A DETAILED PROOF OF THE STRONG GOLDBACH CONJECTURE BASED ON PARTITIONS OF A NEW FORMULATION OF A SET OF EVEN NUMBERS

Abstract

The Strong Goldbach's conjecture, a fundamental problem in Number Theory, asserts that every even integer greater than 2 can be expressed as the sum of two prime numbers. Despite significant efforts over centuries, this conjecture remains unproven, challenging the core of mathematics. The known algorithms for attempting to prove or verify the conjecture on a given interval $[a\, ,\, b]$ consist of finding two sets of primes p_i and p_j such that $p_i + p_j$ cover all the even numbers in the interval $[a\, , b]$. However, the traditional definition of an even number as $2n$ for $n \in \mathbb{N}$ (where $\mathbb{N}$ is the set of natural numbers), has not provided mathematicians with a straightforward method to obtain all Goldbach partitions for any even number of this form. This paper introduces a novel approach to the problem, utilizing all odd partitions of an even number of a new formulation of the form $E_{ij} = n_i + n_j + (n_j - n_i)^n$ for all $n \in \mathbb{N}$. By demonstrating that there exist at least a pair of prime numbers in these odd partitions, the fact that the sum of any two prime numbers is even and there exists infinitely many prime numbers, this paper provides a compelling proof of the conjecture. This breakthrough not only solves a long-standing mathematical mystery but also sheds light on the structure of prime numbers.

Keywords *Goldbach Conjecture, Goldbach partition, Even numbers, Odd numbers, Prime numbers, Natural numbers*

1.Introduction

"The strong version of the Goldbach Conjecture states that for every even integer greater than 2, there exist two primes whose sum is that even number. It is one of the challenging questions in Number Theory yet to be answered. The conjecture is named after Christian Goldbach, an 18th-century mathematician from Prussia, who first proposed it in a letter to the famous mathematician Leonhard Euler in 1742 and despite centuries of effort by mathematicians to prove or disprove the conjecture, it remains unsolved to this day"[1, 2]. "The closest related results are that: (i) there exists an integer *S* such that every integer is the sum of at most *S* primes, and (ii) every sufficiently large even integer may be written as the sum of a prime number and of the product of at most two prime numbers" [4]. Other key milestones towards the proof of Goldbach conjecture in the past 278 years and beyond include: 1) The Euclid's *Fundamental Theorem of Arithmetic* (FTA) that revealed there is always a unique prime factorization for any integer [5] and the Legendre's *Sieve of Eratosthenes* (1808) that provided a foundation for modern sieve theories [6].

While the Goldbach Conjecture is easy to state, all efforts to prove or disprove it have so far been unsuccessful. However, the conjecture has been extensively verified. In July 2000, JörgRichstein used computational methods to demonstrate the conjecture's validity up to 4 × 10^14 [8]. "Subsequently, in November 2013, Thomás Oliveira e Silva, Siegfried Herzog, and SilvoPardi employed advanced computational techniques to establish the truth of the binary form of the Goldbach Conjecture up to 4 × 10^18" [3]. More recently, Daniel Sankei, LoyfordNjagi, and Josephine Mutembei utilized a new formulation of even numbers [7] and the property that such even numbers can be partitioned into pairs of odd numbers [9] to numerically verify the Strong Goldbach Conjecture up to 9 × 10^18 [10]. In a more theoretical vein, Hardy and Littlewood made significant progress towards proving the conjecture in their groundbreaking work on the Hardy-Littlewood circle method [18].

While numerous attempts to prove or disprove it have failed, the conjecture has been extensively verified computationally, with the most recent efforts pushing the boundaries of numerical verification to unprecedented levels. Despite this, the formal proof of the Goldbach Conjecture remains elusive [13]. The belief in its truth is bolstered by statistical considerations and the probabilistic distribution of prime numbers, which suggests that as numbers grow larger, there are increasingly more ways to express them as sums of primes [12]. The conjecture's impact extends beyond its unsolved status, influencing the development of Number Theory and inspiring further mathematical inquiry. Thus, while the Goldbach Conjecture continues to resist formal proof, it remains a fascinating and influential problem in the realm of mathematics.

2.Preliminaries

Efforts to prove the Strong Goldbach's conjecture have so far been inconclusive, with many attempts grounded in analytic number theory approaches, such as the examination of prime number gaps [11]. Proving this conjecture requires a systematic and rigorous approach that synthesizes various mathematical techniques. This paper presents a proof achieved through the integration of different findings, including a novel formulation of even numbers [7] and the established fact that any even number in this new formulation can be decomposed into pairs of odd numbers [9]. Furthermore, the study utilizes Theorem 1, which asserts the existence of infinitely many prime numbers, and Theorem 2, which states that the sum of two odd numbers is even. These foundational concepts lay the groundwork for the subsequent proof of the conjecture.

Theorem 1(Euclid's theorem)

There are infinitely many prime numbers [14,17].

Proof

"Given the list of prime numbers $p_1, p_2, \ldots, p_n$, the number $N = (p_1 * p_2 * p_3 * \cdots * p_n) + 1$ must contain a prime factor not among the primes used in its construction. To see this, notice p_1 does not divide N since it leaves a remainder of 1 (or alternatively N/p_1 is clearly not an integer). Similarly, the other $pi's$ does not divide N.We therefore conclude that any finite list of primes is not complete, and therefore there must be infinitely many primes" [15].

Theorem 2

The sum of two odd numbers is even.

Proof

A number is odd if it can be written as 2x + 1, where x is some integer. “A number is even if it can be written as 2x, where x is some integer. To start, pick any two odd numbers. We can write them as 2n + 1 and 2m + 1. The sum of these two odd numbers is (2n + 1) + (2m + 1). This can be simplified to 2n + 2m + 2 and further simplified to 2(n + m + 1). The number 2(n + m + 1) is even because n + m + 1 is an integer. Therefore, the sum of the two odd numbers is even” [16].

2.1 Partitions of an even number of the new formulation

Let P be the set of all prime numbers, ℕ be the set of all natural numbers and O the set of all odd numbers.

Step 1 : Let P_1 and $P_2 \in P$,then $(P_1 + P_2) + (P_2 - P_1)^n$ is even, $\forall n \in \mathbb{N}$, and $p_2 > p_1$ [7].

Step 2: Let d be even and belong to the half-open interval [1, $[\frac{1}{2}((p_1 + p_2) + (p_2 - p_1)^n]]$.

Step 3: Let z_i *and* $y_i \in 1 \le O \le \frac{1}{2}((P_1 + P_2) + (P_2 - P_1)^n)$ for $i \in O$ belonging to the half-open interval [1, $[\frac{1}{2}((p_1 + p_2) + (p_2 - p_1)^n]]$.

With p_1, p_2, d and z_i, we partition $(p_1 + p_2) + (p_2 - p_1)^n$ as follows:

Partition 1: $((P_1 + P_2) + (P_2 - P_1)^n) - (d + z_1) = y_1$

Partition 2: $((p_1 + p_2) + (p_2 - p_1)^n) - (d + z_3) = y_3$

Partition 3: $((p_1 + p_2) + (p_2 - p_1)^n) - (d + z_5) = y_5$

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Partition i: $((p_1 + p_2) + (p_2 - p_1)^n) - (d + z_{((\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)-1)}) = y_{((\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)-1)}$

The set of pairs $(d + z_1, y_1), (d + z_3, y_3), (d + z_5, y_5), \dots,$ $(d + z_{((\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)-1)}, y_{((\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)-1)})$ of odd numbers are all partitions of the even number $(p_1 + p_2) + (p_2 - p_1)^n$.

If we let the set A contain all these pairs of odd number , that is, A = { $(d + z_1, y_1), (d + z_3, y_3), (d + z_5, y_5), \dots,$ $(d + z_{((\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)-1)}, y_{((\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)-1)})$ },then there exist a proper subset say $B \subset A$ containing at least a pair of prime numbers fulfilling the Strong Goldbach’s Conjecture.

Example 1

Let $p_1 = 13$, $p_2 = 23$ and $n = 1$,then $(p_1 + p_2) + (p_2 - p_1)^1 = (13 + 23) + (23 - 13)^1 = 36 + 10 = 46$ is even.

Remark 1

The multiples of d belonging to the half-open interval $[1, [\frac{1}{2}(46)]$ are $\{2,4,6,8,10,12,14,16,18,20,22\}$. In this example 1 we let $d = 8$.

Step 2: let $d = 8$

Step 3: Take odd numbers in the range $1 \le O \le \frac{1}{2}(46) \rightarrow (1,3,5,7,9,11,13,15,17,19,21,23)$

Then we partition 46 as follows:

I.	$46-(8+1)=37$	VII.	$46-(8+13)=25$
II.	$46-(8+3)=35$	VIII.	$46-(8+15)=23$
III.	$46-(8+5)=33$	IX.	$46-(8+17)=21$
IV.	$46-(8+7)=31$	X.	$46-(8+19)=19$
V.	$46-(8+9)=29$	XI.	$46-(8+21)=17$
VI.	$46-(8+11)=27$	XII.	$46-(8+23)=15$

The partitions of 46 are therefore: $((8+1),37), ((8+3),35), ((8+5),33), ((8+7),31), ((8+$ *9,29,8+11,27,8+13,25,8+15,23,8+17,21,8+19,19,8+21,17,(8+23,15)* $\Rightarrow$ The set A $=\{(9,37), (11,35), (13,33), (15,31), (17,29), (19,27),$

$(21,25), (23,23), (25,21), (27,19), (29,17), (41,5), (31,15)\}$ contains all pairs of odd numbers. It further implies that there exist a proper subset $B \subset A$ containing at least a pair of prime numbers. Notice that in this case the proper subset $B = \{(17,29), (23,23), (41,5)\}$ indicating that $(17+$ *29=23+23=41+5=46*. The prime pairs (17,19), (23,23) and (41,5) satisfies the statement of the Strong Goldbach Conjecture.

Remark 2

Any multiple of d belonging to the half-open interval $[1, [\frac{1}{2}(46)] = \{2,4,6,8,10,12,14,16,18,20,22\}$ can be used to partition 46 into the same number of all pairs of odd numbers for $n = 1$ [10] as shown in example 2.

Example 2

Let $p_1 = 13$, $p_2 = 23$ and $n = 1$, then $(p_1 + p_2) + (p_2 - p_1)^1 = (13+23) + (23-13)^1 = 36 + 10 = 46$ is even.

Step 2: $d = 20$

Step 3: Take odd numbers in the range $1 \le O \le \frac{1}{2}(46) \rightarrow (1,3,5,7,9,11,13,15,17,19,21,23)$

Then we partition 46 as follows:

I.	$46-(20+1)=25$	VII.	$46-(20+13)=13$
II.	$46-(20+3)=23$	VIII.	$46-(20+15)=11$
III.	$46-(20+5)=21$	IX.	$46-(20+17)=9$
IV.	$46-(20+7)=19$	X.	$46-(20+19)=7$
V.	$46-(20+9)=17$	XI.	$46-(20+21)=5$
VI.	$46-(20+11)=15$	XII.	$46-(20+23)=3$

The partitions of 46 are therefore: $\big((20+1),25\big),\big((20+3),23\big),\big((20+5),21\big),\big((20+$ $7,19,20+9,17,20+11,15,20+13,13,20+15,11,20+17,9,20+19,7,20+21,5,(20+23,3)$ $\Rightarrow$ The set A $=\{(21,25),(23,23),(25,21),(31,15),$

$(27,19),(29,17),(31,15),(33,13),(35,11),(37,9),(39,7),(41,5),(43,3)\}$ contains all pairs of odd numbers. The proper subset $B \subset A$containing at least a pair of prime numbers is the set B=$\{(29,17),(23,23),(41,5),(43,3\}$ indicating that $(29+17)=(23+23)=(41+5)=(43+3)=46$ and therefore all the prime pairs in the proper subset satisfies the statement of the Strong Goldbach' Conjecture.

Corollary 1

Let $p_1 and p_2 \in P$, where P is the set of all primes and d be the difference between $p_1 and p_2$ such that $d = p_2 - p_1 > 0$ and Let $z_i \in 1 \le O \le \frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)$ be the set of odd numbers for $i \in 1 \le \quad O \quad \le (\frac{1}{2}(p_1+p_2)+(p_2-p_1)^n)$, then any multiple of d in the range $\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n) < d < (p_1+p_2)+(p_2-p_1)^n$ can be used to generate the same set of pairs of odd numbers or a subset of pairs of odd numbers whose sum is $(p_1+p_2)+(p_2-p_1)^n$. For the set of values of d in the open interval $[\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n),(p_1+p_2)+(p_2-p_1)^n]$, it is expected that as d gets closer to $(p_1+p_2)+(p_2-p_1)^n$,the pairs of odd numbers whose sum is $(p_1+p_2)+(p_2-p_1)^n$ reduces significantly[9].

Example 3

Let $(p_1+p_2)+(p_2-p_1)^n = 46$ for $n=1$

Remark 3

The multiples of d in the interval $\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n) < d < (p_1+p_2)+(p_2-p_1)^n$ are $\{24,26,28,30,32,34,36,38,40,42,44\}$. For example 3 we choose to use $d=40$ to illustrate

Corollary 1.

Step 2:Let $d = 40$

Step 3: Take odd numbers in the range $1 \le O \le \frac{1}{2}(\ 46\ ) \rightarrow (1,3,5,7,9,11,13,15,17,19,21,23)$

Then the following are partitions of 46:

I. $46-(40+1)=5$
II. $46-(40+3)=3$
III. $46-(40+5)=1$
IV. $46-(40+7)=-1$
V. $46-(40+9)=-3$
VI. $46-(40+11)=-5$
VII. $46-(40+13)=-7$
VIII. $46-(40+15)=-9$
IX. $46-(40+17)=-11$
X. $46-(40+19)=-13$
XI. $46-(40+21)=-15$
XII. $46-(40+23)=-17$

The results include both positive and negative partitions of 46. The positive partitions of 46 are therefore: $\big((40+1),5\big),\big((40+3),3\big),\big((40+5),11\big)$, implying that the set A containing all positive pairs of odd numbers, is the set A = $\{\ (41,5),(43,3),(45,1)\}$. We only obtain three positive pairs of odd numbers and the negative partitions are not considered in this case since the statement of the Strong Goldbach's Conjecture does not include negative prime numbers.

The subset B = {(41,5), (43,3)} exemplifies a proper subset containing at least one pair of prime numbers, satisfying the Strong Goldbach's Conjecture by demonstrating that both 41 + 5 and 43 + 3 equal 46. This example illustrates that when an even number is partitioned into a set comprising all pairs of odd numbers, a proper subset of this set can be identified, which contains at least one pair of prime numbers that satisfy the Strong Goldbach's Conjecture. The next step is to establish that this pattern holds true for all even numbers.

3. Construction of the proof

Let us consider two finite sets of prime numbers $P = \{p_1, p_2, \dots, p_n\}$ and $Q = \{p_1, p_2, \dots, p_m\}$ and the fact that according to Theorem 1, there are infinitely many prime numbers. Therefore, it is expected that for the finite set P, if p_n is the large prime number in the set, then, there is another larger prime number than p_n say $N = (p_1 * p_2 * p_3 \cdots * p_n) + 1$ and subsequently, another larger prime number than N say $K = (p_1 * p_2 * p_3 \cdots * p_n * ((p_1 * p_2 * p_3 \cdots * p_n) + 1)) + 1$ and so on. The same argument can also be used with the set Q so that the prime number $M = (p_1 * p_2 * p_3 * \cdots * p_m) + 1$ is a larger prime number than p_m and say the prime number $H = (p_1 * p_2 * p_3 * \cdots * p_m * ((p_1 * p_2 * p_3 * \cdots * p_m) + 1)) + 1$ is larger than M and so on. We therefore obtain two infinite sets of prime numbers $P^* = \{p_1, p_2, \dots, p_{n,} N, K, \dots\}$ and $Q^* = \{p_1, p_2, \dots, p_m, M, H, \dots\}$.

According to Theorem 2, the sum of any two odd numbers is even and since prime numbers greater than 2 are subsets of odd numbers, then, the addition of any two elements from set $P^*$and set $Q^*$is even. To illustrate this concept, let us consider the addition of elements from both infinite sets for various prime numbers ≥ 3 (although 2 is a prime number, we exempt it from each of the infinite sets since the addition of 2 to any prime number results to an odd number and not an even number). Therefore, the infinite set $P^*$is: 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97,..., pn, $(p_1 * p_2 * p_3 \cdots * p_n) + 1, (p_1 * p_2 * p_3 \cdots * p_n * ((p_1 * p_2 * p_3 \cdots * pn + 1) + 1$, and so on. Table 1 illustrates the results of adding elements in set $P*$ to elements in set Q^* with all the possible combinations so that (P^* + Q^*) gives an even number as follows:

Table 1 Strong Goldbach Conjecture table (excluding 2)

Set P^*	Set Q^*	Sum (P^*+ Q^*)	Cont' Set P^*	Cont' Set Q^*	Cont' Sum (P^* + Q^*)
3	3	6	7	13	20
3	5	8	7	17	24
3	7	10	7	19	26
3	11	14	7	23	30
3	13	16	11	11	22
3	17	20	11	13	24
3	19	22	11	17	28
3	23	26	11	19	30
5	5	10	11	23	34
5	7	12	13	13	26
5	11	16	13	17	30
5	13	18	13	19	32
5	17	22	13	23	36
5	19	24	17	17	34
5	23	28	17	19	36
7	7	14	17	23	40
7	11	18	19	19	38
			.	.	.
			.	.	.
			.	.	.
			pn	pm	$2n_{nm}$
			$(p_1 * p_2 * p_3 * \cdots * p_n) + 1$	$(p_1 * p_2 * p_3 * \cdots * p_m) + 1$	$2n_{(N,M)}$
			$(p_1 * p_2 * p_3 \cdots * p_n * ((p_1 * p_2 * p_3 \cdots * p_n) + 1)) + 1$	$(p_1 * p_2 * p_3 * \cdots * p_m * ((p_1 * p_2 * p_3 * \cdots * p_m) + 1)) + 1$	$2n_{(K,H)}$
			.	.	.
			.	.	.
			.	.	.

It is straightforward from table 1 that the even numbers (P^* + Q^*) are results of addition of elements of set P^* to elements of set Q^*. As the process of adding the elements of both infinite sets continues, we eventually get to the addition of very large pairs of arbitrary prime numbers ($((p_1 * p_2 * p_3 * \cdots * p_n) + 1$), $((p_1 * p_2 * p_3 * \cdots * p_m) + 1$)) and (

$$(p_1 * p_2 * p_3 \cdots * p_n * ((p_1 * p_2 * p_3 \cdots * p_n) + 1)) + 1,$$

$(p_1 * p_2 * p_3 * \cdots * p_m * ((p_1 * p_2 * p_3 * \cdots * p_m) + 1)) + 1)$ whose sum is even. It is important to note that these larger pairs of prime numbers are chosen arbitrarily, and therefore the results $2n_{(N,M)}$ and $2n_{(K,H)}$ could be any arbitrary even numbers. The results in table 1 therefore, shows that the addition of any two arbitrarily chosen prime numbers give an even number and since the converse is true, it also implies that any arbitrarily chosen even number can be written as a sum of two prime numbers. Using the results in Table 1, the fact that any even number can be partitioned into all pairs of odd numbers from which we obtain a subset of prime numbers, the proof of the Strong Goldbach's conjecture can be derived.

4. The Proof of the Strong Goldbach Conjecture

Theorem 3 (The Strong Goldbach Conjecture)

Every even integer greater than two is the sum of two prime numbers.

Proof

Using a new formulation of set of even numbers $(p_1 + p_2) + (p_2 - p_1)^n$ [7], it has been shown that it is possible to obtain all pairs of odd numbers whose sum is even as in example 1, example 2 and example 3. The set of all pairs of odd numbers { $(d + z_1, y_1), (d + z_3, y_3), (d + z_5, y_5), \ldots,$ $(d + z_{((\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)-1)}, y_{((\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)-1)})$ } [9] generated from partitioning the even number $(p_1 + p_2) + (p_2 - p_1)^n$ is used to obtain two sets of odd numbers $\{d + z_i\}$ and $\{y_i\}$ such that $(p_1 + p_2) + (p_2 - p_1)^n = \{d + z_i\} + \{y_i\}$ for all pairs of odd numbers of the elements in both sets.

Further, it is also clearfrom example 1, example 2 and example 3 that, from the two sets of odd numbers $\{d + z_i\}$ and $\{y_i\}$,there exist two proper subsets of prime numbers say $\{d + z_k\} \subset \{d + z_i\}$ and $\{y_k\} \subset \{y_i\}$ in such a way that each proper subset contains at least one prime number say $(d + z_j) \in \{d + z_k\} \subset \{d + z_i\}$ and $y_j \in \{y_k\} \subset \{y_i\}$ such that $(p_1 + p_2) + (p_2 - p_1)^n = (d + z_j) + y_j$.

The question however is whether this is true for all even numbers. In order to ensure that all the even numbers are generated, the sums $(P^* + Q^*)$ in table 1 are calculated with all the possible combinations of the sum of all the elements in the infinite set P^* to the elements in the infinite set Q^*. The results in table 1therefore proves two things: 1) The addition of any two arbitrarily chosen odd numbers gives an even number, implying that the sum of any two arbitrarily chosen prime numbers greater than 2 gives an even number which is arbitrary and 2) Any arbitrary even number can be written as a sum of two prime numbers since the converse of 1) is also true. It is therefore true to say that any arbitrary even number, however large it is, can always be written as a sum of two prime numbers since the sum of any two arbitrary prime numbers is even. These results therefore guarantees that the two sets of odd numbers $\{d + z_i\}$ and $\{y_i\}$ obtained from partitioning any arbitrarily chosen even number $(p_1 + p_2) + (p_2 - p_1)^n$ contains two proper subsets of prime numbers $\{d + z_k\} \subset \{d + z_i\}$ and $\{y_k\} \subset \{y_i\}$ in such a way that each proper subset contains at least one prime number say $(d + z_j) \in \{d + z_k\} \subset \{d + z_i\}$ and $y_j \in \{y_k\} \subset \{y_i\}$ such that $(p_1 + p_2) + (p_2 - p_1)^n$ = $(d + z_j)$ + y_j .These results therefore proves the Strong Goldbach's Conjecture restated as every even integer greater than 6 can be expressed as the sum of two prime numbers. Since it is already known that $4 = 2 + 2$ and $6 = 3 + 3$, the proof presented here is considered a general proof of the Strong Goldbach's Conjecture.

5.Conclusion

It has been shown that the elements of two arbitrary sets of prime numbers when added together gives an even number. We also show that the set of all pairs of odd numbers obtained from partitioning an even number of the new formulation $(p_1 + p_2) + (p_2 - p_1)^n$, contains two proper subsets of prime numbers such that in each of these proper subsets of prime numbers there exist prime numbers whose sum equals $(p_1 + p_2) + (p_2 - p_1)^n$. Since the addition of any two arbitrary prime numbers gives an arbitrary even number and the converse has been shown to be true, then these results ensures that there exists at least a prime number in each of these two proper subsets of prime numbers such that the addition of the two prime numbers equals $(p_1 + p_2) + (p_2 - p_1)^n$. Based on these results, the Strong Goldbach's Conjecture is considered to be proven.

Data availability statement

The concepts discussed in this research article is purely abstract and all citations have been included appropriately.

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