrak{r}_1 \frac{\mathfrak{d}(\theta,\Upsilon\theta)\mathfrak{d}(\vartheta,\Delta\vartheta)}{1+\mathfrak{d}(\theta,\vartheta)} + \mathfrak{r}_2\mathfrak{d}(\theta,\vartheta)$$

4. If $\theta = 0, \vartheta = 1$, we have

$$\mathfrak{d}(\Upsilon\theta, \Delta\vartheta) = \mathfrak{d}(\Upsilon 0, \Delta 1) = \mathfrak{d}\left(0, \frac{1}{8}\right) = \frac{1}{8}$$
$$\le \frac{1}{9}\frac{(0)\left(\frac{1}{8}\right)}{1+1} + \frac{1}{2}(1)$$
$$= \mathfrak{r}_1 \frac{\mathfrak{d}(\theta,\Upsilon\theta)\mathfrak{d}(\vartheta,\Delta\vartheta)}{1+\mathfrak{d}(\theta,\vartheta)} + \mathfrak{r}_2\mathfrak{d}(\theta,\vartheta)$$

5. If $\theta = 1, \vartheta \in (0,1)$, we have

$$\mathfrak{d}(\Upsilon\theta, \Delta\vartheta) = \mathfrak{d}(\Upsilon 1, \Delta\vartheta) = \mathfrak{d}\left(\frac{1}{16}, \frac{\vartheta}{2}\right) = \frac{\vartheta}{32} \le \frac{1}{9}\frac{\frac{\vartheta^2}{32}}{1+\vartheta} + \frac{1}{2}\vartheta$$
$$\le \mathfrak{r}_1 \frac{\mathfrak{d}(\theta,\Upsilon\theta)\mathfrak{d}(\vartheta,\Delta\vartheta)}{1+\mathfrak{d}(\theta,\vartheta)} + \mathfrak{r}_2\mathfrak{d}(\theta,\vartheta)$$

In view of Theorem 3.1, we conclude that Υ and Δ have a unique common fixed point $0 \in [0,1]$.

Theorem 3.6 Let $(\mathfrak{D}, \mathfrak{d}, \mathfrak{s})$ be a complete super-metric space and let Υ, Δ be self-mappings of $\mathfrak{X}$. If there exists real number $\mathfrak{r} \in (0,1)$ such that

$$\mathfrak{d}(\Upsilon\theta, \Delta\vartheta) \leq \mathfrak{r}[\mathfrak{d}(\theta, \vartheta) + |\, \mathfrak{d}(\theta, \Upsilon\theta) - \mathfrak{d}(\vartheta, \Delta\vartheta)|] \tag{14}$$

for all $\theta, \vartheta \in \mathfrak{D}$. Then, Υ and Δ have a unique common fixed point in $\mathfrak{X}$.

Proof Following the steps of proof of Theorem 3.1, we construct the sequence $\{\theta_i\}$ by iterating

$$\theta_{i+1} = \Upsilon\theta_i, \theta_{i+2} = \Delta\theta_{i+1} \text{ for all } i \in \mathbb{N}.$$

where $\theta_0 \in \mathfrak{X}$is arbitrary starting point. Then, by (14), we have

$$\begin{aligned} 0 < \mathfrak{d}(\theta_{i+1}, \theta_{i+2}) &= \mathfrak{d}(\Upsilon\theta_i, \Delta\theta_{i+1}) \\ &\leq \mathfrak{r}[\mathfrak{d}(\theta_i, \theta_{i+1}) + |\, \mathfrak{d}(\theta_i, \Upsilon\theta_i) - \mathfrak{d}(\theta_{i+1}, \Delta\theta_{i+1})|] \\ &= \mathfrak{r}[\mathfrak{d}(\theta_i, \theta_{i+1}) + |\mathfrak{d}(\theta_i, \theta_{i+1}) - \mathfrak{d}(\theta_{i+1}, \theta_{i+2})|] \end{aligned} \tag{15}$$

If $\mathfrak{d}(\theta_i, \theta_{i+1}) < \mathfrak{d}(\theta_{i+1}, \theta_{i+2})$ for some i, from (15), we have

$$\mathfrak{d}(\theta_{i+1}, \theta_{i+2}) \leq \mathfrak{r}[\mathfrak{d}(\theta_i, \theta_{i+1}) - \mathfrak{d}(\theta_i, \theta_{i+1}) + \mathfrak{d}(\theta_{i+1}, \theta_{i+2})] = \mathfrak{r}\mathfrak{d}(\theta_{i+1}, \theta_{i+2})$$

which is a contradiction. Hence, $\mathfrak{d}(\theta_i, \theta_{i+1}) > \mathfrak{d}(\theta_{i+1}, \theta_{i+2})$ and so from (15), we have

$$\mathfrak{d}(\theta_{i+1}, \theta_{i+2}) \leq \mathfrak{r}[\mathfrak{d}(\theta_i, \theta_{i+1}) + \mathfrak{d}(\theta_i, \theta_{i+1}) - \mathfrak{d}(\theta_{i+1}, \theta_{i+2})]$$

The last inequality gives

$$0 < \mathfrak{d}(\theta_{i+1}, \theta_{i+2}) \leq \frac{2\mathfrak{r}}{1+\mathfrak{r}} \mathfrak{d}(\theta_i, \theta_{i+1}) = c\, \mathfrak{d}(\theta_i, \theta_{i+1})$$

where $c = \frac{2\mathfrak{r}}{1+\mathfrak{r}}$. From this, we can write

$$0 < \mathfrak{d}(\theta_{i+1}, \theta_{i+2}) \leq c_1\, \mathfrak{d}(\theta_i, \theta_{i+1}) \leq {c_1}^2\, \mathfrak{d}(\theta_{i-1}, \theta_i) \leq \cdots \leq {c_1}^{i+1}\, \mathfrak{d}(\theta_0, \theta_1) \tag{16}$$

On the other hand, one writes

$$0 < \mathfrak{d}(\theta_i, \theta_{i+1}) = \mathfrak{d}(\Upsilon\theta_{i-1}, \Delta\theta_i)$$

$$\leq \mathfrak{r}[\mathfrak{d}(\theta_{i-1}, \theta_i) + |\, \mathfrak{d}(\theta_{i-1}, \Upsilon\theta_{i-1}) - \mathfrak{d}(\theta_i, \Delta\theta_i)|]$$

$$= \mathfrak{r}[\mathfrak{d}(\theta_i, \theta_{i-1}) + |\mathfrak{d}(\theta_{i-1}, \theta_i) - \mathfrak{d}(\theta_i, \theta_{i+1})|] \tag{17}$$

If $\mathfrak{d}(\theta_{i-1}, \theta_i) < \mathfrak{d}(\theta_i, \theta_{i+1})$ for some i, from (17), we have

$$\mathfrak{d}(\theta_i, \theta_{i+1}) \leq \mathfrak{r}[\mathfrak{d}(\theta_i, \theta_{i-1}) - \mathfrak{d}(\theta_{i-1}, \theta_i) + \mathfrak{d}(\theta_i, \theta_{i+1})] = \mathfrak{r}\, \mathfrak{d}(\theta_i, \theta_{i+1})$$

which is a contradiction. Hence, $\mathfrak{d}(\theta_{i-1}, \theta_i) > \mathfrak{d}(\theta_i, \theta_{i+1})$ and so from (17), we have

$$\mathfrak{d}(\theta_i, \theta_{i+1}) \leq \mathfrak{r}[\mathfrak{d}(\theta_i, \theta_{i-1}) + \mathfrak{d}(\theta_{i-1}, \theta_i) - \mathfrak{d}(\theta_i, \theta_{i+1})]$$

which yields that,

$$0 < \mathfrak{d}(\theta_{i+1}, \theta_i) \leq \frac{\mathfrak{r}_1 + \mathfrak{r}_3}{1 - \mathfrak{r}_2 - \mathfrak{r}_4} \mathfrak{d}(\theta_i, \theta_{i-1}) = c\, \mathfrak{d}(\theta_i, \theta_{i-1})$$

where $c_2 = \frac{\mathfrak{r}_1 + \mathfrak{r}_2}{1 - \mathfrak{r}_3 - \mathfrak{r}_4}$. Then, we can write

$$0 < \mathfrak{d}(\theta_i, \theta_{i+1}) \leq c_2\, \mathfrak{d}(\theta_i, \theta_{i-1}) \leq {c_2}^2\, \mathfrak{d}(\theta_{i-1}, \theta_{i-2}) \leq \cdots \leq {c_2}^i\, \mathfrak{d}(\theta_0, \theta_1) \tag{18}$$

By appealing to (18) and (19), we find that

$$0 < \mathfrak{d}(\theta_i, \theta_{i+1}) \leq c^i\, \mathfrak{d}(\theta_0, \theta_1) \tag{19}$$

Taking the limit i tends to infinity in inequality (19), we get

$$\lim_{i \to \infty} \mathfrak{d}(\theta_i, \theta_{i+1}) = 0. \tag{20}$$

As already elaborated in the proof of Theorem 3.1, the classical procedure leads to $\{\theta_i\}$ is a Cauchy sequence in a complete super-metric space $(\mathfrak{D}, \mathfrak{d}, \mathfrak{s})$. Hence, the sequence $\{\theta_i\}$ converges to $\theta^* \in \mathfrak{D}$ and then $\lim_{i \to \infty} \mathfrak{d}(\theta_i, \theta^*) = 0$. Further, we show that θ^* is the fixed point of Υ and Δ. If not, $\theta^* \neq \Upsilon\theta^* \neq \Delta\theta^*$, and then $\mathfrak{d}(\theta^*, \Upsilon\theta^*) > 0$ and $\mathfrak{d}(\theta^*, \Delta\theta^*) > 0$. From (14), we have

$$0 < \mathfrak{d}(\theta_{i+2}, \Upsilon\theta^*) = \mathfrak{d}(\Upsilon\theta^*, \theta_{i+2}) = \mathfrak{d}(\Upsilon\theta^*, \Delta\theta_{i+1})$$

$$\leq \mathfrak{r}[\mathfrak{d}(\theta^*, \theta_{i+1}) + |\mathfrak{d}(\theta^*, \Upsilon\theta^*) - \mathfrak{d}(\theta_{i+1}, \Delta\theta_{i+1})|]$$

$$\leq \mathfrak{r}[\mathfrak{d}(\theta^*, \theta_{i+1}) + |\mathfrak{d}(\theta^*, \Upsilon\theta^*) - \mathfrak{d}(\theta_{i+1}, \theta_{i+2})|]$$

Taking $i \to \infty$, we derive $\limsup_{i\to\infty} \mathfrak{d}(\theta_{i+2}, \Upsilon\theta^*) \leq \mathfrak{r}\mathfrak{d}(\theta^*, \Upsilon\theta^*)$. Thus, we have,

$$0 < \mathfrak{d}(\theta^*, \Upsilon\theta^*) \leq \limsup_{i\to\infty} \mathfrak{d}(\theta_{i+2}, \Upsilon\theta^*) \leq \mathfrak{r}\mathfrak{d}(\theta^*, \Upsilon\theta^*) \quad (21)$$

and one can conclude that $\mathfrak{d}(\theta^*, \Upsilon\theta^*) = 0$, which implies that $\Upsilon\theta^* = \theta^*$.On the other hand,

$$0 < \mathfrak{d}(\theta_{i+2}, \Delta\theta^*) = \mathfrak{d}(\Upsilon\theta_{i+1}, \Delta\theta^*)$$

$$\leq \mathfrak{r}[\mathfrak{d}(\theta_{i+1}, \theta_{i+1}) + |\mathfrak{d}(\theta_{i+1}, \Upsilon\theta_{i+1}) - \mathfrak{d}(\theta^*, \Delta\theta^*)|]$$

$$\leq \mathfrak{r}[\mathfrak{d}(\theta^*, \theta_{i+1}) + |\mathfrak{d}(\theta_{i+1}, \theta_{i+2}) - \mathfrak{d}(\theta^*, \Delta\theta^*)|]$$

Taking $i \to \infty$, we derive $\limsup_{i\to\infty} \mathfrak{d}(\theta_{i+2}, \Delta\theta^*) \leq \mathfrak{r}\mathfrak{d}(\theta^*, \Delta\theta^*)$. Thus, we have,

$$0 < \mathfrak{d}(\theta^*, \Delta\theta^*) \leq \limsup_{i\to\infty} \mathfrak{d}(\theta_{i+2}, \Delta\theta^*) \leq \mathfrak{r}\mathfrak{d}(\theta^*, \Delta\theta^*) \quad (22)$$

and one can conclude that $\mathfrak{d}(\theta^*, \Upsilon\theta^*) = 0$, which implies that $\Delta\theta^* = \theta^*$. Hence, θ^* is a common fixed point of Υ and Δ. We shall now prove the uniqueness of θ^*. Suppose there exists another point $\vartheta^* \in \mathfrak{D}$ such that $\Upsilon\vartheta^* = \Delta\vartheta^* = \vartheta^*$. Then, by inequality (14), we have

$$0 < \mathfrak{d}(\theta^*, \vartheta^*) = \mathfrak{d}(\Upsilon\theta^*, \Delta\vartheta^*) \leq \mathfrak{r}[\mathfrak{d}(\theta^*, \vartheta^*) + |\mathfrak{d}(\theta^*, \Upsilon\theta^*) - \mathfrak{d}(\vartheta^*, \Delta\vartheta^*)|]$$

$$\leq \mathfrak{r}\mathfrak{d}(\theta^*, \vartheta^*) < \mathfrak{d}(\theta^*, \vartheta^*) \quad (23)$$

which is a contradiction. This completes the proof.

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