Novel idea on Edge-Ultrafilter and Edge-Tangle

Abstract: The study of width parameters holds significant interest in both graph theory and algebraic settings. Among these, the tree-cut decomposition stands out as a key metric. The "Edge-tangle" concept is closely related to the "tree-cut width" width parameter in graph theory. This obstruction is often seen as vital for creating effective algorithms to calculate graph width, with the edge-tangle being the specific obstruction for tree-cut width. Meanwhile, the idea of an "Ultrafilter" is well-established in topology and algebra. Due to their versatile nature, ultrafilters hold significant and broad-ranging importance. In this paper, we introduce a new concept called Edge-Ultrafilters for graphs and demonstrate how they are equivalent to Edge-tangles.
Keyword: filter, ultrafilter, tangle, edge-tangle, tree-cut-width, tree-cut-decomposition

1. Introduction

In recent years, there has been a notable surge in research interest surrounding the exploration of width parameters within both graph theory and algebraic contexts [1-4]. These parameters are intricately linked to metrics grounded in tree-like structures, often referred to as graph decompositions. One such concept deeply intertwined with width parameters is the "Edge-tangle," associated with the graph's "tree-cut width" [1,14,15,16]. In studies involving parameters like tree-cut-width, the concept of "obstruction" is commonly utilized (cf.[8,9,10]).This "obstruction" is often deemed crucial for devising efficient graph width algorithms, and in the case of tree-cut-width, the corresponding obstruction is the edge-tangle. Therefore, investigations into Edge-tangles, which bear a profound connection to width parameters, carry significant importance.

The concept of a "Filter" is firmly established in the realms of topology and algebra. In simple terms, a filter can be understood as a collection of sets containing a particular element, serving as a valuable tool for discussing convergence properties in mathematics. In the realm of Boolean algebra, maximal filters are known as "ultrafilters." Due to their versatile nature, ultrafilters hold significant and broad-ranging importance, finding applications across various fields including topology, algebra, logic, set theory, lattice theory, matroid theory, graph theory, combinatorics, measure theory, model theory, and functional analysis (cf.[11,12,13]). Essentially, a filter can be seen as a collection of sets containing a specific element, playing a crucial role in discussing convergence properties in mathematics.

Based on the aforementioned, research on edge-tangles and ultrafilters is crucial. In this paper, we introduce a novel concept called Edge-ultrafilters on graphs and demonstrate their correlation with Edge-tangles. It is noteworthy to observe that despite their apparent differences, filters and Edge-tangles establish a profound connection under specific conditions. Our aim is to advance this research further in the future by thoroughly investigating the properties of ultrafilters on graphs, particularly focusing on exploring the distinctive characteristics that arise in different classes of graphs.

2. **Notations**

This section delineates the mathematical constructs and symbols employed throughout the discourse. Note that a graph in graph theory is a mathematical structure comprising vertices and edges to represent relationships between objects.

- *V(G)* denotes the ensemble of vertices, often referred to as nodes, within a graph *G.*
- *E(G)* encapsulates the collection of edges in the graph *G*, establishing the connectivity among vertices.
- The notation *G=(V,E)* formally defines *G* as a graph constituted by two distinct sets: *V*, representing vertices, and *E,* representing edges. This notation underscores the foundational structure of a graph as an ordered pair of these sets.

- The paper consistently employs k, a natural number, as a parameter in various definitions and theorems, serving as a pivotal mathematical constant or threshold within the context of graph theory explored herein.

3. Filters on Boolean Algebras

We explore the concept of Filters within Boolean Algebras. Boolean Algebras are mathematical structures utilized for operations such as AND, OR, and NOT, commonly employed in logic and computer science.

The definition of a filter in a Boolean algebra $(X,\cup,\cap)$ is succinctly articulated below. As elucidated in the introductory segment, both Filters and Ultrafilters are pivotal in the mathematical realm. The counterpart of a filter in this algebraic structure is termed an ideal.

Definition 1: In a Boolean algebra $(X,\cup,\cap)$, a set family $F \subseteq 2^X$ satisfying the following conditions is called a filter on the carrier set X.

(FB1) $A, B \in F \Rightarrow A \cap B \in F,$

(FB2) $A \in F, A \subseteq B \subseteq X \Rightarrow B \in F,$

(FB3) $\varnothing$ is not belong to F.

In a Boolean algebras $(X,\cup,\cap)$, A maximal filter is called an ultrafilter and satisfies the following axiom (FB4):

(FB4) $\forall A \subseteq X$, *either* $A \in F$ *or* $X / A \in F$.

4. Edge-tangle on the graph

We explain about an edge-tangle on the graph. Note that an edge-cut $[A, B]$ of a graph G is an ordered pair of disjoint subsets of $V(G)$ such that $A \cup B = V(G)$. And the order of an edge-cut $[A, B]$ of G is the number of edges of G with one end in A and one end in B (cf: [1]). Be sure to note that A may also be the empty set.

The definition of a tangle on the graph is provided below. Edge-tangles are well-known for their deep connection with tree-cut-decompositions [1].

Definition 2 [1]: Let G be a graph. An *Edge-tangle* E of order k is a set of edge-cuts of G such that the following hold.

(E1) For all edge-cut $[A, B]$ of G of order less than k, either $[A, B] \in E$ or $[B, A] \in E$.

(E2) If $[A_1, B_1],[A_2, B_2],[A_3, B_3] \in E$ then $A_1 \cup A_2 \cup A_3 \neq G.$

(E3) If $[A, B] \in E$, then G has at least k edges incident with vertices in B.

5. Ultrafilter of edge-cuts : Relation to tree-cut decomposition

The definition of an Edge-ultrafilter on the graph is given below. We naturally extend the definition from Boolean algebras to a set of edge-cuts. The complement of an edge-ultrafilter is referred to as an edge-maximal ideal on a graph.

Definition 3: Let G be a graph. An edge-Ultrafilter of order k is a set of edge-cuts of G such that the following hold.

(F1) For all edge-cut $[A, B]$ of G of order less than k, either $[A, B] \in F$ or $[B, A] \in F$.

(F2) $[A_1, B_1] \in F, A_1 \subseteq A_2$, $[A_2, B_2]$ of order less than k $\Rightarrow$ $[A_2, B_2] \in$ F, (F3) $[A_1, B_1] \in$ F, $[A_2, B_2] \in$ F, $[A_1 \cap A_2, B_1 \cup B_2]$ of order less than k

$\Rightarrow [A_1 \cap A_2, B_1 \cup B_2] \in$ F,

(F4)If $[A, B] \in E$, then G has at least k edges incident with vertices in B.

(F5) *If* $V(A) = \varnothing$, *then* $[A, B]$ is not belong to F.

Proving the main theorem of this paper, which establishes the equivalence between edge-ultrafilters and edge-tangles.

Theorem 4: Let G be a graph. E is aedge-Tangle of separations of order k in graph iff $F = \{[A,B] \mid [B,A] \in E\}$ is an edge-Ultrafilter of separations of order k in graph.

Proof. To prove this theorem, we need to establish a bidirectional implication: if E is an edge-tangle of order k, then F defined by $F = \{[A,B] \mid [B,A] \in E\}$ is an edge-ultrafilter of order k, and vice versa. We will prove both directions separately.

We show that F satisfies axiom (F1). Let $[A, B]$ be any edge-cut of G of order less than k. Since E is an edge-tangle of order k, either $[A, B]$ or $[B, A]$ is in E. Thus, either $[A, B]$ or $[B, A]$ is in F, satisfying condition (F1).

To demonstrate that F satisfies axiom (F2), let's consider the scenario where $[A_1, B_1]$ is an element of F, with A_1 being a subset of A_2, and $[A_2, B_2]$ forming an edge-cut of G with an order less than k. For the purpose of this proof, we'll employ a proof by contradiction. Thus, we initially assume that $[A_2, B_2]$ does not belong to F. According to axiom (F1), this assumption would imply that $[B_2, A_2]$ is in F, and by the definition of F, $[A_2, B_2]$ must then be in E.

However, given that $[B_1, A_1]$ is in E and assuming $[A_2, B_2]$ is also in E, we arrive at a point where the union of B_1 and A_2 equals G. This situation, however, contradicts the edge-tangle condition (E2), which posits that no three edge-cuts in E can cover the entire graph G.

This contradiction leads us to conclude that our initial assumption that $[A_2, B_2]$ is not in F is false. Therefore, $[A_2, B_2]$ must indeed belong to F, thereby confirming that F adheres to axiom (F2).

We show that F satisfies axiom (F3). Suppose $[A_1, B_1]$ and $[A_2, B_2]$ are in F. If $[A_1 \cap A_2, B_1 \cup B_2]$ is not in F, then by definition, $[B_1 \cup B_2, A_1 \cap A_2]$ is not in E. Yet, by the properties of an edge-tangle, $[A_1 \cap A_2, B_1 \cup B_2]$ should be in E. This contradiction implies F satisfies (F3).

We show that F satisfies axiom (F4). This directly follows from the fact that if $[A, B]$ is in E, then G has at least k edges incident with vertices in B.

The axiom (F5) is clearly satisfied by F.

We show that E satisfies axiom (E1). Let $[A, B]$ be any edge-cut of G of order less than k. By (F1), either $[A, B]$ or $[B, A]$ belongs to F. Hence, either $[B, A]$ or $[A, B]$ must be in E, satisfying condition (E1).

We show that E satisfies (E2). Suppose $[B_1, A_1]$, $[B_2, A_2]$, and $[B_3, A_3]$ are in E. By the definition of F, $[A_1, B_1]$, $[A_2, B_2]$, and $[A_3, B_3]$ are in F. If we assume $B_1 \cup B_2 \cup B_3 = G$, then $A_1 \cap A_2 \cap A_3$ would be empty, contradicting axiom (F5). Hence, $B_1 \cup B_2 \cup B_3 \neq G$, verifying axiom (E2).

We show that E satisfies axiom (E3). For any edge-cut $[B, A]$ in E, $[A, B]$ is in F. By (F4), G has at least k edges incident with vertices in B, confirming E satisfies (E3).

This completes the proof of Theorem 4.

6. Conclusion and future tasks

In this paper, we introduced a novel definition termed Edge-Ultrafilters on graphs and demonstrated their equivalence to Edge-tangles. Furthermore, we intend to investigate the relationship between linear tangles [5,6,7] and edge-tangles in future research.

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