Generating functions for a new class of recursive polynomials

Abstract

In this paper, we examine a family of recursively defined polynomials with four-variables on a fourth order recurrence relation and build their generating function. These generating functions enable us to derive several properties of the four-variable polynomials. Finally, we deduce new identities for the new class of polynomials with four-variables and also, we define the Q-matrix.

Keywords: Fibonacci polynomials; Recurrence relation; Recursive polynomials; Generating functions
2010 Mathematics Subject Classification: 11B39;11B37;05A15;05A19

1 Introduction

Polynomials are ubiquitous. Polynomials are used in every branch of mathematics. There are many well-known polynomials like Bernstein-Sato polynomials, Lagrange polynomials, Hermite polynomials, characteristic polynomials, minimal polynomials, and invariant polynomials play a vital role in the field of mathematics and engineering. The Fibonacci, Lucas, and Pell polynomials contribute significantly to the study of combinatorial structures. In particular various types of tiling problems [(1),(2)] can be analyzed using properties of Fibonaci-like polynomials.
Several authors [(3)-(15)] focus on generalizations of Fibonacci polynomials, their generating functions, Binet formula,roots, and related identities. In [(16)], the authors defined Gaussian Fibonacci polynomials and obtained the relation with the Fibonacci polynomials. The extension of Fibonacci and Lucas polynomials with two variables was defined by Catalani [(17)]. Ozdemir and Simsek [(18)] gave the family of two-variable polynomials and investigated the relationship between Fibonacci, Jacobsthal and Chebyshev polynomials. Kizilates [(19)] discussed the properties of the family of three-variable Fibonacci polynomials. Motivated by the work of Kizilates and Ozdemir we construct four-variable polynomials from generalized (r, s, t, u)-numbers [(20)] and three different types of generating functions. In Section 2, we discuss a generating function G(g) of the four-variable recurrence equation which is related to Fibonacci polynomials. In Section 3, we construct other two generating functions of the new generalized polynomials, and some properties have been investigated. In Sections 4 and 5, we discuss new identities of the four-variable polynomials.

2 Generating function of four-variable recursive polynomials

The recurrence relation of Fibonacci polynomial, two variables, three variables polynomials and the first five terms are presented in the following table.

Table 1: **Recurrence relation of Fibonacci-like polynomials**

Recurrence Relation	Initial values
$F_n(x_1) = x_1F_{n-1}(x_1) + F_{n-2}(x_1)$	$F_0(x_1) = 0$,$F_1(x_1) = 1$
$F_n(x_1, x_2) = \sum_{i=1}^{2} x_iF_{n-i}(x_1, x_2), n \geq 2$	$F_0(x_1, x_2) = 0$,$F_1(x_1, x_2) = 1$
$F_n(X) = \sum_{i=1}^{3} x_iF_{n-i}(X), n \geq 3$	$F_0(X) = 0$, $F_1(X) = 1$,$F_2(X) = x_1$
where $X = (x_1, x_2, x_3)$	

Table 2: **First five terms of Fibonacci-like polynomials**

n	One Variable	Two variables	Three Variables
0	0	0	0
1	1	1	1
2	x_1	x_1	x_1
3	$x_1^2 + 1$	$x_1^2 + x_2$	$x_1^2 + x_2$
4	$x_1^3 + 2x_1$	$x_1^3 + 2x_1x_2$	$x_1^3 + 2x_1x_2 + x_3$
5	$x_1^4 + 3x_1^2 + 1$	$x_1^4 + 3x_1^2x_2 + x_2^2$	$x_1^4 + 3x_1^2x_2 + 2x_1x_3 + x_2^2$

Based on suitable x_i and initial values [(21)], we can generate many well known polynomials like Lucas polynomials, Pell polynomials, Padovan polynomials.
Now, we define a four-variable recurrence polynomial. Let x_1, x_2, x_3 and x_4 be the positive integer variables. Then

$$F_n(X) = \sum_{i=1}^{4} x_iF_{n-i}(X); \quad n \geq 4 \tag{2.1}$$

where $X = (x_1, x_2, x_3, x_4)$ with initial conditions

$$F_0(X) = 0 \quad F_1(X) = 1 \quad F_2(X) = x_1 \quad F_3(X) = x_1^2 + x_2$$

From the equation (2.1), we generate Fibonacci-like polynomials in four variables and present in table(3).

We observe that, if the Fibonacci-like polynomial terms are arranged in dictionary order, then the degree of $F_n(X) = n - 1$.
Also, we can obtain many standard well-known sequences listed in OEIS [(22)] as the particular cases of x_i and by giving suitable initial values. Lan Qi [(23)] discussed the identities and state the generating functions of fourth order linear recurence sequence.On the basis of that, we present the following theorem.

Table 3: **The sequence of Fibonacci like polynomials in four-variables**

n	$F_n(x_1, x_2, x_3, x_4)$
0	0
1	1
2	x_1
3	$x_1^2 + x_2$
4	$x_1^3 + 2x_1x_2 + x_3$
5	$x_1^4 + 3x_1^2x_2 + 2x_1x_3 + x_2^2 + x_4$
6	$x_1^5 + 4x_1^3x_2 + 3x_1^2x_3 + 3x_1x_2^2 + 2x_2x_3 + 2x_1x_4$
⋮	⋮

Theorem 2.1. *Let $X = (x_1, x_2, x_3, x_4)$. The generating function of Fibonacci-like polynomial is given by*

$$G(g) = \frac{g}{1 - \sum_{i=1}^{4} x_i g^i} \tag{2.2}$$

Proof.

$$G(g) = \sum_{n=0}^{\infty} F_n(X) g^n$$
$$= g + x_1 g^2 + (x_1^2 + x_2) g^3 + \sum_{n=4}^{\infty} F_n(X) g^n$$

Now, we consider

$$\sum_{n=4}^{\infty} F_n(X) g^n = \sum_{n=4}^{\infty} (x_1 F_{n-1}(X) + x_2 F_{n-2}(X) + x_3 F_{n-3}(X) + x_4 F_{n-4}(X)) g^n$$

$$\sum_{n=4}^{\infty} F_n(X) g^n = x_1 g[G(g) - g - x_1 g^2] + x_2 g^2 [G(g) - g] + x_3 g^3 G(g) + x_4 g^4 G(g)$$

Hence, we obtain

$$G(g) = g + x_1 g^2 + (x_1^2 + x_2) g^3 + x_1 g[G(g) - g - x_1 g^2] + x_2 g^2 [G(g) - g] + x_3 g^3 G(g) + x_4 g^4 G(g)$$

From this, we get the desired result. □

3 Generating functions of New family of four-variable polynomials

In this section, we construct two different ordinary generating functions of the four-variable recursive polynomials and investigate some results of these functions.

Now we consider family of four variable polynomials $S_j(\Theta; \Lambda)$ where the variables are denoted by $\Theta = (x_1, x_2, x_3, x_4)$ and $\Lambda = (\alpha, \beta, \gamma, \delta, \zeta) \in N_0 = \{0, 1, 2, \ldots\}$ Then we define the generating function $S_j(\Theta;\Lambda)$

$$U(g;\Theta;\Lambda)=\sum_{j=0}^{\infty}S_j(\Theta;\Lambda)g^j=\frac{1}{1-x_1^{\alpha}g-x_2^{\beta}g^{\beta+\gamma}-x_3^{\delta}g^{\beta+\gamma+\delta}-x_4^{\zeta}g^{\beta+\gamma+\delta+\zeta}} \tag{3.1}$$

$$and \mid x_1^{\alpha}g+x_2^{\beta}g^{\beta+\gamma}+x_3^{\delta}g^{\beta+\gamma+\delta}+x_4^{\zeta}g^{\beta+\gamma+\delta+\zeta}\mid<1.$$

Now, Using Taylor's series expansion we derive the explicit representation of polynomials $S_j(\Theta;\Lambda)$

Theorem 3.1.

$$S_j(\Theta;\Lambda)=\sum_{s=0}^{\lambda}\sum_{u=0}^{\mu}\sum_{v=0}^{\xi}\binom{A}{s+u+v}\binom{s+u+v}{u+v}\binom{u+v}{v}(x_1)^{B}(x_2)^{\beta s}(x_3)^{\delta u}(x_4)^{\zeta v} \tag{3.2}$$

where

$\lambda=\left\lfloor\frac{j}{(\beta+\gamma)}\right\rfloor,\quad \mu=\left\lfloor\frac{j-(\beta+\gamma)s}{\beta+\gamma+\delta}\right\rfloor,\quad \xi=\left\lfloor\frac{j-(\beta+\gamma+\delta)u}{(\beta+\gamma+\delta+\zeta)}\right\rfloor$

$A=j-(\beta+\gamma-1)s-(\beta+\gamma+\delta-1)u-(\beta+\gamma+\delta+\zeta-1)v$

$B=j-(\beta+\gamma)s-(\beta+\gamma+\delta)(u+v)-\zeta v$

Proof. Using Taylor's series and binomial expansion, we expand the generating function, equation (3.1) becomes

$$U(g;\Theta;\Lambda)=\sum_{j=0}^{\infty}S_jg^j=\sum_{J=0}^{\infty}(x_1^{\alpha}g+x_2^{\beta}g^{\beta+\gamma}+x_3^{\delta}g^{\beta+\gamma+\delta}+x_4^{\zeta}g^{\beta+\gamma+\delta+\zeta})^j$$

$$U(g;\Theta;\Lambda)=\sum_{j=0}^{\infty}\sum_{s=0}^{\infty}\binom{j}{s}(x_1^{\alpha}g)^{j-s}(x_2^{\beta}g^{\beta+\gamma}+x_3^{\delta}g^{\beta+\gamma+\delta}+x_4^{\zeta}g^{\beta+\gamma+\delta+\zeta})^s$$

Replace j by $j+s$, we get

$$U(g;\Theta;\Lambda)=\sum_{j=0}^{\infty}\sum_{s=0}^{\infty}\binom{j+s}{s}(x_1^{\alpha}g)^j(x_2^{\beta}g^{\beta+\gamma}+x_3^{\delta}g^{\beta+\gamma+\delta}+x_4^{\zeta}g^{\beta+\gamma+\delta+\zeta})^s$$

Now, we expand $(x_2^{\beta}g^{\beta+\gamma}+x_3^{\delta}g^{\beta+\gamma+\delta}+x_4^{\zeta}g^{\beta+\gamma+\delta+\zeta})^s$

$$U(g;\Theta;\Lambda)=\sum_{j=0}^{\infty}\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\binom{j+s}{s}\binom{s}{u}(x_1^{\alpha}g)^j(x_2^{\beta}g^{\beta+\gamma})^{s-u}(x_3^{\delta}g^{\beta+\gamma+\delta}+x_4^{\zeta}g^{\beta+\gamma+\delta+\zeta})^u$$

Replace s by $s+u$, the above expression will become,

$$U(g;\Theta;\Lambda)=\sum_{j=0}^{\infty}\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\binom{j+s+u}{s+u}\binom{s+u}{u}(x_1^{\alpha}g)^j(x_2^{\beta}g^{\beta+\gamma})^{s}(x_3^{\delta}g^{\beta+\gamma+\delta}+x_4^{\zeta}g^{\beta+\gamma+\delta+\zeta})^u$$

Finally, we expand $(x_3^{\delta}g^{\beta+\gamma+\delta}+x_4^{\zeta}g^{\beta+\gamma+\delta+\zeta})^u$

$$U(g;\Theta;\Lambda)=\sum_{j=0}^{\infty}\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\binom{j+s+u}{s+u}\binom{s+u}{u}\binom{u}{v}$$
$$(x_1^{\alpha}g)^j(x_2^{\beta}g^{\beta+\gamma})^{s}(x_3^{\delta}g^{\beta+\gamma+\delta})^{u-v}(x_4^{\zeta}g^{\beta+\gamma+\delta+\zeta})^v$$

taking u as $u+v$, we get

$$U(g;\Theta;\Lambda)=\sum_{j=0}^{\infty}\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\binom{j+s+u+v}{s+u+v}\binom{s+u+v}{u+v}\binom{u+v}{v}(x_1^{\alpha}g)^j(x_2^{\beta}g^{\beta+\gamma})^s$$
$$(x_3^{\delta}g^{\beta+\gamma+\delta})^u(x_4^{\zeta}g^{\beta+\gamma+\delta+\zeta})^v$$

In the last expression taking $j-(\beta+\gamma+\delta+\zeta)v$ instead of j, taking $j-(\beta+\gamma+\delta)u$ instead of j, taking $j-(\beta+\gamma)s$ instead of j respectively, and using the following notation,
$\lambda=\lfloor\frac{j}{(\beta+\gamma)}\rfloor,\quad \mu=\lfloor\frac{j-(\beta+\gamma)s}{\beta+\gamma+\delta}\rfloor,\quad \xi=\lfloor\frac{j-(\beta+\gamma+\delta)u}{(\beta+\gamma+\delta+\zeta)}\rfloor$
$A=j-(\beta+\gamma-1)s-(\beta+\gamma+\delta-1)u-(\beta+\gamma+\delta+\zeta-1)v$
$B=j-(\beta+\gamma)s-(\beta+\gamma+\delta)(u+v)-\zeta v$ and apply the lemma (**?**), we have the desired result.
□

Note that the suitable values of $\Theta=(x_1,x_2,x_3,x_4)$ and $\Lambda=(\alpha,\beta,\gamma,\delta,\zeta)$ gives the relation between $S_j(\Theta;\Lambda)$ and standard polynomials like Legendre polynomials, Humbert polynomials. The Legendre polynomials have the generating function given by

$$\sum_{n=0}^{\infty}P_n(x)g^n=\frac{1}{\sqrt{1-2xg+g^2}}$$

If $\Theta=(2x,-1,0,0)$ and $\Lambda=(1,1,1,1,1)$, then equation (3.1) becomes

$$U(g;\Theta;\Lambda)=\sum_{j=0}^{\infty}S_j(\Theta;\Lambda)g^j=\frac{1}{1-2xg+g^2}$$

Hence the relation between the polynomials $S_j(\Theta,\Lambda)$ and the Legendre polynomials $P_j(x)$ is
$S_j(2x,-1,0,0;1,1,1,1,1)=\sum_{r=0}^{j}P_{j-r}(x)P_r(x)$
where $P_r(x)$ are the Legendre polynomials.
The Generalized Humbert polynomials(24) are defined by

$$\sum_{n=0}^{\infty}P_n(m,x,y,p,c)g^n=(c-mxg+yg^m)^p$$

If $c=1,m=1,p=-1$, then the generalized Humbert polynomials becomes

$$\sum_{n=0}^{\infty}P_n(1,x,y,-1,1)g^n=(1-xg+yg)^{-1}$$

If $\Theta=(x,-y,0,0)$ and $\Lambda=(1,1,0,1,1)$, then equation (3.1) becomes

$$U(g;\Theta;\Lambda)=\sum_{j=0}^{\infty}S_j(\Theta;\Lambda)g^j=\frac{1}{1-xg+yg}$$

From the above relation easily we can derive the relation between the polynomials $S_j(\Theta,\Lambda)$ and the Humbert polynomials $P_j(1,x,y,-1,1)$.

Now, we consider another family of polynomials $W_j:=W_j(\Theta,\Lambda)$ and define generating function in this manner

$$R:=R(g;\Theta,\Lambda)=U(g;\Theta,\Lambda)g^n=\frac{g^n}{1-x_1^{\alpha}g-x_2^{\beta}g^{\beta+\gamma}-x_3^{\delta}g^{\beta+\gamma+\delta}-x_4^{\zeta}g^{\beta+\gamma+\delta+\zeta}}$$
$$=\sum_{j=0}^{\infty}W_jg^j \quad (3.3)$$

where $\alpha, \beta, \gamma, \delta, \zeta \in N$ and $| x_1^{\alpha} g + x_2^{\beta} g^{\beta+\gamma} + x_3^{\delta} g^{\beta+\gamma+\delta} + x_4^{\zeta} g^{\beta+\gamma+\delta+\zeta} | < 1.$

Taking $\alpha = \beta = \gamma = \delta = \zeta = 1$ then,

$$\sum_{j=0}^{\infty} W_j g^j = \frac{g}{1 - x_1 g - x_2 g^2 - x_3 g^3 - x_4 g^4}$$

In particular, when $x_4 = 0,$

$$\sum_{j=0}^{\infty} W_j g^j = \frac{g}{1 - x_1 g - x_2 g^2 - x_3 g^3}$$

which is generating function of three variables Fibonacci-like polynomials.
In particular, $\alpha = \beta = \gamma = \delta = \zeta = 1$ and $x_1 \to x^2, x_2 \to x, x_3 \to 1, x_4 = 0$, then we obtain the generating function of Tribonacci polynomials.
that is,

$$\sum_{j=0}^{\infty} W_j g^j = \frac{g}{1 - x^2 g - x g^2 - g^3}$$

In the following, we introduce another family of polynomials denoted by $K_j := K_j(\Theta, \Lambda)$ via the generating function

$$\sum_{j=0}^{\infty} K_j g^j = \frac{A(g : x_1, x_2, x_3) - B(g : x_1, x_2, x_3)}{1 - x_1^{\alpha} g - x_2^{\beta} g^{\beta+\gamma} - x_3^{\delta} g^{\beta+\gamma+\delta} - x_4^{\zeta} g^{\beta+\gamma+\delta+\zeta}} \tag{3.4}$$

where $\alpha, \beta, \gamma, \delta, \zeta \in N,$
$A(g : x_1, x_2, x_3), B(g : x_1, x_2, x_3)$ are arbitrary polynomials depending on g, x_1, x_2, x_3 and

$$| x_1^{\alpha} g + x_2^{\beta} g^{\beta+\gamma} + x_3^{\delta} g^{\beta+\gamma+\delta} + x_4^{\zeta} g^{\beta+\gamma+\delta+\zeta} |< 1.$$

Suppose $\Lambda = (1, 1, 1, 1, 1)$, Then
$3U(g; \Theta, \Lambda) - 2x_1 R(g; \Theta, \Lambda) - x_2 g R(g; \Theta, \Lambda) - x_3 g^2 = \frac{3 - 2x_1 g - x_2 g^2 - x_3 g^3}{1 - x_1 g - x_2 g^2 - x_3 g^3 - x_4 g^4}$
In particular, $x_4 = 0$, and

$$\begin{aligned} 3U(g; \Theta, \Lambda) - 2x_1 R(g; \Theta, \Lambda) - x_2 g R(g; \Theta, \Lambda) &= \frac{3 - 2x_1 g - x_2 g^2}{1 - x_1 g - x_2 g^2 - x_3 g^3} \\ &= \sum_{j=0}^{\infty} K_j(x_1, x_2, x_3, 0) g^j \end{aligned}$$

where $K_j(x_1, x_2, x_3, 0)$ are trivariate Lucas polynomials.

4 Partial Derivatives For the Generating Functions

In this section, using partial derivatives [(25)] we derive new relations for polynomials.
Taking the derivative with regard to x_1, x_2, x_3, x_4, g of the generating function (3.1), they hold

$$\frac{\partial U}{\partial x_1} = \alpha x_1^{\alpha-1} g U^2 \tag{4.1}$$

$$\frac{\partial U}{\partial x_2} = \beta x_2^{\beta-1} g^{\beta+\gamma} U^2 \tag{4.2}$$

$$\frac{\partial U}{\partial x_3} = \delta x_3^{\delta-1} g^{\beta+\gamma+\delta} U^2 \tag{4.3}$$

$$\frac{\partial U}{\partial x_4} = \zeta x_4^{\zeta-1} g^{\beta+\gamma+\delta+\zeta} U^2 \tag{4.4}$$

$$\frac{\partial U}{\partial g} = (x_1^\alpha + x_2^\beta(\beta+\gamma) g^{\beta+\gamma-1} + x_3^\delta(\beta+\gamma+\delta) g^{\beta+\gamma+\delta-1} + x_4^\zeta(\beta+\gamma+\delta+\zeta) g^{\beta+\gamma+\delta+\zeta-1}) U^2 \tag{4.5}$$

From equation (3.1),

$$U^2 = S_0^2 + (S_0S_1 + S_1S_0)g + (S_0S_2 + S_1S_1 + S_2S_0)g^2 + (S_0S_3 + S_1S_2 + S_2S_1 + S_3S_0)g^3 + \cdots \tag{4.6}$$

Theorem 4.1. *For positive integer j,*

$$\frac{\partial S_j}{\partial x_1} = \alpha x_1^{\alpha-1} \sum_{i=0}^{j-1} S_i S_{j-1-i}$$

Proof. Using equations (4.1) and (3.1),

$$\sum_{j=1}^{\infty} \frac{\partial S_j}{\partial x_1} g^j = \alpha x_1^{\alpha-1} g U^2$$

Using equation (4.6), in the above equation and comparing the coefficients of g^j, we get the result. □

Theorem 4.2. *Let $j \geq \beta+\gamma$. Then we have*

$$\frac{\partial S_j}{\partial x_2} = \beta x_2^{\beta-1} \sum_{i=0}^{j-(\beta+\gamma)} S_i S_{j-(\beta+\gamma)-i}$$

Proof. Using equations (4.2) and (3.1) ,

$$\sum_{j=1}^{\infty} \frac{\partial S_j}{\partial x_2} g^j = \beta x_2^{\beta-1} g^{\beta+\gamma} U^2$$

Using equation (4.6) in the above expression and comparing the coefficients of g^j we get the result. □

Theorem 4.3. *Let $j \geq \beta+\gamma+\delta$. Then we have*

$$\frac{\partial S_j}{\partial x_3} = \delta x_3^{\delta-1} \sum_{i=0}^{j-(\beta+\gamma+\delta)} S_i S_{j-(\beta+\gamma+\delta)-i}$$

Proof. Using equations (4.3) and (3.1),

$$\sum_{j=1}^{\infty} \frac{\partial S_j}{\partial x_3} g^j = \delta x_3^{\delta-1} g^{\beta+\gamma+\delta} U^2$$

Using equation (4.6), comparing the coefficients of g^j we get the result. □

Theorem 4.4. *Let $j \geq \beta+\gamma+\delta+\zeta$. Then we have*

$$\frac{\partial S_j}{\partial x_4} = \zeta x_4^{\zeta-1} \sum_{i=0}^{j-(\beta+\gamma+\delta+\zeta)} S_i S_{j-(\beta+\gamma+\delta+\zeta)-i}$$

Proof. Using equations (4.4) and (3.1),

$$\sum_{j=1}^{\infty} \frac{\partial S_j}{\partial x_4} g^j = \zeta x_4^{\zeta-1} g^{\beta+\gamma+\delta+\zeta} U^2 +$$

Using equation (4.6), comparing the coefficients of g^j we get the result. □

Theorem 4.5. *If $U = \sum_{j=0}^{\infty} S_j g^j$, then*

1. $For j \leq \alpha+\beta-1$,

$$(j+1)S_{j+1} = x_1^k \sum_{i=0}^{j} S_i S_{j-i}$$

2. *For $\alpha+\beta-1 \leq j \leq \alpha+\beta+\gamma-1$,*

$$(j+1)S_{j+1} = x_1^{\alpha} \sum_{i=0}^{j} S_i S_{j-i} + (\beta+\gamma) x_2^{\beta} \sum_{i=0}^{j-(\beta+\gamma)-1} S_i S_{j-(\beta+\gamma)-1-i}$$

3. *For $\alpha+\beta+\gamma-1 \leq j \leq \alpha+\beta+\gamma+\delta-1$,*

$$(j+1)S_{j+1} = x_1^{\alpha} \sum_{i=0}^{j} S_i S_{j-i} + (\beta+\gamma) x_2^{\beta} \sum_{i=0}^{j-(\beta+\gamma)-1)} S_i S_{j-(\beta+\gamma)-1-i} + \\ (\beta+\gamma+\delta) x_3^{\gamma} \sum_{i=0}^{j-(\beta+\gamma+\delta)-1} S_i S_{j-(\beta+\gamma+\delta)-1-i}$$

4. *For $j \geq (\alpha+\beta+\gamma+\delta)-1$,*

$$(j+1)S_{j+1} = x_1^{\alpha} \sum_{i=0}^{j} S_i S_{j-1} + (\beta+\gamma) x_2^{\beta} \sum_{i=0}^{j-(\beta+\gamma)-1)} S_i S_{j-(\beta+\gamma)-1-i} + \\ (\beta+\gamma+\delta) x_3^{\gamma} \sum_{i=0}^{j-(\beta+\gamma+\delta)-1} S_i S_{j-(\beta+\gamma+\delta)-1-i} + \\ (\beta+\gamma+\delta+\zeta) x_4^{\zeta} \sum_{i=0}^{j-(\beta+\gamma+\delta+\zeta)-1} S_i S_{j-(\beta+\gamma+\delta+\zeta)-1-i}$$

Proof. By definition of U,
$U = \sum_{i=0}^{\infty} S_i g^i$ differentiate partially with respect to g,

$$\frac{\partial U}{\partial g} = \sum_{i=1}^{\infty} i S_i g^{i-1}$$

Take $i-1=j$, then the above expression will become, $\frac{\partial U}{\partial g} = \sum_{j=0}^{\infty} (j+1) S_{j+1} g^j$
Using equation (4.5) and (4.6), and comparing the coefficient of g^j, we get the required result. □

Theorem 4.6. $For j \geq 0,$ *we have*

$$jS_j = \frac{x_1}{\alpha}\frac{\partial S_j}{\partial x_1} + \left(\frac{\beta+\gamma}{\beta}\right)x_2\frac{\partial S_j}{\partial x_2} + \left(\frac{\beta+\gamma+\delta}{\delta}\right)x_3\frac{\partial S_j}{\partial x_3} + \left(\frac{\beta+\gamma+\delta+\zeta}{\zeta}\right)x_4\frac{\partial S_j}{\partial x_4}$$

Proof. Using equations (4.1),(4.2),(4.3),(4.4) and (4.5), we obtain

$$\frac{x_1}{\alpha}\frac{\partial U}{\partial x_1} = x_1^{\alpha} g U^2 \tag{4.7}$$

$$\frac{x_2}{\beta}\frac{\partial U}{\partial x_2} = x_2^{\beta} g^{\beta+\gamma} U^2 \tag{4.8}$$

$$\frac{x_3}{\delta}\frac{\partial U}{\partial x_3} = x_3^{\delta} g^{\beta+\gamma+\delta} U^2 \tag{4.9}$$

$$\frac{x_4}{\zeta}\frac{\partial U}{\partial x_4} = x_4^{\zeta} g^{\beta+\gamma+\delta+\zeta} U^2 \tag{4.10}$$

combining the above equations and using (4.6), we get the desired result. □

5 New Identities

In this section, we derive some identities connected with these polynomials.
Taking $g = \frac{1}{a}$ for $a > 1$ in equation (3.3),

$$\sum_{j=0}^{\infty} \frac{W_j}{a^j} = \frac{a^{\beta+\gamma+\zeta}}{a^{\beta+\gamma+\delta+\zeta} - x^{\alpha}a^{\beta+\gamma+\delta+\zeta-1} - y^{\beta}a^{\delta+\zeta} - z^{n}a^{\zeta} - t^{\zeta}} \tag{5.1}$$

Identity 1:
Setting $a = 2, x \to x^2, y \to x, z \to 1, t \to 0, \alpha = \beta = \gamma = \delta = \zeta = 1$

$$\sum_{j=0}^{\infty} \frac{W_j}{2^j} = \frac{4}{7 - 4x^2 - 2x}$$

In particular $x = 1,$

$$\sum_{j=0}^{\infty} \frac{W_j}{2^j} = 4$$

Identity 2:
Setting $a = 2, x \to x^2, y \to x, z \to 0, t \to 0, \alpha = \beta = \gamma = \delta = \zeta = 1$

$$\sum_{j=0}^{\infty} \frac{W_j}{2^j} = \frac{2}{4 - 2x^2 - x}$$

In particular $x = 1,$

$$\sum_{j=0}^{\infty} \frac{W_j}{2^j} = 2$$

Identity 3:
Let k be the odd positive integer and $k \geq 5$. Let F_n be the Fibonacci Four-Variate polynomials.Then

$$\sum_{i=4}^{k} F_i F_{i+1} = \sum_{i=4,6,8}^{k-1} F_{i+1}\{x_1[F_{i-1} + F_{i+1}] + x_2[F_{i-2} + F_i] + x_3[F_{i-3} + F_{i-1}] + x_4[F_{i-4} + F_{i-2}]\}$$

Proof. Consider the sum $F_4F_5 + F_5F_6$

$$F_4F_5 = [x_1F_3 + x_2F_2 + x_3F_1 + x_4F_0]F_5$$
$$F_5F_6 = [x_1F_5 + x_2F_4 + x_3F_3 + x_4F_2]F_5$$

Therefore,

$$[F_4 + F_6]F_5 = [x_1\{F_3 + F_5\} + x_2\{F_2 + F_4\} + x_3\{F_1 + F_3\} + x_4\{F_0 + F_2\}]F_5$$

In general,

$$[F_i + F_{i+2}]F_{i+1} = [x_1\{F_{i-1} + F_{i+1}\} + x_2\{F_{i-2} + F_i\} + x_3\{F_{i-3} + F_{i-1}\} + x_4\{F_{i-4} + F_{i-2}\}]F_{i+1} \quad (5.2)$$

Now,

$$\sum_{i=4}^{k} F_iF_{i+1} = F_4F_5 + F_5F_6 + \cdots + F_kF_{k+1} = [F_4 + F_6]F_5 + [F_6 + F_8]F_7 + \cdots + [F_{K-1} + F_{K+1}]F_k$$

Using (4.6), we get the desired result. □

Identity 4:
For $n \geq 4$,

1. If n is odd,

$$\frac{1}{2}\sum_{i=0}^{n} F_iF_{n-i} = x_1 \sum_{i=0}^{\frac{n-1}{2}} F_iF_{n-1-i} + x_2 \sum_{i=0}^{\frac{n-1}{2}} F_iF_{n-2-i} + x_3 \sum_{i=0}^{\frac{n-1}{2}} F_iF_{n-3-i} + x_4 \sum_{i=0}^{\frac{n-1}{2}} F_iF_{n-4-i}$$

2. If n is even,

$$\frac{1}{2}\sum_{i=0}^{n} F_iF_{n-i} = x_1 \sum_{i=0}^{\frac{n}{2}-1} F_iF_{n-1-i} + x_2 \sum_{i=0}^{\frac{n}{2}-1} F_iF_{n-2-i} + x_3 \sum_{i=0}^{\frac{n}{2}-1} F_iF_{n-3-i} + x_4 \sum_{i=0}^{\frac{n}{2}-1} F_iF_{n-4-i} + F^2_{\frac{n}{2}}$$

Proof. Case (i): If n is odd,

$$\sum_{i=0}^{n} F_iF_{n-i} = F_0F_n + F_1F_{n-1} + \cdots + F_nF_0$$
$$= 2[F_0F_n + F_1F_{n-1} + \cdots + F_{\frac{n-1}{2}}F_{\frac{n+1}{2}}]$$

Using (2.1),$F_n, F_{n-1}, \dots F_{\frac{n+1}{2}}$ we get the desired result.
Case(ii): If n is even,

$$\sum_{i=0}^{n} F_i F_{n-i} = F_0 F_n + F_1 F_{n-1} + \cdots + F_n F_0$$
$$= 2[F_0 F_n + F_1 F_{n-1} + \cdots + F_{\frac{n}{2}-1} F_{\frac{n}{2}+1}] + F_{\frac{n}{2}}^2$$

Using the definition in (2.1) on the polynomial $F_n(X)$ we get the desired result.

□

6 The Q-matrix and its properties

Hoggat (5) introduced the concept of generating matrix of the Tribonacci polynomial. Kocer (7) studied the generating matrix of trivariate Fibonacci polynomials. Based on that we define the matrix Q, which generates a fibonacci-like polynomial with Four-Variable.
The matrix Q is defined as

$$\begin{pmatrix} x_1 & 1 & 0 & 0 \\ x_2 & 0 & 1 & 0 \\ x_3 & 0 & 0 & 1 \\ x_4 & 0 & 0 & 0 \end{pmatrix}$$

Let $F_n = F_n(X)$ be the n^{th} four-variable fibonacci like polynomial.
By taking positive powers of the matrix Q and $n \geq 5$, we get

$$Q^n = \begin{pmatrix} F_{n+1} & F_n & F_{n-1} & F_{n-2} \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} \end{pmatrix}$$

where
$Q_{21} = x_2 F_n + x_3 F_{n-1} + x_4 F_{n-2}$
$Q_{22} = x_2 F_{n-1} + x_3 F_{n-2} + x_4 F_{n-3}$
$Q_{23} = x_2 F_{n-2} + x_3 F_{n-3} + x_4 F_{n-4}$
$Q_{24} = x_2 F_{n-3} + x_3 F_{n-4} + x_4 F_{n-5}$
$Q_{31} = x_3 F_n + x_4 F_{n-1}$
$Q_{32} = x_3 F_{n-1} + x_4 F_{n-2}$
$Q_{33} = x_3 F_{n-2} + x_4 F_{n-3}$
$Q_{34} = x_3 F_{n-3} + x_4 F_{n-4}$
$Q_{41} = x_4 F_n$
$Q_{42} = x_4 F_{n-1}$
$Q_{43} = x_4 F_{n-2}$
$Q_{44} = x_4 F_{n-3}$

Since determinant of Q is $-x_4$ the determinant of Q^n is $(-x_4)^n$. Clearly, the determinant of Q^n is non-zero, so the matrix is invertible. If we consider suitable x_i and n, Q will play a vital role in the concept of encrypting and decrypting the cipher text.
Using this fact and we apply mathematical induction, we can obtain the following result.

Theorem 6.1. *Let $F_n = F_n(X)$ be the n^{th} four-variable fibonacci like polynomial, then*

$$\begin{vmatrix} F_{n+3} & F_{n+2} & F_{n+1} & F_n \\ F_{n+2} & F_{n+1} & F_n & F_{n-1} \\ F_{n+1} & F_n & F_{n-1} & F_{n-2} \\ F_n & F_{n-1} & F_{n-2} & F_{n-3} \end{vmatrix} = (-1)^{n-1} x_4^{n-1}$$

7 Conclusions

The fields of encryption and decryption employ multi-variable polynomials. We have defined four-variable polynomials using the four-term recurrence relation, as opposed to utilizing some random polynomials here. We have constructed their generating function. The generator matrix of the sequence is derived. Crypto-systems can be developed by employing matrices associated with these polynomials. Finally we conclude that the Q-matrix is invertible matrix.

References

[1] Benjamin, A.T, Quinn, J.J. (2003).Proofs that Really Count The art of Combinatorial Proof, twenty seven ed. The mathematical Association of America.

[2] Brigham, R., Caron, R.M., Chinn, P.Z., Grimaldi, R.P. (1997). A Tiling scheme for the Fibonacci numbers. Journal of Recreational Mathematics, 28, 10-17.

[3] Amdeberhan, T., Chen, X., Moll, V.H., Sagan, B.E. (2014).Generalised fibonacci polynomials and fibonomial coefficients. Annals of Combinatorics, 18,(doi:10.1007/s00026-014-0242-9), 541–562.

[4] Chen, L., Wang, X. (2019). The power sums involving Fibonacci polynomials and their applications. Symmetry, 11,(doi:10.3390/sym11050635), 635.

[5] Hoggat, V.E., Bicknell, M. (1973). Roots of Fibonacci polynomials. Fibonacci Quaterly, 11, 271–274.

[6] Hoggat, V.E., Bicknell, M. (1973). Generalized Fibonacci polynomials. Fibonacci Quaterly, 11, 457–465.

[7] Kocer, E.G., Gedikce, H. (2016). Trivariate Fibonacci and Lucas polynomials. Konuralap Journal of Math., 4, 247–254.

[8] Koshy, T. (2001). Fibonacci and Lucas Numbers with Applications, First edition. John Wiley and Sons Inc, New York.

[9] Ma, Y., Zhang, W. (2018). Some Identities Involving Fibonacci Polynomials and Fibonacci Numbers. Mathematics, 6, (doi:10.3390/math6120334), 334.

[10] Ozkan, E., Altun, I. (2019). Generalized Lucas polynomials and relationships between the Fibonacci polynomials and Lucas polynomials. Communications in Algebra, 47, (doi:10.1080/00927872.2019.1576186), 4020–4030.

[11] Panwar, Y.K., Singh, B., Gupta, V.K. (2013). Generalized Fibonacci Polynomials. Turkish Journal of Analysis and Number Theory, 1, (doi:10.12691/tjant-1-1-9), 43–47.

[12] Prodinger, H. (2009). Sums of powers of Fibonacci polynomials. Proc.Indian Acad.sci., 119, 567–570.

[13] Simsek, Y. (2021). Relations between Certain Combinatorial Numbers and Fibonacci Numbers. 3rd and 4th Mediterranean International Conference of Pure and Applied Mathematics and Related Areas, Turkey, 11–12.

[14] Tan, M., Zhang, Y.A. (2005). A Note on Bivariate and Trivariate Fibonacci Polynomials. Southeast Asian Bulletin of Math., 29, 975–990.

[15] Tastan, M., Ozkan, E., Shannon, A.G. (2021). The generalized k-Fibonacci polynomials and generalized k-Lucas polynomials. Notes on Number Theory and Discrete Mathematics, 27, (doi:10.7546/nntdm.2021.27.2.148-158), 148–158.

[16] Ozkan, E., Taston. (2019). On Gauss Fibonacci polynomials,On Gauss Lucas polynomials and their applications. Communications in Algebra, 48, (doi:10.1080/00927872.2019.1670193), 952–960.

[17] Catalani, M. (2004). Some formula for bivariate Fibonacci and Lucas polynomials. (doi:10.48550/arXiv.math/0406323).

[18] Ozdemir, G., Simsek, Y. (2016). Generating Functions For Two-Variable Polynomials Related to a Family of Fibonacci Type Polynomials and Numbers. Filomat, 30, (doi:10.2298/FIL1604969O), 969–975.

[19] Kizilates, C., Cekim, B., Tuglu, N., Kim, T. (2019). New Families of Three Variable Polynomials Coupled with Well-Known Polynomials and Numbers. Symmetry, 11, (doi:10.3390/sym11020264), 264–276.

[20] Soykan, Y. (2021). Properties of Generalised (r,s,t,u)-Numbers. Earthline Journal of Mathematical Sciences, 5, (doi:/10.34198/ejms.5221.297327), 297–327.

[21] Soykan, Y. (2019). Summation Formulas for Generalized Tetranacci Numbers. Asian Journal of Advanced Research and Reports, 7(2) (https://doi.org/10.9734/ajarr/2019/v7i230170), 1—12.

[22] Sloane, N. (1995). The Encyclopedia of Integer sequences, First edition, Elsevier science publishing.

[23] Lan Qi., Zhuoyu Chen. (2019). Identities Involving the Fourth-Order Linear Recurrence Sequence. Symmetry, 11, (doi:10.3390/sym11121476) 1476.

[24] Wang, W., Wang, H. (2017). Generalized Humbert polynomials via generalized Fibonacci polynomials. Applied Mathematics and computation, 307, (doi: 10.1016/j.amc.2017.02.050) 204–216.

[25] Cakmak, T., Karaduman, E. (2018). On the derivatives of bivariate fibonacci polynomials. Notes on number theory and Discrete mathematics, 24, (doi: 10.7546/nntdm.2018.24.3.37-46), 37–46.