__Original Research Article__

A DETAILED PROOF OF THE STRONG GOLDBACH CONJECTURE BASED ON PARTITIONS OF A NEW FORMULATION OF A SET OF EVEN NUMBERS

Abstract

The Strong Goldbach's conjecture also known as the Binary Goldbach conjecture (BGC) is one of the oldest and best-known unsolved problems in Number Theory and all of mathematics. It states that every even integer greater than 2 can be expressed as the sum of two primes. The BGC has set a persistent challenge to the exploration of the foundations of mathematics in general and Number Theory in particular as it remains unproven for almost 250 years despite considerable efforts by mathematicians throughout history. The known algorithms for attempting to prove or verify the BGC on a given interval consist of finding two sets of prime numbers that cover all the even numbers in that interval. This study utilizes a new formulation of a set of even numbers and the fact that an even number of this formulation can be partitioned into all pairs of all odd numbers. From this set of all pairs of odd numbers, it has been shown that there exists at least one Goldbach partition whose sum is the given even number. Finally, using these results and the facts that the sum of any two prime numbers is even and there exist infinitely many prime numbers, a detailed proof of the Strong Goldbach's conjecture is provided.

Keywords *Goldbach Conjecture, Goldbach partition, Even numbers, Odd numbers, Prime numbers, Natural numbers*

1.Introduction

The strong version of the Goldbach Conjecture states that for every even integer greater than 2, there exist two primes whose sum is that even number. It is one of the challenging questions in Number Theory yet to be answered. The conjecture is named after Christian Goldbach, an 18th-century mathematician from Prussia, who first proposed it in a letter to the famous mathematician Leonhard Euler in 1742[1] and despite centuries of effort by mathematicians to prove or disprove the conjecture, it remains unsolved to this day[2].The closest related results are that: (i) there exists an integer *S* such that every integer is the sum of at most *S* primes, and (ii) every sufficiently large even integer may be written as the sum of a prime number and of the product of at most two prime numbers [4]. Other key milestones towards the proof of Goldbach conjecture in the past 278 years and beyond include: 1) The Euclid's *Fundamental Theorem of Arithmetic* (FTA) that revealed there is always a unique prime factorization for any integer [5] and the Legendre's *Sieve of Eratosthenes* (1808) that provided a foundation for modern sieve theories [6].

Although this conjecture is simple to state, all attempts to prove it or find a counter-example have failed.Further, this conjecture has been verified to an astonishing degree. In July of 2000 Jörg Richstein published a paper using computational techniques showing that the Goldbach Conjecture was valid up to 4×10^{14}[8]. In November of 2013 a paper was published by Thomás Oliveira e Silva, Siegfried Herzog, and Silvo Pardi which also used advances in computational computing proving that the binary form of the Goldbach Conjecture is true up to 4×10^{18}[3]. The most recent computational approach is that of Daniel Sankei, Loyford Njagi and Josephine Mutembei who using a new formulation of a set of even numbers [7] and the fact that an even number of this formulation can be partitioned into all pairs of odd numbers [9] presented a numerical verification of the Strong Goldbach Conjecture up to 9×10^{18} [10].

The majority of mathematicians believe that Goldbach's conjecture is true, especially under statistical considerations based on the probabilistic distribution of prime numbers. The larger the number, the more manners available to represent it as a sum of two or more primes [12]. Despite the lack of proof, the Goldbach conjecture has been influential in the development of Number Theory and has inspired many other important results. While the Goldbach conjecture has not been proved, many mathematicians believe that it is true for all even integers. However, a formal proof of the conjecture remains an open problem in mathematics [13].

2.Preliminaries

All attempts to prove the Strong Goldbach's conjecture have failed and many of these attempts rely on an analytic number theory approach such as analyzing the gaps between primes [11]. Proving the strong Goldbach conjecture involves a systematic and rigorous approach that combines various mathematical techniques. The proof presented here is derived at by combining various results of a new formulation of a set of even numbers [7] and the fact that it has been proven that any even number of this new formulation can be partitioned into all pairs of odd numbers [9]. The study further uses the facts that from the following theorem 1 and theorem 2, there exist infinitely many prime numbers and the sum of two odd umbers is even respectively.

Theorem 1(Euclid's theorem)

There are infinitely many prime numbers [14,17].

Proof

Given the list of prime numbers $p_1, p_2, \ldots, p_n$, the number $N = (p_1 * p_2 * p_3 * \cdot \cdot * p_n) + 1$ must contain a prime factor not among the primes used in its construction. To see this, noticep_1 does not divide N since it leaves a remainder of 1 (or alternatively N/p_1 is clearly not an integer). Similarly, the other $pi\,'s$ does not divide N. We therefore conclude that any finite list of primes is not complete, and therefore there must be infinitely many primes [15].

Theorem 2

The sum of two odd numbers is even.

Proof

A number is odd if it can be written as 2x + 1, where x is some integer. A number is even if it can be written as 2x, where x is some integer. To start, pick any two odd numbers. We can write them as 2n + 1 and 2m + 1. The sum of these two odd numbers is (2n + 1) + (2m + 1). This can be simplified to 2n + 2m + 2 and further simplified to 2(n + m + 1). The number 2(n + m + 1) is even because n + m + 1 is an integer. Therefore, the sum of the two odd numbers is even [16].

2.1 Partitioning an even number of the new formulation into all pairs of odd numbers

Using the new formulation of a set of even numbers as $(P_1 + P_2) + (P_2 - P_1)^n$ [7], it has been shown that it is always possible to partition any even number into all pairs of odd numbers using the following algorithm [9]:

Let P be the set of all prime numbers, ℕ be the set of all natural numbers and O the set of all odd numbers.

Step 1 : Let P_1 and $P_2 \in P$,then $(P_1 + P_2) + (P_2 - P_1)^n$ is even, $\forall n \in \mathbb{N}$, and $p_2 > p_1$ [7].

Step 2: Let d be even and belong to the half-open interval [1, $[\frac{1}{2}((p_1 + p_2) + (p_2 - p_1)^n]]$.

Step 3: Let z_i *and* $y_i \in 1 \leq O \leq \frac{1}{2}((P_1 + P_2) + (P_2 - P_1)^n)$ for $i \in O$ and belong to the half-open interval [1, $[\frac{1}{2}((p_1 + p_2) + (p_2 - p_1)^n]]$.

With p_1, p_2, d and z_i, we partition $(p_1 + p_2) + (p_2 - p_1)^n$ as follows:

Partition 1: $((P_1 + P_2) + (P_2 - P_1)^n) - (d + z_1) = y_1$

Partition 2: $((p_1 + p_2) + (p_2 - p_1)^n) - (d + z_3) = y_3$

Partition 3: $((p_1 + p_2) + (p_2 - p_1)^n) - (d + z_5) = y_5$

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Partition i: $((p_1 + p_2) + (p_2 - p_1)^n) - (d + z_{((\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)-1)}) = y_{((\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)-1)}$

The set of pairs $(d + z_1, y_1), (d + z_3, y_3), (d + z_5, y_5), \ldots,$ $(d + z_{((\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)-1)}, y_{((\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)-1)})$ of odd numbers are all partitions of the even number $(p_1 + p_2) + (p_2 - p_1)^n$. Since prime numbers greater than 2 are subsets of odd numbers, from this set of pairs of odd numbers, the possibility is that, there exist at least one pair of primes [10]. We are to show that when an even number is partitioned into a set say A containing all pairs of odd numbers, that is, A = { $(d + z_1, y_1), (d + z_3, y_3), (d + z_5, y_5), \ldots,$ $(d + z_{((\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)-1)}, y_{((\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)-1)})$ }, then, there exist a proper subset say $B \subset A$ containing at least a pair of prime numbers fulfilling the Strong Goldbach's Conjecture.

Example 1

Step 1: Let $p_1 = 13$, $p_2 = 23$ and $n = 1$,then $(p_1 + p_2) + (p_2 - p_1)^1 = (13 + 23) + (23 - 13)^1 = 36 + 10 = 46$ is even.

Remark 1

The multiples of d belonging to the half-open interval $[1, [\frac{1}{2}(46)]$ are $\{2,4,6,8,10,12,14,16,18,20,22\}$. In this example 1 we let $d = 8$.

Step 2: $d = 8$

Step 3: Take odd numbers in the range $1 \le O \le \frac{1}{2}(46) \rightarrow (1,3,5,7,9,11,13,15,17,19,21,23)$

Then we partition 46 as follows:

I.	$46-(8+1)=37$	VII.	$46-(8+13)=25$
II.	$46-(8+3)=35$	VIII.	$46-(8+15)=23$
III.	$46-(8+5)=33$	IX.	$46-(8+17)=21$
IV.	$46-(8+7)=31$	X.	$46-(8+19)=19$
V.	$46-(8+9)=29$	XI.	$46-(8+21)=17$
VI.	$46-(8+11)=27$	XII.	$46-(8+23)=15$

The partitions of 46 are therefore: $((8+1), 37), ((8+3), 35), ((8+5), 33), ((8+7), 31), ((8+$ *9,29,8+11,27,8+13,25,8+15,23,8+17,21,8+19,19,8+21,17,(8+23,15)* $\Rightarrow$ The set A $=\{(9,37), (11,35), (13,33), (15,31), (17,29), (19,27),$

$(21,25), (23,23), (25,21), (27,19), (29,17), (41,5), (31,15)\}$ contains all pairs of odd numbers. It further implies that there exist a proper subset $B \subset A$ containing at least a pair of prime numbers. Notice that in this case the proper subset $B = \{(17,29), (23,23), (41,5)\}$ indicating that $(17+$ *29=23+23=41+5=46*. The prime pairs (17,19), (23,23) and (41,5) satisfies the statement of the Strong Goldbach Conjecture.

Remark 2

Any multiple of d belonging to the half-open interval $[1, [\frac{1}{2}(46)] = \{2,4,6,8,10,12,14,16,18,20,22\}$ can be used to partition 46 into the same number of all pairs of odd numbers for $n = 1$ [10] as shown in example 2.

Example 2

Step 1 : Let $p_1 = 13$, $p_2 = 23$ and $n = 1$, then $(p_1 + p_2) + (p_2 - p_1)^1 = (13+23) + (23-13)^1 = 36+10 = 46$ is even.

Step 2: $d = 20$

Step 3: Take odd numbers in the range $1 \le O \le \frac{1}{2}(46) \rightarrow (1,3,5,7,9,11,13,15,17,19,21,23)$

Then we partition 46 as follows:

I.	$46-(20+1)=25$	VII.	$46-(20+13)=13$
II.	$46-(20+3)=23$	VIII.	$46-(20+15)=11$
III.	$46-(20+5)=21$	IX.	$46-(20+17)=9$
IV.	$46-(20+7)=19$	X.	$46-(20+19)=7$
V.	$46-(20+9)=17$	XI.	$46-(20+21)=5$
VI.	$46-(20+11)=15$	XII.	$46-(20+23)=3$

The partitions of 46 are therefore: $((20+1),25),((20+3),23),((20+5),21),((20+7,19,20+9,17,20+11,15,20+13,13,20+15,11,20+17,9,20+19,7,20+21,5,(20+23,3)$ $\Rightarrow$ The set A $=\{(21,25),(23,23),(25,21),(31,15),$

$(27,19),(29,17),(31,15),(33,13),(35,11),(37,9),(39,7),(41,5),(43,3)\}$ contains all pairs of odd numbers. The proper subset $B \subset A$ containing at least a pair of prime numbers is the set B $=\{(29,17),(23,23),(41,5),(43,3\}$ indicating that $(29+17)=(23+23)=(41+5)=(43+3)=46$ and therefore all the prime pairs in the proper subset satisfies the statement of the Strong Goldbach' Conjecture.

Corollary 1

Let $p_1 and p_2 \in P$, where P is the set of all primes and d be the difference between $p_1 and p_2$ such that $d = p_2 - p_1 > 0$ where $p_2 > p_1$ and Let $z_i \in 1 \le O \le \frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)$ be the set of odd numbers for $i \in 1 \le O \le (\frac{1}{2}(p_1+p_2)+(p_2-p_1)^n)$, then any multiple of d in the range $\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n) < d < (p_1+p_2)+(p_2-p_1)^n$ can be used to generate the same set of pairs of odd numbers or a subset of pairs of odd numbers whose sum is $(p_1+p_2)+(p_2-p_1)^n$. For the set of values of d in the open interval $[\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n),(p_1+p_2)+(p_2-p_1)^n]$, it is expected that as the value of d gets closer to $(p_1+p_2)+(p_2-p_1)^n$, the pairs of odd numbers whose sum is $(p_1+p_2)+(p_2-p_1)^n$ reduces significantly[9].

Example 3

Step 1: Let the $Even\ number\ be = 46$ and $n=1$

Remark 3

The multiples of d in the range $\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n) < d < (p_1+p_2)+(p_2-p_1)^n$ are $24,26,28,30,32,34,36,38,40,42$ *and* 44. For example 3 we choose to use $d = 40$ to illustrate Corollary 1.

Step 2: $d = 40$

Step 3: Take odd numbers in the range $1 \le O \le \frac{1}{2}(46) \rightarrow (1,3,5,7,9,11,13,15,17,19,21,23)$

Then we partition 46 as follows:

I. $46-(40+1)=5$
II. $46-(40+3)=3$
III. $46-(40+5)=1$
IV. $46-(40+7)=-1$
V. $46-(40+9)=-3$
VI. $46-(40+11)=-5$
VII. $46-(40+13)=-7$
VIII. $46-(40+15)=-9$
IX. $46-(40+17)=-11$
X. $46-(40+19)=-13$
XI. $46-(40+21)=-15$
XII. $46-(40+23)=-17$

The positive partitions of 46 are therefore: $((40+1),5),((40+3),3),((40+5),11)$ $\Rightarrow$ The set A $=\{(41,5),(43,3),(45,1)\}$. We only obtain three positive pairs of odd numbers. Notice that the negative partitions are not considered in this case since the statement of the Strong Goldbach's Conjecture does not include negative prime numbers. The proper subset $B \subset A$ containing at least a pair of prime numbers is the set $B=\{(41,5),(43,3)\}$ indicating that $(41+5)=(43+3)=46$ and therefore satisfies the statement of the Strong Goldbach' Conjecture. From the three examples, it is evident that when an even number is partitioned into a set containing all pairs of odd numbers, then there exist a proper subset of the set containing all pairs of odd numbers containing at least a pair of prime numbers that satisfies the Strong Goldbach' Conjecture. What is now left is to show that this applies to all even numbers.

3. Construction of the proof

Let us consider two finite sets of prime numbers $P = \{p_1, p_2, \ldots, p_n\}$ and $Q = \{p_1, p_2, \ldots, p_m\}$ and the fact that according to Theorem 1, there are infinitely many prime numbers. Therefore, it is expected that for the finite set P, if p_n is the large prime number in the set, then, there is another larger prime number than p_n say $N = (p_1 * p_2 * p_3 \cdots * p_n) + 1$ and subsequently, there will always be another larger prime number than N say $K = (p_1 * p_2 * p_3 \cdots * p_n * \big((p_1 * p_2 * p_3 \cdots * pn + 1) + 1$ and so on. The same argument can also be used with the set Q so that the prime number $M = (p_1 * p_2 * p_3 * \cdots * p_m) + 1$ is a larger prime number than p_m and say the prime number $H = (p_1 * p_2 * p_3 * \cdots * p_m * \big((p_1 * p_2 * p_3 * \cdots * p_m) + 1\big)) + 1$ is larger than M and so on. This shows the possibility of obtaining two infinite sets of prime numbers $P^* = \{p_1, p_2, \ldots, p_n, N, K, \ldots\}$ and $Q^* = \{p_1, p_2, \ldots, p_m, M, H, \ldots\}$.

Theorem 2 confirms that the sum of any two odd numbers is even and since prime numbers greater than 2 are subsets of odd numbers, then, the addition of elements of set P^* to elements of set Q^*results to an even number. For the sake of illustration, let us consider the addition of elements of both infinite sets for values of prime numbers ≥ 3 (although 2 is a prime number, we exempt it from each of the infinite sets since the addition of 2 to any prime number results to an odd number and not an even number). Therefore, the elements in the infinite set P^* for example will begin from: 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97,..., pn, $(p_1 * p_2 * p_3 \cdots * p_n) + 1, (p_1 * p_2 * p_3 \cdots * p_n * \big((p_1 * p_2 * p_3 \cdots * p_n) + 1\big)) + 1$, and so on. We therefore create table 1 to illustrate the results of adding elements in set P^* to elements in set Q^* and show the result ($P^* + Q^*$) gives an even number as follows:

Table 1 Strong Goldbach Conjecture table (excluding 2)

Set P^*	**Set** Q^*	**Sum** (P^*+ Q^*)	**Cont' Set** P^*	**Cont' Set** Q^*	**Cont' Sum** (P^* + Q^*)
3	3	6	7	13	20
3	5	8	7	17	24
3	7	10	7	19	26
3	11	14	7	23	30
3	13	16	11	11	22
3	17	20	11	13	24
3	19	22	11	17	28
3	23	26	11	19	30
5	5	10	11	23	34
5	7	12	13	13	26
5	11	16	13	17	30
5	13	18	13	19	32
5	17	22	13	23	36
5	19	24	17	17	34
5	23	28	17	19	36
7	7	14	17	23	40
7	11	18	19	19	38
			.	.	.
			.	.	.
			.	.	.
			pn	pm	$2n_{nm}$
			$(p_1 * p_2 * p_3 * \cdots * p_n) + 1$	$(p_1 * p_2 * p_3 * \cdots * p_m) + 1$	$2n_{(N,M)}$
			$(p_1 * p_2 * p_3 \cdots * p_n * ((p_1 * p_2 * p_3 \cdots * p_n) + 1)) + 1$	$(p_1 * p_2 * p_3 * \cdots * p_m * ((p_1 * p_2 * p_3 * \cdots * p_m) + 1)) + 1$	$2n_{(K,H)}$
			.	.	.
			.	.	.
			.	.	.

It is straightforward from table 1 that the even numbers (P^* + Q^*) are results of addition of elements of set P^* to elements of set Q^*. As the process of adding the elements of both infinite sets continues, we eventually get to the addition of very large pairs of arbitrary prime numbers ($((p_1 * p_2 * p_3 * \cdots * p_n) + 1$), $((p_1 * p_2 * p_3 * \cdots * p_m) + 1$)) and (

$$(p_1 * p_2 * p_3 \cdots * p_n * ((p_1 * p_2 * p_3 \cdots * p_n) + 1)) + 1,$$

$(p_1 * p_2 * p_3 * \cdots * p_m * ((p_1 * p_2 * p_3 * \cdots * p_m) + 1)) + 1)$ whose sum is even. It is important to note that these larger pairs of prime numbers are chosen arbitrarily, and therefore the results $2n_{(N,M)}$ and $2n_{(K,H)}$ could be any arbitrary even numbers. The results in table 1 therefore, shows that the addition of any two arbitrarily chosen prime numbers give an even number and since the converse is true, it also implies that any arbitrarily chosen even number can be written as a sum of two prime numbers. For that reason, using the understating of the results in Table 1, the fact that any even number can be partitioned into all pairs of odd numbers and from this set of all pairs of odd numbers, there exist a proper subset of prime numbers, we provide a detailed proof of the Strong Goldbach's conjecture as below.

4. The Proof of the Strong Goldbach Conjecture

Theorem 3 (The Strong Goldbach Conjecture)

Every even integer greater than two is the sum of two prime numbers.

Proof

Using a new formulation of set of even numbers $(p_1 + p_2) + (p_2 - p_1)^n$ [7], it has been shown that it is possible to obtain all pairs of odd numbers whose sum is even as in example 1, example 2 and example 3. The set of all pairs of odd numbers { $(d+z_1, y_1), (d+z_3, y_3), (d+z_5, y_5), \ldots,$ $(d+z_{((\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)-1)}, y_{((\frac{1}{2}((p_1+p_2)+(p_2-p_1)^n)-1)})$ } [9] generated from partitioning the even number $(p_1 + p_2) + (p_2 - p_1)^n$ could be used to obtain two sets of odd numbers $\{d + z_i\}$ and $\{y_i\}$ such that $(p_1 + p_2) + (p_2 - p_1)^n = \{d + z_i\} + \{y_i\}$ for all pairs of odd numbers of the elements in both sets.

Further, it is also clearly seen from example 1, example 2 and example 3 that, from the two sets of odd numbers $\{d + z_i\}$ and $\{y_i\}$,there exist two proper subsets of prime numbers say $\{d + z_k\} \subset \{d + z_i\}$ and $\{y_k\} \subset \{y_i\}$ in such a way that each proper subset contains at least one prime number say $(d + z_j) \in \{d + z_k\} \subset \{d + z_i\}$ and $y_j \in \{y_k\} \subset \{y_i\}$ such that $(p_1 + p_2) + (p_2 - p_1)^n = (d + z_j) + y_j$. The question however is whether this is always the case for all even numbers. Well in order to ensure that all the even numbers are generated, the sums $(P^* + Q^*)$ in table 1 are calculated with all the possible combinations of the sum of all the elements in the infinite set P^* to the elements in the infinite set Q^*. The results in table 1therefore proves two things: 1) The addition of any two arbitrarily chosen odd numbers gives an even number, implying that the sum of any two arbitrarily chosen prime numbers greater than 2 will always give an even number which is also arbitrary since the set of all prime numbers excluding 2 is a proper subset of the set of all odd numbers, and 2) Any arbitrary even number can be written as a sum of two prime numbers since the converse of 1) is also true. It is therefore true to say that any arbitrary even number, however large it is, can always be written as a sum of two prime numbers. These results further guarantees that the two sets of odd numbers $\{d + z_i\}$ and $\{y_i\}$ obtained from partitioning any arbitrarily chosen even number $(p_1 + p_2) + (p_2 - p_1)^n$ contains two proper subsets of prime numbers $\{d + z_k\} \subset \{d + z_i\}$ and $\{y_k\} \subset \{y_i\}$ in such a way that each proper subset contains at least one prime number say $(d + z_j) \in \{d + z_k\} \subset \{d + z_i\}$ and $y_j \in \{y_k\} \subset \{y_i\}$ such that $(p_1 + p_2) + (p_2 - p_1)^n$ = $(d + z_j)$ + y_j .These results therefore proves the Strong Goldbach's Conjecture restated as every even integer greater than 6 can be expressed as the sum of two prime numbers. Since it is already known that $4 = 2 + 2$ and $6 = 3 + 3$, the proof presented here is considered a general proof of the Strong Goldbach's Conjecture.

5.Conclusion

Using the results in table 1, it is shown that two arbitrary sets of prime numbers when added together gives an even number. We also show that the set of all pairs of odd numbers obtained from partitioning an even number of the new formulation $(p_1 + p_2) + (p_2 - p_1)^n$, contains two proper subsets of prime numbers such that in each of these proper subsets of prime numbers there exist prime numbers whose sum equals $(p_1 + p_2) + (p_2 - p_1)^n$. Since the addition of any two arbitrary prime numbers gives an arbitrary even number and the converse has been shown to be true, then these results ensures that there exists at least a prime number in each of these two proper subsets of prime numbers such that the addition of the two prime numbers equals $(p_1 + p_2) + (p_2 - p_1)^n$. Based on these results, the Strong Goldbach's Conjecture is considered to be proven.

Data availability statement

The concepts discussed in this research article is purely abstract and all citations have been included appropriately.

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