Common Fixed Point of Contractive – Type Fuzzy self Mapping in real Banach Spaces

ABSTRACT

Fuzzy fixed point theorems for self mappings of contractive type in real Banach Spaces are taken into consideration on this paper. The out-flip hypothesize and increase the sequel because of Fisher and Gregus. Mapping which considers right here isn't always commuting and given a few examples to aid the final results of the work.

Keywords: Fuzzy, fixed point, Self mapping, Banach Space

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INTRODUCTION:

Many authors were accomplished in reading the affect of fixed point theorems self-mappings of a closed subset of a Banach area each at single-valued and multi-valued maps[1][4][5][9]. In assessment maximum of the programs do now no longer contain self-mapping of a closed set. A non-expansive mapping includes contraction mappings and is contained below all non-stop mappings. Some authors have proved a hard and fast factor theorem for non-expansive mappings on a closed, bounded and convex subset of a uniformly convex Banach area and in areas with richer structure [2],[3],[6][10]. In this paper, planned using fixed point theorems for self-mappings of Banach area with particular not unusual place constant factor.

Definition:[7],[8]

Let P and Q be two self-mappings of a fuzzy Banach space $\mathbb{X}$.The pair $\{P,Q\}$ to be weakly commuting if $\mathbb{N}(PQx-QPx,kt)\leq\mathbb{N}(Qx-Px,t)$, for all $x\in\mathbb{X},t>0$.

Let $\mathbb{X}$ is a Banach space and $\mathbb{C}$, a closed convex subset of $\mathbb{X}$.

Lemma 1:

Let P, Q be self-maps of $\mathbb{C}$ such that $\mathbb{N}(Px-x,kt)\leq\mathbb{N}(Py-y,t)$ ....(1)

If and only if $\mathbb{N}(Qx-x,kt)\leq\mathbb{N}(Qy-y,t)$ for all $x,y\in\mathbb{C}$.

Then $\inf\{\vee(\mathbb{N}(Px-x,kt),\mathbb{N}(Qx-x,kt)):x\in\mathbb{C}=\vee\{\inf(\mathbb{N}(Px-x,kt):x\in\mathbb{C},\ \inf(\mathbb{N}(Qx-x,kt):x\in\mathbb{C})\}$.

Proof:

If for any $x \in \mathbb{C}$,

Put $R(x) = \vee\{\mathbb{N}(Px - x, kt), \mathbb{N}(Qx - x, kt)\}$,

$m = \inf\{R(x): x \in \mathbb{C}\}$ and

$$p = \inf\{\mathbb{N}(Px - x, kt): x \in \mathbb{C}\},$$

$$q = \inf\{\mathbb{N}(Px - x, kt): x \in \mathbb{C}\}.$$

Since $\max\{p, q\} < B(x), x \in \mathbb{C}$,

$$\Longrightarrow \max\{s, t\} \leq b.$$

Suppose $\max\{p, q\} < b$.

Then there exist $x \in \mathbb{C}, y \in \mathbb{C}$ such that

$\mathbb{N}(Px - x, kt) < s + b - s = b,$ ....(2)

And $\mathbb{N}(Qy - y, kt) < t + b - t = b$.................(3)

$$\Longrightarrow B(x) = \mathbb{N}(Qx - x, kt), \qquad B(y) = \mathbb{N}(Qy - y, kt).$$

$$B(x) \geq b \ and \ B(y) \geq b,$$

From equation 2 & 3,

$$\Longrightarrow \mathbb{N}(Px - x, kt) < N(Qy - y, kt) and \ \mathbb{N}(Qy - y, kt) < N(Qx - x, kt),$$

This is contradiction to equation 1.

$$\Longrightarrow \max\{s, t\} = b.$$

Accordingly the end result follows, Contractive circumstance taken into consideration here's a mild variation of that studied through Hardy and Rogers.

MAIN RESULTS

Theorem 1:

Let P, Q be self-mappings of $\mathbb{C}$ satisfying Equation 1 and

$N(Px - Qy, kt) \leq a\, N(x - y, t) + b \vee \{N(Px - x, t), N(Py - y, t) + c \vee \{N(Px - x, t) + N(x - y, t), N(Qy - y, t) + N(x - y, t)\}$ ....(4)

For all $x, y \in \mathbb{C}$, a, b, c are such that $0 < a < 1, 0 < b < 1, 0 < c < 1$.

$a + b + 2c - 1$ and $4c(2 - b) < a(1 - a)$.

Then P and Q have a completely unique not unusual place fuzzy fixed point theorem , which is likewise a completely unique constant factor of each P and Q.

Proof:

Let $x \in \mathbb{C}$ be arbitrary. From equation 4, that

$$\mathbb{N}(PQx - Px, kt) \leq a\,\mathbb{N}(Qx - x, t) + b \bigvee .\{\mathbb{N}(PQx - Px, t), \mathbb{N}(Px - x, t)\} + c \bigvee .\ \{\mathbb{N}(PQx - Px, t) + \mathbb{N}(Px - x, t), \mathbb{N}(Px - x, t) + \mathbb{N}\mathbb{N}(Px - x, t)\},$$

$\Longrightarrow \mathbb{N}(PQx - Px, kt) \leq \mathbb{N}(Qx - x, t)$ ....(5)

Analogously, $\mathbb{N}(QPx - Px, kt) \leq \mathbb{N}(Px - x, t)$ ....(6)

Since equation 5 & 6 detail for any $x \in \mathbb{C}$.

Also acquire,

$\mathbb{N}(PQPx - QPx, kt) \leq \mathbb{N}(QPx - x, kt) \leq \mathbb{N}(Px - x, kt)$ and

$$\mathbb{N}(QPQx - xPQx, kt) \leq \mathbb{N}(PQx - Qx, kt) \leq (Px - x, kt)$$

Also in equation 1, yield

$\mathbb{N}(QQPx - QPx, kt) \leq \mathbb{N}(Px - x, t)$ .... (7)

And $\mathbb{N}(PPQx - QPx, kt) \leq \mathbb{N}(Px - x, t)$ ....(8)

prescribe a point z as

$z = \frac{1}{2}\ QPx + \frac{1}{2}\ QQPx$. ....(9)

From Equation 7 & 9,

$\Longrightarrow 2\mathbb{N}(QPx - z, kt) = 2\ \mathbb{N}\ (PQPx - x, kt) = \mathbb{N}(PPQx - xPQx, kt) \leq \mathbb{N}(Px - x, t)$..........(10)

$\mathbb{C} -$ convex, $z \in \mathbb{C}$ and using equation 4,6,7,& 10,

$\Longrightarrow 2\ N(Pz - z) \leq N(Pz - QPx) + N(Pz - QQPx, kt)$ ....(11)

$$\leq a\,N(Px - z, kt) + b \bigvee\{\ N(Pz - z, kt), N(QPx - Qx, kt)\} + c \bigvee\{\ N(Px - Px, kt), N(QPx - z, kt) + a\{N(z - QPx, kt)\} + b \bigvee\{\ N(Pz - z, kt), N(QQPx - QPx, kt) + c \bigvee\{\ N(Pz - QPx, kt), N(QQPx - z)\}$$

$$\leq a\ \mathbb{N}(Px - z, kt) + \mathbb{N}(z - QPx, kt) + 2b \bigvee\{\mathbb{N}(Pz - z, kt), \mathbb{N}(Px - x, kt), \mathbb{N}(Qx - x, kt)\} + 2c \bigvee\{\ \mathbb{N}(Pz - z, kt) + \mathbb{N}(Px - z, kt), \mathbb{N}(Pz - z, kt) + \mathbb{N}(z - QPx, kt)\}$$

Further, using equation 4,6,&7,

$\Longrightarrow 2\ \mathbb{N}(Px - z, kt) \leq \mathbb{N}(Px - QPx, kt) + \mathbb{N}(px - QQPx, kt)$ ....(12)

$$\leq \mathbb{N}(Px - x, kt) + a\,\mathbb{N}(x - QPx, kt) + b \bigvee\{\mathbb{N}(Px - x, kt), \mathbb{N}(QQPx - QPx, kt)$$
$$+ c \bigvee\{\mathbb{N}(QPx - Px, kt), \mathbb{N}(x - QQPx, kt)\}$$

$$\leq (2a + 1)\mathbb{N}(Px - x, kt) + b \bigvee\{B(x), \mathbb{N}(Pz - z, kt)\}$$

$$\leq (2 + a + 2c)B(x).$$

Equation 11 & 12 jointly that is,

$$2\mathbb{N}(Pz - z, kt) \leq a\left(\frac{3}{2} + \frac{a}{2} + c\right)B(x) + 2b \bigvee\{\mathbb{N}(Pz - z, kt) + \left(1 + \frac{a}{2} + c\right)B(x), \mathbb{N}(Pz - z, kt)$$
$$+ \frac{1}{2}\,B(x)\} \ldots \ldots \ldots \ldots .. (13)$$

Then $\mathbb{N}(Pz - z, kt) < B(x)$,

Otherwise equation 13 yield,

$$\mathbb{N}(Pz - z, kt) < \frac{1}{2}\left(3\frac{a}{2} + \frac{a^2}{2} + 2ac + 2c2 + 2b + 4c\right)\mathbb{N}(Pz - z, kt)$$

$$= \lambda\,\mathbb{N}(Pz - z, kt) < N(Pz - z, t)$$

Where $0 < \lambda$

$$= \frac{1}{2}\left(2 + \frac{a^2}{2} - \frac{a}{2} + 4c - 2bc\right) < 1,$$

By the conjecture on constants a, b, c.

$\Longrightarrow \mathbb{N}(Pz - z, kt) \leq \lambda\, B(x)$ ....(14)

Putting $h = \inf\{\mathbb{N}(Pz - z, kt): z = \frac{1}{2}\,QPx + \frac{1}{2}\,QQPx, x \in \mathbb{C}\}$

By virtue of the lemma 1 and from equation 14, we deduce that,

$$h \leq \lambda . b = \lambda . \max\{p, q\}.$$

Thus $h \leq \lambda . q$ ....(15)

Obviously $s \leq h$. ....(16)

Similarly by construe $z' = \frac{1}{2}PQx + \frac{1}{2}\,PPQx$ and using equation 8,

$\Longrightarrow 2\,\mathbb{N}\left(PQx - z'\right) = 2\,\mathbb{N}(PPQx - z')$ ....(17)

$$= \mathbb{N}(QQPx - QPx) < N(Px - x, t)$$

By setting:

$$K = \inf\left(\mathbb{N}\left(Qz' - z'\right): z' = \frac{1}{2}\,PQx + \frac{1}{2}\,QQPx, x \in \mathbb{C}\right\},$$

By handling equation 4, 5,8 &17,

We acquire the inequality:

$k \leq \lambda.p$ ....(18)

Resulting evidently

$k \geq q.$ ....(19)

Thus equation 15, 16,18,and 19 that,

$$p \leq h \leq \lambda.q \leq \lambda.k \leq \lambda^2.p$$

$p = 0$ because $0 < \lambda < 1$, and consequently $q = 0$, from equation 18 & 19,

So each of the sets $G\sigma$ and $H\sigma$ for every $\sigma > 0$ must be nonempty, where

$$G\sigma = \{x \in : \mathbb{N}(Px - x, t) \leq \sigma\}.$$

$$H\sigma = x \in : \ \mathbb{N}(Qx - x.t) < \sigma\}.$$

Further $diam\ G\sigma < (4 + c).\frac{\sigma}{b}$............................(20)

From equation 4 & 6, and for any $x, y \in G\sigma$,

We acquire,

$$\mathbb{N}(x - y, kt) \leq \mathbb{N}(x - Px, t) + \mathbb{N}(y - Py, t) + \mathbb{N}(Px - QPx, t) + \mathbb{N}(Py - QPx, t)$$

$$\leq 3\sigma + a\ \mathbb{N}(Px - x, t) + a\ \mathbb{N}(x - y, t) + b \bigvee\{\mathbb{N}(Py - y, t), \mathbb{N}(Px - x, t)\} + c \bigvee\{\mathbb{N}(y - Px, t) + \mathbb{N}(Px - QPx, t), \mathbb{N}(Px - y, t) + \mathbb{N}(Py - y, t)\}$$

$$\leq (3 + a + b)\sigma + a\mathbb{N}(x - y, t) + c\{(\mathbb{N}(x - y, t) + \mathbb{N}(x - Px, t) + \sigma)\}$$

$$\leq (4 + c)\sigma + (a + c)\mathbb{N}(x - y, t).$$

From the last inequality equation (20) follows, since $a + c = 1 - b$.

Let $H\sigma$ denote the closure of $H\sigma$ for any $\sigma > 0$, choose $x \in H\sigma$.

Arbitrary > 0, thre exists a point $y \in H\sigma$ such that $\mathbb{N}(x - y, t) \leq \epsilon$.

Applying equation 4,

$\Longrightarrow \mathbb{N}(Px - x, kt) \leq \mathbb{N}(Px - Qy, t) + \mathbb{N}(Qy - y, t) + \mathbb{N}(x - y, t)$ ....(21)

$$\leq a\mathbb{N}(x - y, t) + b \bigvee\{\mathbb{N}(Px - x, t), \mathbb{N}(Qy - y, t)\} + c \bigvee\{\mathbb{N}(Px - x, t) + \mathbb{N}(y - Qy, t), \mathbb{N}(x - y, t) + \mathbb{N}(Px - x, t)\} + \sigma + \varepsilon$$

$$\leq (1+a)\varepsilon + b\bigvee\{\mathbb{N}(Px-x,t),\sigma\} + c\bigvee\{\varepsilon+\sigma,\varepsilon+\mathbb{N}(Px-x,t)\}+\sigma\,.$$

If $\mathbb{N}(Px-x,t) \leq \sigma$, then $x \in G\tau \subset G\tau/a$ since $0 < a < 1$.

If $\mathbb{N}(Px-x,t) > \sigma$ infer from equation 21 that,

$$\mathbb{N}(Px-x,t) < (1+a+c)\varepsilon + (b+c)\mathbb{N}(Px-x,t) + \tau$$

$\Longrightarrow \mathbb{N}(Px-x,t) \leq \frac{\tau}{a}$, ε being arbitrary and $b+c = 1-a$,

$\Longrightarrow x \in G\frac{\tau}{a}$, that is $H\tau \subset \frac{G\tau}{a}$ in each case.

Let $\{\tau_n\}$ be a decreasing sequence of reals for which $\tau_{\{n\}} = \tau_n \to 0 \; as \; n \to \infty$.

So $\left\{H\tau_{\{n\}}\right\} \leq \frac{diamG\;\tau_{\{n\}}}{a} \leq \frac{(4+c)\tau_{\{n\}}}{ab}$.

Clearly, $diam\; H\tau_{\{n\}} \to 0$ as $n \to \infty$.

As $\mathbb{X}$ is complete, by the Cantor's intersection theorem,

There is a $w \in \mathbb{X}$ such that,

$$\{w\} = \bigcap_{n=1}^{\infty} H \quad \tau_{\{n\}} \subset \bigcap_{n=1}^{\infty} G\;\tau_{\{n\}}/a$$

$\Longrightarrow \mathbb{N}(Pw-w,t) \leq a.7/a$ for every $n = 1,2,3, \dots \dots$ and so $Sw = w$.

From equation 4 , acquire

$$\mathbb{N}(w-Qw,t) = \mathbb{N}(Pw-Qw,t)$$

$$\leq b\;\bigvee\{\mathbb{N}(Pw-w,t)\,,\mathbb{N}(Qw-w,t)\} + c\;\bigvee\{\mathbb{N}(w-Qw,t),\mathbb{N}(w-Pw,t)$$

$$= (1-a)\mathbb{N}(Qw-w,t).$$

$$\Longrightarrow Qw = w.$$

So w is a common fixed point of P and Q.

Let w' be another fixed point of P.

Then, applying equation 4 for $x = w$ and $y = w'$,

$$\Longrightarrow \mathbb{N}\big(w' - w, t\big) = \mathbb{N}(Rw' - Sw', t)$$

$$\leq a\, \mathbb{N}(w' - w, t)$$
$$+ b \bigvee \{ \mathbb{N}(Pw' - w', t), \mathbb{N}(Tw - w, t)$$
$$+ c \bigvee \{\mathbb{N}(w' - Sw, t), \mathbb{N}(Pw - w, t)\} - (1 - b)\mathbb{N}(w' - w, t).$$

$\Rightarrow w' = w.$

$\Rightarrow w$ is the unique fixed point of P.

Similarly one can show that w is the unique fixed point of Q.

⇒Complete the proof.

By theorem 1 for some iterates of Q and P.

We have the following.

Theorem 2:

Let $P, Q : \mathbb{C} \to \mathbb{C}$ satisfying $\mathbb{N}(x - R^m x, t) \leq \mathbb{N}(y - P^m y, t)$ if and only if $\mathbb{N}(x - Q' x, t) \leq \mathbb{N}(y - Q' y, t)$, and

$\mathbb{N}(P^m x - Q' y, t) < a\, \mathbb{N}(x - y, t) + b. \vee\{\mathbb{N}(R^m x - x, t), \mathbb{N}(Q' y - y, t)\} + c\ \vee\{\mathbb{N}(P^m x - y, t), \mathbb{N}(Q^m y - x, t)$ for all $x, y \in \mathbb{C}$.

Where l, m are positive integers and a, b, c are as in theorem 1. Then P and Q have a unique common fuzzy fixed point, which is also the unique fuzzy fixed point of both P and Q.

Proof :

By theorem 1, the maps $P^m : \mathbb{C} \to \mathbb{C}$ and $Q' : \mathbb{C} \to \mathbb{C}$ have a unique common fuzzy fixed point w.

Since $Pw = P(P^m w) = P^m(Pw)$, infer that Pw is also a fixed point of P^m.

Theorem 1, assures that w is also the unique fuzzy fixed point of P^m, necessarily have $Pw = w$.

Similarly, one can show that $Qw = w$.

So w is the unique common fuzzy fixed point of P and Q.

If w' is another fixed point of P, we have $P^m w' = w'$, but the uniqueness of w implies $w = w'$.

Therefore, w is also the unique fuzzy fixed point of P as well as for the map Q.

Example 1:

Let $\mathbb{X}$ be the Banach space of reals with Euclidean norm and $\mathbb{C} = [0,2]$. Define $P, Q: \mathbb{C} \to \mathbb{C}$ by putting, $P(x) = 0\ if\ 0 \leq x < 1, P(x) = \frac{3}{5} if\ 1 \leq x < 2, Q(x) = 0\ if\ 0 \leq x < 2, Q(x) = \frac{9}{5}$.

Then condition equation 4 of theorem 1 does not hold.

Otherwise taking $x = 1$ and $y = 2$.

We have:

$\mathbb{N}(P_1 - Q_2, t) = 6/5$

$$\leq a(2-1) + b \bigvee\left\{\left(1-\frac{3}{5}\right)\left(2-\frac{9}{5}\right)\right\} + c \bigvee\{(\frac{9}{5}-1),(2-\frac{3}{5})\}$$

$$= a + \frac{2}{5b} + 7/5c$$

$\leq \frac{3}{5}a + 2/5$+c.

By the assumptions of theorem 1,

$$\Rightarrow 4c < a(1-a).(2-b)^{-1} < 1/2,$$

$$\Rightarrow \frac{6}{5} \leq 1 + \frac{1}{8} = 9/8,$$

This is a contradiction.

However , theorem 2 is trivially satisfied for $1 = m = 2$,

Since $Q^2(x) = P^2(x) = 0$ for any $x \in \mathbb{C}$.

Remark 1:

By assuming $c = 0$ in theorem 1, we obtain the theorem of Fisher. The evidence exhibited inherently assumed the commutativity of the mappings below consideration, despite the fact that the writer does now no longer explicitly point out such hypothesis. However, you could drop this more requirement through enhancing the arguments of as indicated through the evidence of our theorem1.

Remark 2:

Assuming $P = S$ in theorem 1, we obtain a result more general than that of under a different set of conditions on the mapping Q.

Conclusion:

Thus, fuzzy fixed point theorem for self mapping of a convex subset in Banach area is analyzed. The mapping taken into consideration and analyzed isn't always commuting and features a completely unique not unusual place fuzzy constant factor. The instance located from the result.

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