UNIFORM ESTIMATES ON LENGTH OF PROGRAMS AND COMPUTING ALGORITHMIC COMPLEXITIES FOR QUANTITATIVE INFORMATION MEASURES

Abstract

Shannon entropy and Kolmogorov complexity are two conceptually distinct information metrics since the latter is based on probability distributions while the former is based on program size. All recursive probability distributions, however, are known to have an expected Up to a constant that solely depends on the distribution, the Kolmogorov complexity value is equal to its Shannon entropy. We investigate if a comparable correlation exists between Renyi and Havrda- Charvat Entropy entropies order α, indicating that it is consistent solely with Renyi and Havrda-Charvat entropies of order 1.

Kolmogorov noted that the characteristics of Shannon entropy and algorithmic complexity are comparable. We examine a single facet of this resemblance. Specifically, linear inequalities that hold true for Shannon entropy and for Kolmogorov complexity. As it happens, the following are true: (1) all linear inequalities that hold true for Shannon entropy and vice versa for Kolmogorov complexity; (2) all linear inequalities that hold true for ranks of finite subsets of linear spaces for Shannon entropy; and (3) the reverse is untrue.

Keywords: Kolmogorov complexity, Shannon entropy , non-Shannon entropy.

1. Introduction

"The size of the smallest program that can construct an object x is used to assess its exact information content. This is known as Kolmogorov complexity, or C(x). Assigning a probability of $2^{-C(x)}$ to any given string x, it constructs a probability distribution over $\int *$ naturally. μ stands for the universal probability distribution, which is the name given to this probability distribution introduced in"[3].

"A random variable X's average uncertainty is expressed as its Shannon entropy, or S(X). In average, it is the least amount of bits needed to express x, the random variable X's output. Shannon entropy and Kolmogorov complexity are conceptually distinct from one another since the former is dependent on program length. Three separate generalizations ofnon Shannonentropy are of importance to us and that was provided" in [6], [8], [9],[12] and [[16].

Shannon's [12] original paper introduced the concepts of discrete and differential entropy. Kolmogorov [6, 7, 8] established "the general rigorous definition of relative entropy and mutual information for arbitrary random variables. They defined mutual information as $\text{Sup}_{P,Q} I\big([X]_P : [Y]_Q\big)$, where the supremum is over all finite partitions P and Q. The Asymptotic Equipartition Property (AEP), which asserts that for a sequence of random variables $P(X_1, X_2, \dots \dots \dots, X_n)$is closed to $2^{-nH(X)}$ with high probability, is one of the key roles of entropy for discrete random variables. This allows us to characterize the behavior of typical sequences and establish the typical set. The same applies to continuous random variables".

For continuous random variables, the usual set's features are similar to those of discrete random variables. The volume of the typical set for continuous random variables is the analogue of the cardinality of the typical set for the discrete case.

Shannon Entropy :"Let X be a finite or infinitely countable set and let X be a random variable taking values in X with distribution P introduced" in [12] . The Shannon entropy of the random variable X is

S(X) = $-\sum_{x\epsilon\chi} p(x) \log p(x)$

Renyi Entropy : Let X be a finite or infinitely countable set and let X be a randomvariable taking values in X with distribution P introduced in[13] ,[14] and let α ≠ 1 be a non-negative real number. The Renyi ´entropy of order α of the random variable X is defined as:

$$R_\alpha\,(X) = \frac{1}{1-\alpha} \log\left(\sum_{x\in\chi} p(x)^\alpha\right)$$

Havrda and Charvat Entropy: Let X be a finite or infinitely countable set and Let X be a random variable taking values in X with distribution P introduced in [15], [23] and let α ≠ 1 be a non negative real number .The Harveda –Charvat entropy of order α of the random variable X is defined as:

$HC_\alpha(X) \; = \frac{1}{1-\alpha} \sum_{x\in X} p(x)^\alpha$ -$\frac{1}{1-\alpha}$

It is easy to prove that $\lim_{\alpha\to1} R_\alpha\,(X) = \;\lim_{\alpha\to1} HC_\alpha(X)$ = S(X).

Note that we also use the notation $R_\alpha\,(P)$, $HC_\alpha(P)$ and S(P) to denote the Renyi,Harveda-Charvat and Shannon entropies of distribution P,respectively.

2. Our Results:

Kolmogorov Complexity and Non-Shannon entropy

Considering the conceptual distinctions between Shannon entropy and Kolmogorov complexity, it's noteworthy to note that under certain loose constraints on the string distribution, They share a relationship. Up to a constant term that solely depends on the distribution, the Shannon entropy value actually matches the expected value of Kolmogorov complexity that we provided in [4],[5],[9] and [16].

Proposition: There are recursive probability distribution P such that:

i) $\sum_x P(x)\; K(x) - \; R_\alpha(P) > K(P)$,where $\alpha > 1$

ii) $\sum_x P(x)\; K(x) - \; R_\alpha(P) < 0$, where $\alpha < 1$

iii) $\sum_x P(x)\ K(x) - HC_\alpha(P) > K(P)$, where $\alpha > 1$
iv) $\sum_x P(x)\ K(x) - \ HC_\alpha(P) < 0$, where $\alpha < 1$

Proof:.consider the following probability distribution:

$$D_n\ (x) = \begin{cases} \frac{1}{2}\ \ if\ x = \ 0^n \\ 2^{-n}\ if\ x = 1x^{\ '}, x^{'} \in \{0,1\}^{n-1} \\ 0 \quad otherwise \end{cases}$$

i)First we observe that

$$\mathrm{S}(D_n) = \ -\sum_x D_n\ (x)\ logD_n\ (x)$$

$$= -\left(\frac{1}{2}\log\frac{1}{2} + \ \frac{1}{2^n}2^{n-1}\log\frac{1}{2^n}\right)$$

$$= -\left(-\frac{1}{2} - \frac{1}{2^n}2^{n-1}\ n\right)$$

$$= \frac{n+1}{2}$$

By theorem we have

Let $\mathrm{P}(x)$ be a recursive probability distribution ,then:

$0 \leq \sum_x D_n(x)\, K(x) - \ \mathrm{S}(D_n) \leq K(D_n)$ “

Which implies that

$0 \leq \sum_x D_n(x)\, K(x) - \mathrm{S}(D_n)$

$\sum_x D_n\ (x) K(x) \geq \mathrm{S}(D_n)$

$$\sum_x D_n(x)\, K(x) \geq \frac{n+1}{2}$$

On the other hand,by definition

$$R_\alpha\ (D_n) = \ \frac{1}{1-\alpha}\log\left(\sum_{x\in\chi} D_n(x)^\alpha\right)$$

$$= \frac{1}{1-\alpha}\log\left(\frac{1}{2^n} + \ 2^{n-1} \times \frac{1}{2^{n\alpha}}\right)$$

$$= \frac{1}{1-\alpha}\left(\log\left(2^{(n-1)\alpha} + \ 2^{n-1}\right) - n\alpha\right)$$

To prove that $\sum_x D_n(x)\ K(x) - \ R_\alpha(D_n) > K(D_n)$, it is sufficiently to prove that

$$\lim_{n}\left(\sum_{x} D_n(x)\ K(x) - R_\alpha(D_n) > K(D_n)\right) > 0$$

i.e. $\lim_n \left(\frac{n+1}{2} - \frac{1}{1-\alpha}\left(\log\left(2^{(n-1)\alpha} + 2^{n-1}\right) - n\alpha\right) - c\log n\right) > 0$

But, $\lim_n \left(\frac{n+1}{2} - \frac{\log\left(\left(2^{(n-1)\alpha} + 2^{n-1}\right)\right)}{1-\alpha} + \frac{n\alpha}{1-\alpha} - c\log n\right)$

$$\geq \lim_{n}\left(\frac{n+1}{2} - \frac{\log\left(\left(2^{(n-1)\alpha}\right)\right)}{\alpha-1} + \frac{n\alpha}{\alpha-1} - c\log n\right)$$

$$= \lim_{n}\left(\frac{n+1}{2} + \frac{(n-1)\alpha}{\alpha-1} - \frac{n\alpha}{\alpha-1} - c\log n\right)$$

$$= \lim_{n}\left(\frac{n+1}{2} - \frac{\alpha}{\alpha-1} - c\log n\right)$$

$= +\infty$

ii) To prove this item we use the other inequality

$$\sum_{x} D_n(x)\, K(x) - S(D_n) \leq K(D_n)$$

Which implies that

$$\sum_{x} D_n(x)\, K(x) \leq K(D_n) + S(D_n)$$

$\leq \frac{n+1}{2} + c\log n$

So, $\sum_x D_n(x)\ K(x) - R_\alpha(D_n)$

$\leq \frac{n+1}{2} + c\log n - \frac{1}{1-\alpha}\left(\log\left(2^{(n-1)\alpha} + 2^{n-1}\right) - n\alpha\right)$

$\leq \frac{n+1}{2} + c\log n - \frac{\log\left(2^{n-1}\right)}{1-\alpha} + \frac{n\alpha}{1-\alpha}$

$= \frac{n+1}{2} + c\log n - \frac{1}{1-\alpha}(n-1-n\alpha)$

$= -\frac{n}{2} + \frac{1}{2} + c\log n + \frac{1}{1-\alpha}$

Thus, taking n sufficiently large.

iii) The Havrda and Charvat Entropy of order α of distribution P_n is :

$$HC_\alpha(D_n) = \frac{1}{1-\alpha}\sum_{x\in X} D_n(x)^\alpha \text{ -1}$$

$$= \frac{1}{1-\alpha}\left(\frac{1}{2^\alpha} + 2^{n-1} + \frac{1}{2^{n\alpha}}\right)\text{-1}$$

$$= \frac{1}{1-\alpha}\left(2^{(n-1)\alpha} + 2^{(n-1)}\right) - 1$$

$$= \frac{2^{(n-1)\alpha}}{1-\alpha} + \frac{2^{(n-1)}}{1-\alpha} \text{ -1}$$

Using the inequality,we get

$$\sum_x D_n(x)\ K(x) - HC_\alpha(D_n) = \sum_x D_n(x)\ K(x) - \frac{2^{(n-1)\alpha}}{1-\alpha} - \frac{2^{(n-1)}}{1-\alpha} + 1$$

$$\geq \frac{n+1}{2} - \frac{2^{(n-1)\alpha}}{1-\alpha} - \frac{2^{(n-1)}}{1-\alpha} + 1$$

$$\geq \frac{n+3}{2} - \frac{2^{(n-1)\alpha}}{1-\alpha} - \frac{2^{(n-1)}}{1-\alpha}$$

For n sufficiently large

$$\geq \frac{n+3}{2} - \frac{2^{(n-1)\alpha}}{1-\alpha} - \frac{2^{(n-1)}}{1-\alpha}$$

$$> c\log n$$

$$= kD_n$$

iv) Using the inequality, we get:

$$\sum_x D_n(x)\ K(x) - HC_\alpha(D_n) \leq \frac{n+1}{2} + \text{c}\log n - \frac{2^{(n-1)\alpha}}{1-\alpha} - \frac{2^{(n-1)}}{1-\alpha} + 1$$

Since n is sufficiently large,thuswe conclude that:

$$< \frac{n+1}{2} + \text{c}\log n - \frac{2^{(n-1)\alpha}}{1-\alpha} - \frac{2^{(n-1)}}{1-\alpha} + 1$$

$$< 0$$

Uniform Continuity of the Shannon and Non Shannon entropies

i) Shannon entropy

In order to prove the uniform continuity of the Shannon entropy[18], [19], we need some technical lemmas and theorems.

Theorem: let X and Y be two probability distribution over $\sum n$,let $\text{S}(X)$ and $\text{S}(Y)$ be the Shannon entropy of distribution X and Y respectively. then,

$$\forall \mathcal{E} > 0, \exists \delta > 0, \forall X, Y: max\ |X(u) - Y(u)| \leq \delta \Rightarrow |S(X) - S(Y)| \leq \varepsilon$$

Proof. By lemma the function $u \log u$ is uniformly continuous in[0,1],we have that:

$$\forall \gamma > 0 , \exists \beta > 0 , \forall u, v : max\, |u - v| \leq \beta \Rightarrow |u \log u - v \log v| \leq \gamma$$

So,

$$|S(X) - S(Y)| = \left| -\sum_u X(u) \log X(u) + \sum_u Y(u) \log Y(u) \right|$$

$$= \left| \sum_u X(u) \log X(u) - \sum_u Y(u) \log Y(u) \right|$$

$$\leq \sum_u |X(u) \log X(u) - Y(u) \log Y(u)|$$

$$\leq \sum_u \gamma$$

$= 2^n \gamma$

It is sufficient to consider $\gamma = \frac{\varepsilon}{2^n}$ and consider $\delta = \beta$

ii)Non-Shannon Entropy

a)Renyi Entropy

Let X and Y be two probability distribution over $\sum n$, let $R_\alpha(X)$ and $R_\alpha(Y)$ be the Renyi entropy[21], [22] of distribution X and Y respectively. then ,

$$\forall \mathcal{E} > 0, \exists \delta > 0 , \forall X, Y : max\, |X(u) - Y(u)| \leq \delta \Rightarrow |R_\alpha(X) - R_\alpha(Y)| \leq \varepsilon$$

Proof: By Lemma,the function x^α is uniformly continuous in [0,1] .Thus,

$$\forall \gamma > 0, \exists \beta > 0 , \forall x, y : max\, |x - y| \leq \beta \Rightarrow |x^\alpha - y^\alpha| \leq \varepsilon (2.1)$$

We have to show that

If $max\, |X(u) - Y(u)| \leq \delta \Rightarrow |R_\alpha(X) - R_\alpha(Y)| \leq \varepsilon$

$$|R_\alpha(X) - R_\alpha(Y)| = \left| \frac{1}{1-\alpha} \log \left(\sum_{x \in \chi} X(x)^\alpha \right) - \frac{1}{1-\alpha} \log \left(\sum_{x \in \chi} Y(x)^\alpha \right) \right|$$

$$= \frac{1}{1-\alpha} \left| \log \left(\sum_{x \in \chi} X(x)^\alpha \right) - \log \left(\sum_{x \in \chi} Y(x)^\alpha \right) \right|$$

$\leq \frac{2}{1-\alpha} |\sum_{x \in X} X(x)^\alpha - \sum_{x \in X} Y(x)^\alpha |$, by Lemma we have" $|\log a - \log b| \leq 2|a-b|$"

$$\leq \frac{2}{1-\alpha}\sum_{X}|X(x)^{\alpha} - Y(x)^{\alpha}|$$

From (2.1) we consider $\gamma = \frac{\epsilon(1-\alpha)}{2^{n+1}}$

$$\leq \frac{2}{1-\alpha}\sum_{x}\frac{\epsilon(1-\alpha)}{2^{n+1}}$$

$$\leq \frac{2^{n+1}}{1-\alpha}\frac{\epsilon(1-\alpha)}{2^{n+1}}$$

$$\leq \epsilon$$

(b)Havrda and Charvat Entropy

Let X and Y be two probability distribution over $\sum n$, let $HC_{\alpha}(X)$ and $HC_{\alpha}(Y)$ be the **Havrda-Charvat** entropy [17], [20]of distribution X and Y respectively. then ,

$$\forall \varepsilon > 0, \exists \delta > 0, \forall X, Y: max\,|X(u) - Y(u)| \leq \delta \Rightarrow |HC_{\alpha}(X) - HC_{\alpha}(Y)| \leq \varepsilon$$

Proof: We have,

$$|HC_{\alpha}(X) - HC_{\alpha}(Y)| = \left|\frac{1}{1-\alpha}\sum_{x\in X}X(x)^{\alpha} - \frac{1}{1-\alpha} - \frac{1}{1-\alpha}\sum_{x\in X}Y(x)^{\alpha} + \frac{1}{1-\alpha}\right|$$

$$= \left|\frac{1}{1-\alpha}\sum_{x\in X}X(x)^{\alpha} - \frac{1}{1-\alpha}\sum_{x\in X}Y(x)^{\alpha}\right|$$

$$= \frac{1}{1-\alpha}\left|\sum_{x\in X}X(x)^{\alpha} - \sum_{x\epsilon X}Y(x)^{\alpha}\right|$$

$$\leq \frac{1}{1-\alpha}\sum_{x\in X}|X(x)^{\alpha} - Y(x)^{\alpha}|$$

$$\leq \frac{1}{1-\alpha}\gamma$$

We consider $\gamma = \in (1-\alpha) \leq \in$

Conclusion

Among the three entropies that we have examined, we have demonstrated that only the Shannon entropy meets the conditions for the relationship with the expected value of Kolmogorov complexity stated in by presenting a probability distribution for which the relationship breaks down for certain values of α of the Havrda - Charvat and Renyi ʁ entropies. This relationship also applies for the Shannon entropy in a time-bounded manner, assuming that the cumulative probability distribution can be computed in the given amount of time. We investigated the convergence of this distribution under the Havrda- Charvat and Renyí entropies, which are two generalizations of the Shannon entropy. Additionally, we demonstrated the uniform continuity of the three entropies taken into consideration in this work.

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