Short Research Article

Edge-UltraFilter and Edge-Tangle for graph

Abstract: The investigation of width parameters in both graph theory and algebraic contexts has attracted considerable attention. Among these parameters, tree-cut-decomposition has emerged as a crucial metric. The "Edge-Tangle" concept is deeply tied to the width parameter known as "tree-cut width" in graphs. In this paper, we establish a new definition called Edge-Ultrafilters on graphs and demonstrate their equivalence to Edge-Tangles.

Keyword: filter, ultrafilter, tangle, edge-tangle, tree-cut-width, tree-cut-decomposition

1. Introduction

In recent years, there has been a significant increase in research interest surrounding the exploration of width parameters in both graph theory and algebraic contexts [1-45]. Width parameters relate to metrics rooted in tree-like structures, often termed graph decompositions. The "Edge-Tangle" concept is deeply tied to the width parameter known as "tree-cut width" in graphs[1]. As such, investigations into Edge-Tangles, which have a profound connection with width parameters, hold critical significance. The term "Filter" is well-recognized within the spheres of topology and algebra. At its core, a filter can be understood as a collection of sets that include a particular element, acting as an invaluable instrument for discussing convergence properties in mathematics. In this paper, we introduce a novel definition named Edge-Ultrafilters on graphs and showcase their congruence with Edge-Tangles.

2. Definitions and Notations in this paper

This section provides mathematical definitions of each concept.
V(G) represents the set of vertices (nodes) in a graph *G, E(G)* represents the set of edges in the same graph *G*, and *G=(V,E)* signifies that *G* is a graph defined by a pair of sets, *V* for vertices and *E* for edges.Additionally, in this paper, we utilize the natural number *k*.

2.1 Filters on Boolean Algebras

We provide an explanation of Filters in Boolean Algebras.
The definition of a filter in a Boolean algebra $(X, \cup, \cap)$ is given below.As mentioned in the introduction, Filters and Ultrafilters are fundamental concepts in mathematics. The complement of an filter in a Boolean algebra $(X, \cup, \cap)$ is referred to as an ideal in a Boolean algebra $(X, \cup, \cap)$.

Definition 1: In a Boolean algebra $(X, \cup, \cap)$, a set family $F \subseteq 2^X$ satisfying the following conditions is called a filter on the carrier set *X*.
(FB1) $A, B \in F \Rightarrow A \cap B \in F,$
(FB2) $A \in F, A \subseteq B \subseteq X \Rightarrow B \in F,$
(FB3) $\emptyset$ is not belong to *F*.

In a Boolean algebras$(X, \cup, \cap)$, A maximal filter is called an ultrafilter and satisfies the following axiom (FB4):
(FB4) $\forall A \subseteq X$, *either* $A \in F$ *or* $X / A \in F$.

2.2 Edge-Tangle on the graph

We explain about an edge-tangle on the graph.Note that An edge-cut *[A, B]* of a graph G is an ordered pair of disjoint subsets of *V(G)* such that $A \cup B = V(G)$. And the order of an edge-cut *[A, B]* of G is the number of edges of *G* with one end in *A* and one end in *B*(cf: [1]).

The definition of a tangle on the graph is provided below. Edge-Tangles are well-known for their deep connection with tree-cut-decompositions [1].The tree-cut-decompositions have been studied by various researchers [24-31].

Definition 2 [1]: Let G be a graph. A*Edge-tangle* E of order k is a set of edge-cuts of G such that the following hold.

(E1) For all edge-cut $[A, B]$ of G of order less than k, either $[A, B] \in E$ or $[B, A] \in E$.

(E2) If $[A_1, B_1], [A_2, B_2], [A_3, B_3] \in E$ then $A_1 \cup A_2 \cup A_3 \neq G$.

(E3) If $[A, B] \in E$, then G has at least k edges incident with vertices in B.

3 Ultrafilterof edge-cuts : Relation to tree-cut decomposition

The definition of anEdge-Ultrafilter on the graph is given below. We naturally extend the definition from Boolean algebras to a set of edge-cuts. The complement of an edge-ultrafilter is referred to as anedge-maximal ideal on a graph.

Definition 3: Let G be a graph. Anedge-Ultrafilter of order k is a set of edge-cuts of G such that the following hold.

(F1) For all edge-cut $[A, B]$ of G of order less than k, either $[A, B] \in F$ or $[B, A] \in F$.

(F2) $[A_1, B_1] \in F, A_1 \subseteq A_2$, $[A_2, B_2]$ of order less than $k \Rightarrow [A_2, B_2] \in$F,

(F3) $[A_1, B_1] \in$F, $[A_2, B_2] \in$F, $[A_1 \cap A_2, B_1 \cup B_2]$ of order less than k

$\Rightarrow [A_1 \cap A_2, B_1 \cup B_2] \in$F,

(F4)If $[A, B] \in E$, then G has at least k edges incident with vertices in B.

(F5)(F4) *If* $V(A) = V(G)$, *then* $[A, B] \in F$.

Proving the Main Theorem of this paper, which establishes the equivalence between edge-Ultrafilters and Edge-Tangles.

Theorem 4. Let G be a graph. E is aedge-Tangle of separations of order k in graph iff $F = \{[A,B] \mid [B,A] \in E\}$ is an edge-Ultrafilter of separations of order k in graph.

Proof. To prove this theorem, we need to establish a bidirectional implication: if E is an edge-tangle of order k, then F defined by $F = \{[A,B] \mid [B,A] \in E\}$ is an edge-ultrafilter of order k, and vice versa. We will prove both directions separately.

We show that F satisfies axiom (F1). Let $[A, B]$ be any edge-cut of G of order less than k. Since E is an edge-tangle of order k, either $[A, B]$ or $[B, A]$ is in E. Thus, either $[A, B]$ or $[B, A]$ is in F, satisfying condition (F1).

We show that F satisfies axiom (F2). Suppose $[A_1, B_1] \in F$ and $A_1 \subseteq A_2$ and $[A_2, B_2]$is an edge-cut of G of order less than k. Since $[B_1, A_1] \in E$ and $A_1 \subseteq A_2$, by the definition of an edge-tangle, we can infer that $[B_2, A_2]$ belongs to E. This means that $[A_2, B_2]$ belongs to F, establishing condition (F2).

We show that F satisfies axiom (F3). Suppose $[A_1, B_1]$and $[A_2, B_2]$ are in F. If $[A_1 \cap A_2, B_1 \cup B_2]$ is not in F, then by definition, $[B_1 \cup B_2, A_1 \cap A_2]$ is not in E. Yet, by the properties of an edge-tangle, $[A_1 \cap A_2, B_1 \cup B_2]$should be in E. This contradiction implies F satisfies (F3).

We show that F satisfies axiom (F4). This directly follows from the fact that if $[A, B]$ is in E, then G has at least k edges incident with vertices in B.

The axiom (F5) is clearly satisfied by F.

We show that E satisfies axiom (E1). Let $[A, B]$ be any edge-cut of G of order less than k. By (F1), either $[A, B]$ or $[B, A]$ belongs to F. Hence, either $[B, A]$ or $[A, B]$ must be in E, satisfying condition (E1).

We show that E satisfies (E2). Suppose $[B_1, A_1]$, $[B_2, A_2]$, and $[B_3, A_3]$ are in E. By the definition of F, $[A_1, B_1]$, $[A_2, B_2]$, and $[A_3, B_3]$ are in F. If we assume $B_1 \cup B_2 \cup B_3 = G$, then $A_1 \cap A_2 \cap A_3$ would be empty, contradicting axiom (F5). Hence, $B_1 \cup B_2 \cup B_3 \neq G$, verifying axiom (E2).

We show that E satisfies axiom (E3). For any edge-cut $[B, A]$ in E, $[A, B]$ is in F. By (F4), G has at least$k$edges incident with vertices in B, confirming E satisfies (E3).

This completes the proof of Theorem 4.

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