ON THE WAVELET-BASED GALERKIN FINITE ELEMENT TECHNIQUE FOR THE SOLUTION OF TIME-FRACTIONAL ADVECTION-DIFFUSION PROBLEMS

Abstract

This paper revisited the newly constructed wavelet-based Galerkin finite element technique by Iweobodo *et al* (2023) together with its application in seeking approximate solutions to time-fractional advection-diffusion problems. Orthogonal polynomials Mamadu-Njoseh polynomials and finite element method were discussed in relation to Iweobodo-Mamadu-Njoseh wavelet (IMNW) and the step-by-step application of the wavelet-based Galerkin finite element technique using the (IMNW) as basis function was iterated. Also, a convergence investigation of the IMNW wavelet-based Galerkin finite element technique was undertaken, and the resulting evidence exhibited uniformity in convergence.

Keywords: Wavelets, Iweobodo-Mamadu-Njoseh Wavelet, Mamadu-Njoseh Polynomials, Weight functions, Orthogonality and Orthonormality, Galerkin finite element technique

2010 Mathematics Subject Classification: 53C25; 83C05; 57N16

1 Introduction

Time fractional advection-diffusion equations have different physical interpretations, some of these interpretations are identified in terms of heat transportation with external force or additional velocity field, Brownian motion, the process of transportation in a porous medium, diffusion of charges in the electrical field on comb structures, hydrology of ground water, etc (Arkhincheev 2000 [1], and Neild and Bejan 2006 [2]). It is defined as

$$\frac{\partial^{\alpha} w}{\partial t^{\alpha}} = a\Delta w - v \bigtriangledown w, 0 < \alpha \leqslant 1. \quad (1.1)$$

Where a is the diffusion coefficient, v is the velocity vector, $\bigtriangledown$ is a gradient operator which is the first derivative of w, Δ is another gradient operator which is the second derivative of w, w is the

dependent variable, $0 \leqslant \zeta \leqslant 1$, and $0 < t \leqslant T$.
Some studies have shown that differential equations are very significant in wide varieties of real life situations today. For example, in physics, differential equations can be applicable in modeling movement of particles in fluids or in the trajectory of a projectile; in biology, differential equations are applicable in modeling population growth or the spread of diseases, and lots more. Also, Iweobodo *et al* (2021)[3] asserted that many problems involving chemical reactions, wave propagation, heat flow, stock market predictions, etc; are modeled with differential equations. The capacity to model complex physical situations with differential equations makes them valuable and useful instruments to scientists and engineers, according to Povstenko (2014) [4], investigating these physical phenomena with complex structures led to the consideration of fractional differential equations. Differential equations can be useful in predicting and studying the future behavior of certain systems and how they can be manipulated in order to achieve expected and helpful results in designing new technologies and foreseeing the outcome of experiments.
Since its inception, the contributions of the Mamadu-Njoseh polynomial has continued to deepen in the field of science engineering and technology as authors continue to apply it in making positive impact. For instance, Njoseh (2018)[5] with Variational Iteration method (VIM), Mamadu and Ojarikre (2021)[6] for the approximation of fractional integro-differential equations, Njoseh and Musa (2019)[7] for solving the Pantograph-type delay differential equations, Al-Humedi and KadhimMunaty (2021)[8] on the solution of first kind integral equation by spectral petro-Galerkin method, Tsetimi and Mamadu (2021)[9] for the solutions of Cauchy-partial differential equations, etc.
Some researchers have considered different methods for time-fractional differential equations. However, the wavelet-based Galerkin finite element method is among the recently developed methods. Also, some wavelets such as the Haar wavelets(Stankovic and Falkowski 2003)[10], Daubechies wavelets[3], Chebyshev wavelets (Babolian and Fattahzadeh 2007)[11], Languerre wavelets (Pathak and Pandey 2009) [12], Hermite wavelets (Shiralashetti *et al* 2019)[13], etc exist in literature today. Some studies have shown that some of the existing wavelets were developed from existing orthogonal polynomials. The Mamadu-Njoseh polynomial is an orthogonal polynomial; however, no author developed a wavelet from the Mamadu-Njoseh polynomial until Iweobodo *et al* (2023) [14]. Thus we term it Iweobodo-Mamadu-Njoseh wavelet (IMNW). Being orthogonal, our motivation in this work is the urge towards observing more of the performance of this new orthonomal wavelet. Hence, in this work, we intend to apply (IMNW) together with the Galerkin finite element technique in seeking solutions to time-fractional advection-diffusion problems (1) in the Caputo sense. To achieve our results, we shall reconsider the development of the (IMNW) and revisit the formulation of the wavelet-based Galerkin finite element technique with (IMNW) as the basis function, then test for the convergence of the new method.

2 Materials and Methods

2.1 Orthogonal Polynomials

Orthogonal polynomials are a class of polynomials $\eta_n(x)$ defined over an interval $[a, b]$, satisfying the orthogonal function

$$\int_a^b \Lambda(x)\eta_i(x)\eta_j(x)dx = h_i\delta_{ij}$$

where $\Lambda(x)$ is the weight function, and δ_{ij} is the Kronecker delta defined as

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

2.2 Mamadu-Njoseh Polynomials

Mamadu-Njoseh polynomials are polynomials which are constructed by Njoseh and Mamadu (2016) [15] within the interval $[-1,1]$ wrt the weight function x^2+1. According to Mamadu and Njoseh (2016) [16], their realization was based on these three properties:

(1.) $\eta_n(x)=\sum_{i=0}^{n} C_i^{(n)} x^i$

(2.) $<\eta_m(x),\eta_n(x)>=0, m\neq n$

(3.) $\eta_n(x)=1$

where $\eta_i, i=0,1,2,\cdots$ are orthogonal polynomials.
Therefore, the first seven Mamadu-Njoseh polynomials are given as

$$\begin{aligned}
\eta_0(x)&=1\\
\eta_1(x)&=x\\
\eta_2(x)&=\frac{1}{3}(5x^2-2)\\
eta_3(x)&=\frac{1}{5}(14x^3-9x)\\
\eta_4(x)&=\frac{1}{648}(333-289x^2+3213x^4)\\
\eta_5(x)&=\frac{1}{136}(325x-1410x^3+1221x^5)\\
\eta_6(x)&=\frac{1}{1064}(-460+8685x^2-24750x^4+17589x^6)
\end{aligned}\tag{2.1}$$

2.3 Wavelet Transform

Wavelets are made up of a family of functions which are formulated from the dilation and translation of a single function known as the mother wavelet (Amaratunga and William 1994) [17]. If the dilation and translation parameters a and b, respectively, vary continuously, we have its mathematical representation as

$$\psi_{a,b}(x)=|a|^{-\frac{1}{2}}\psi\left(\frac{x-b}{a}\right), \forall a,b\in\Re, a\neq 0 \tag{2.2}$$

If the parameters are discrete values, that is, considering $a=a_0^{-k}$, and $b=nb_0a_0^{-k}$, $a_0>1, b_0>0$, then the family of discrete wavelets is given as

$$\psi_{k,n}(x)=|a|^{-\frac{1}{2}}\psi\left(a_0^k-nb_0\right), \forall a,b\in\Re, a\neq 0. \tag{2.3}$$

And $\psi_{k,n}(x)$ forms a wavelet basis for $L_2(\Re)$. To be particular, when $a_0=2$ and $b_0=1$, then $\psi_{k,n}(x)$ forms an orthonormal basis.

$$\psi_{j,k}(x)=2^{\frac{j}{2}}\psi(2^jx-k). \tag{2.4}$$

According to Saeed and Rehmann [18], the set $\psi_{j,k}(x)$ forms an orthogonal basis of $L_2(\Re)$, which implies that

$$<\psi_{j,k}(x),\psi_{l,m}(x)>=\delta_{jl}\delta_{km}. \tag{2.5}$$

2.4 Iweobodo-Mamadu-Njoseh Wavelet(IMNW)

This is a new orthonormal wavelet developed by [5], it is defined as

$$\chi_{n,m}(x)=\begin{cases}2^{\frac{k}{2}}\left(\overline{MN}\right)_m(2^kx-2n+1), & \frac{n-1}{2^{k-1}}\leqslant x\leqslant\frac{n}{2^{k-1}}\\ 0, & \text{Otherwise}\end{cases} \tag{2.6}$$

where

$$(\overline{MN})_m = \sqrt{\frac{2}{\pi}} MN_m,$$

$m = 0, 1, \cdots, M-1$, $n = 1, 2, \cdots, 2^{k-1}$, k is any positive integer, and MN_m are the Mamadu-Njoseh polynomials of degree m with respect to the weight function $x^2 + 1$ on the interval $[-1, 1]$. For $n = 1$ and $k = 1$, the first five (IMNW) is obtained as

$\chi_{1,0} = 2^{\frac{1}{2}} \sqrt{\frac{2}{\pi}} \approx \frac{2}{\sqrt{\pi}}$

$\chi_{1,1} = \frac{2}{\sqrt{\pi}}(2x-1)$

$$\chi_{1,2} = \frac{2}{\sqrt{\pi}} \frac{1}{3} \left[5(2x-1)^2 - 2\right] = \frac{2}{\sqrt{\pi}} \frac{1}{3} \left[5(4x^2 - 4x + 1) - 2\right]$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{3} \left[20x^2 - 20x + 3\right]$$

$$\chi_{1,3} = \frac{2}{\sqrt{\pi}} \frac{1}{5} \left[14(2x-1)^3 - 9(2x-1)\right]$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{5} \left[14(8x^3 - 12x^2 + 6x - 1) - 9(2x-1)\right]$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{5} \left[112x^3 - 168x^2 + 84x - 14 - 18x + 9\right]$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{5} \left[112x^3 - 168x^2 + 66x - 5\right]$$

$$= \frac{2}{\sqrt{\pi}} \left[\frac{112}{5}x^3 - \frac{168}{5}x^2 + \frac{66}{5}x - 1\right]$$

$$\chi_{1,4} = \frac{2}{\sqrt{\pi}} \frac{1}{648} \left[3213(2x-1)^4 - 289(2x-1)^2 + 333\right]$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{648} [3213(16x^4 - 32x^3 + 24x^2 - 8x + 1)$$

$$- 289(4x^2 - 4x + 1) + 333]$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{648} [51408x^4 - 102816x^3 + 77112x^2 - 25704x$$

$$+ 3213 - 1156x^2 + 1156x - 289 + 333]$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{648} [51408x^4 - 102816x^3 + 75956x^2 - 26860x$$

$$+ 333]$$

$$= \frac{2}{\sqrt{\pi}} [\frac{51408}{648}x^4 - \frac{102816}{648}x^3 + \frac{75956}{648}x^2 - \frac{26860}{648}x$$

$$- \frac{333}{648}]$$

2.5 Properties of (IMNW)

Some of the important properties of the (IMNW) as stated in [14] include

i. **Orthogonality and Orthonormality**: (IMNW) inherited its orthonormality property from the point of it development, being that the Mamadu-Njoseh Polynomials which it emanated from are orthogonal polynomials. Also, orthogonality implies that

$$< \chi_{n,m} R(x) >= \int_0^1 \chi_{n,m} R(x) dx = 0$$

for $n = 1$, and $m = 1, 2, \cdot$.
The achievement of this is with the weight function $x - x^2$.
This orthonormality has given existence to orthogonality and normalization because
Orthogonality $\Rightarrow$ Orthonomality + Normalization
Normalization is a way of multiplying a function by a constant so that a result can be obtained.
2. **Admissibility**
3. **Regularity**
Their justification is outlined in [14].

2.6 Finite Element Method

Njoseh and Ayoola (2008a)[19] described the finite element method (FEM) as a numerical tool suitable for problems posed in variational form in a given space, eg Hilbert space. They continued by identifying some of its applicable areas such as electric and magnetic field, heat transfer, structural mechanics and dynamics, heat flow, acoustic, etc. They asserted that the method is reliable and easy to analyze in many situations because of its strong theoretical basis. In another work, Njoseh and Ayoola (2008b)[20] applied the finite element scheme in seeking solution of a strongly damped stochastic wave equation which is driven by space time noise. They were able to achieve the estimated error of optimal order for both the semidiscrete and the full discrete methods with the use of L_2 projections of the initial data as the first value. In further exploration on the finite element method, Njoseh and Atonuje (2003)[21] identified the computer as one of the instruments causing the finite element method further growth and wider visibility. They listed some applicable areas of the finite element method which include stress flowing around a reinforced opening, aircraft wing, aerodynamics, piston and fin, belleville spring, etc. The following are the major steps involved in solving a differential problem with the finite element method.

i. Formulate an equivalent variational equation.

ii. Implement the discretization process by constructing a finite dimensional space.

iii. Find the solution of the obtained discrete equation.

iv. Use computer software to implement the solution.

3 Application of the New Scheme on TFADE

3.1 Implementing Solutions to TFADE

Given a time-fractional advection-diffusion equation
(TFADE) of the form (1), to solve this with the new wavelet-based Galerkin finite element technique, we consider the following steps

1. Apply time discretization on the term ${}_0^cD_t^\alpha u(x,t)$ and substitute into the equation.

2. choose a trial solution in form of (IMNW) $u(x,t) = \sum_{n=1}^{2^{k-1}} \sum_{r=0}^{j=M-1} c_{n,r}\chi_{n,r}$.

3. Obtain the variational formulation of the given equation by formulating the residual equation $R(x)$. This is done by rewriting equation (1) in the form

$$R(x) =_0^c D_t^\alpha u(x,t) - a\Delta u(x,t) + v \bigtriangledown u(x,t), 0 < \alpha \leqslant 1. \tag{3.1}$$

4. Differentiate the basis function $\sum_{n=1}^{2^{k+1}} \sum_{r=0}^{M-1} \chi_{n,r}$ to get the terms $\left[\sum_{n=1}^{2^{k+1}} \sum_{r=0}^{M-1} \chi_{n,r}\right]_{xx}$ and $\left[\sum_{n=1}^{2^{k+1}} \sum_{r=0}^{M-1} \chi_{n,r}\right]_{x}$.

5. Substitute the terms back into the original equation.

6. Apply space-discretization by taking the inner product of the equation together with the residual equation.
7. Apply the orthogonality condition,
steps 6 and 7 are obtained by integrating on the boundary values together with the residual function and equating it to 0 as considered in [5], this will amount to

$$\int_0^1 \chi_{1,m}(x)R(x)dx = 0, m = 0, 1, 2, \cdots \tag{3.2}$$

which is a system of linear equations, we then solve the obtained linear equations for the unknown parameters which will be substituted into the solution function to achieve the desired numerical solutions.

3.2 Convergence of the proposed method

The convergence behavior of an algorithm is one of the interesting properties that determines its application in solving problems (Apanapudor and Otunta 2005) [22]. Hence, we examine the convergence behavior of the IMNW Galerkin finite element method with the theorem below.

Theorem [15]
Suppose $\zeta(x) \in L^2(\Re)$ defined on the $[0,1)$ is a continuous function, and $\zeta(x)$ is bounded, $\Rightarrow$ $\zeta(x) \leq M$, $M > 0$. Expanding $\zeta(x)$ with the IMNW produces a wavelet which converges uniformly to $\zeta(x)$.

Proof
Assuming $\zeta(x)$ is continuous and defined on $[0,1)$, if we expand $\zeta(x)$ with the IMNW, we will obtain a coefficient $C_{n,m}$, which is defined as

$$C_{n,m} = \int_0^1 \zeta(x)\chi_{n,m}(x)dx$$

But

$$\chi_{n,m}(x) = \begin{cases} 2^{\frac{k}{2}} \left(MN\right)_m (2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \leqslant x \leqslant \frac{n}{2^{k-1}} \\ 0, & \text{Otherwise} \end{cases}$$

Also, if we define

$$I = \frac{n-1}{2^{k-1}} \leqslant x \leqslant \frac{n}{2^{k-1}}$$

with $t = 2^k x - 2n + 1$ and $\zeta(x) = u$. It becomes

$$\begin{aligned} &\frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} \int_{-1}^{1} u\left(\frac{t-1+2n}{2^k}\right) MN(t)2^{-k}dx \\ &= \frac{2^{\frac{-k+1}{2}}}{\sqrt{\pi}} \int_{-1}^{1} u\left(\frac{t-1+2n}{2^k}\right) MN(t)dx \end{aligned}$$

Applying the Gauss Mean Value Theorem (GMVT) on integrals as in [1], for some $p \in (-1,1)$ gives

$$\frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} u\left(\frac{p-1+2n}{2^k}\right) \int_{-1}^{1} MN(t)dx.$$

Assuming $h = \int_{-1}^{1} MN(t)$, then it becomes

$$\frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} u\left(\frac{p-1+2n}{2^k}\right) h$$

$$\Rightarrow |C_{n,m}| = |\frac{2^{-k+1}}{\sqrt{\pi}}||u\left(\frac{p-1+2n}{2^k}\right)|h$$

But u is bounded and $u = \zeta(x)$, thus $\sum_n^\infty \sum_m^\infty C_{n,m}$ converges absolutely. Therefore, the IMNW Galerkin finite element technique converges uniformly.

4 CONCLUSIONS

We have studied the Iweobodo-Mamadu-Njoseh wavelet-based Galerkin finite element technique, and outlined its application in seeking approximate solutions to time fractional advection-diffusion problems. An investigation of the convergence capacity of the technique was also carried out and it displayed a uniform convergence.

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