Original Research Article

SomeCommon Fuzzy FixedPointTheorems onG-MetricSpace

ABSTRACT

The motive of this paper is to give a few not unusual place constant factor theorems in G-Metric spaces, with the aid of using the perception of Common restrict with inside the variety belongings and to illustrate appropriate examples. These outcomes make bigger and generalizes numerous widely known outcomes with inside the literature.

Keywords: Coincidence point, fuzzy set, Fixed point, G-Metric Space, CLRg Property,Weaklycompatiblemaps.

2020 AMS subject classifications: 54H25, 46S40, 47H10

I. INTRODUCTIONANDPRELIMINARIES

Inspired through the truth that metric fixed point idea has a huge utility in nearly all fields of quantitative sciences, many authors have directed their interest to generalize the belief of a metric space. In this respect, several generalized metric spaces have come thru through many authors, with inside the remaining decade. Among all of the generalized metric areas, the belief of G-Metric space has attracted substantial interest from fixed point theorists. The idea of a G-Metric space become brought through Mustafa and Sims in [3], in which the authors mentioned the topological residences of this area and proved the analog of the Banach contraction precept with inside the context of G-metric spaces. Following those outcomes, many authors have studied and advanced numerous not unusual place fixed point theorems on this framework. In 2002, M.Aamri and D.ElMoutawakil [12] brought the belongings (E.A), which is a real generalization of non-compatible maps in metric spaces. Under this belief many not unusual place constant factor theorems have been studied with inside the literature (see [1,2,4,9,10, 13] and the references therein). The idea of Common restriction with inside the variety of g (CLRg) belongings for a couple of self mappings in Fuzzy metric area [5,7,8,11, 14]. The significance of this belongings is, it guarantees that one does now no longer require the closeness of the variety subspaces and hence, now a days, authors are giving an awful lot interest to this belongings for generalizing the outcomes gift with inside the literature(see [11] and the references therein). Very recently, this become prolonged to 2 pairs of self mappings as CLR(S,T) belongings. In the existing paper, through using the notions of not unusual place restriction with inside the variety belongings for 2 in addition to 4 self maps and susceptible compatibility, which is a good device in offering the not unusual placefixed points, we derive a few not unusual placefixed point theorems with inside the realm of G-metric space, which generalizes diverse similar outcomes in [2]and others. At the equal time, we gift appropriate examples to showcase the application of the principle outcomes presented. The following are the primary definitions wanted with inside the primary outcomes.

Definition 1.1:

Let $\mathbb{X}$ be a non empty set and let $\mathcal{G}: \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$ be a function satisfying the following properties

(i) $\mathcal{G}(a, b, c) = 0$ if $a = b = c$.

(ii) $0 < \mathcal{G}(a,a,b)$ for all $a,b \in \mathbb{X}$ with $a \neq b$.
(iii) $\mathcal{G}(a,a,b) \leq \mathcal{G}(a,b,c)$ for all $a,b,c \in \mathbb{X}$ with $b \neq c$.
(iv) $\mathcal{G}(a,b,c) = \mathcal{G}(a,c,b) = \mathcal{G}(b,c,a) = \cdots \ldots \ldots$, symmetry all three variables.
(v) $\mathcal{G}(a,b,c) \leq \mathcal{G}(a,x,x) + \mathcal{G}(x,b,c)$ for all $a,b,c,x \in \mathbb{X}$.

Then the function $\mathcal{G}$ is called a generalized metric or a $\mathcal{G}$- metric on $\mathbb{X}$ and the pair $(\mathbb{X},\mathcal{G})$ is called a $\mathcal{G}$ – metric space.

Example 1.1: Let$(\mathbb{X},d)$betheusualmetricspaceThenthefunction

$\mathcal{G}: \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0,\infty)$definedby$\mathcal{G}(a,b,c) = max\{d(a,b), d(b,c), d(c,a)\}$

forall$a,b,c \in \mathbb{X}$,isa$\mathcal{G}$-Metricspace.

IntheirinitialpaperMustafaandSims[3]alsoprovedthefollowingproposition.

Proposition:Let $(\mathbb{X},\mathcal{G})$be a $\mathcal{G}$ -metric space.Then, for any $a,b,c,x,y,z \in \mathbb{X}$,itfollowsthat

1. if$\mathcal{G}$ $(a,b,c) = 0$, then$a = b = c$
2. $\mathcal{G}$ $(a,b,c) \leq \mathcal{G}$ $(a,a,b) + \mathcal{G}(a,a,c)$
3. $\mathcal{G}$ $(a,b,b) \leq 2\mathcal{G}$ (b,a,a).
4. $\mathcal{G}$ $(a,b,c) \leq \mathcal{G}$ $(a,x,c) + \mathcal{G}$ (x,b,c).
5. $\mathcal{G}(a,b,c) \leq \frac{2}{3}\{\mathcal{G}(a,b,x) + \mathcal{G}(a,x,c) + \mathcal{G}(x,b,c)\}$.
6. $\mathcal{G}(a,b,c) \leq \mathcal{G}(a,x,x) + \mathcal{G}(b,x,x) + \mathcal{G}(c,x,x)$.

Definition 1.2:

The G- metric space $(\mathbb{X},\mathcal{G})$ is called symmetry if $\mathcal{G}(a,a,b) = \mathcal{G}(a,b,b)$ for all $a,b \in \mathbb{X}$.

Definition 1.3:

A 3 – tuple $(\mathbb{X},\mathcal{G},*)$ is called a $\mathcal{G}$ – fuzzy metric space if $\mathbb{X}$ is an arbitrary non empty set, $*$ is a continuous t- norm and $\mathcal{G}$ is a Fuzzy set on $\mathbb{X}^3 \times (0,\infty)$ satisfying the following conditions for each $t,s > 0$.

(i) $\mathcal{G}(a,a,b,t) > 0$ for all $a,b \in \mathbb{X}$ with $a \neq b$.
(ii) $\mathcal{G}(a,a,b,t) \geq \mathcal{G}(a,b,c,t)$ for all $a,b,c \in \mathbb{X}$ with $b \neq c$.
(iii) $\mathcal{G}(a,b,c,t) = 1$ if and only if $a = b = c$.
(iv) $\mathcal{G}(a,b,c,t) = \mathcal{G}(p(a,b,c),t)$, where p is a permutation function.
(v) $\mathcal{G}(a,b,c,t+s) \geq \mathcal{G}(x,b,c,t) * \mathcal{G}(a,x,x,s)$ for all $a,b,c,x \in \mathbb{X}$.
(vi) $\mathcal{G}(a,b,c,.): (0,\infty) \to [0,1]$ is continuous.

Definition 1.4:

A $\mathcal{G}$- fuzzy metric space $(\mathbb{X},\mathcal{G},*)$ is said to be symmetric if $\mathcal{G}(a,a,b,t) = \mathcal{G}(a,b,b,t)$ for all $a,b \in \mathbb{X}$ and for each $t > 0$.

Definition 1.5:

A pair of self – mappings (g, h) of a fuzzy metric spaces $(\mathbb{X},\mathcal{M},*)$ is said to satisfy E.A property, if there exists a sequence $\{a_n\}$ in $\mathbb{X}$ such that $\lim_{n\to\infty} \mathcal{M}(fa_n, ga_n, t) = 1$ for some $t \in \mathbb{X}$.

Definition 1.6:

Let $g, h, \mathcal{K}, \mathcal{L}$ be four self mappings defined on a symmetric space $(\mathbb{X}, d)$. Then the pairs $(g, \mathcal{K})$ and $(h, \mathcal{L})$ are said to have the common limit range property (with respect to $\mathcal{K}$ and $\mathcal{L}$) often denoted by $CLR_{(\mathcal{K},\mathcal{L})}$ property, if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{X}$ such that

$lim_n g x_n = lim_n \mathcal{K} x_n = lim_n g y_n = lim_n \mathcal{L} x_n = t$ with

$t = \mathcal{K}u = \mathcal{L}w$ for some $t, u, w \in \mathbb{X}$.

If $g = h$ and $\mathcal{K} = \mathcal{L}$, then the above definition implies $CLR_{(\mathcal{K},\mathcal{L})}$ property due to Sintunavarat et al.

II. MainResult

The first end result is a not unusualplace fuzzy constant factor theorem for a couple of self mappings the usage of a generalized strict contractive condition, which expand Theorem 1 of [12].

Theorem 2.1:

Let $(\mathbb{X}, \mathcal{G})$ be a symmetric $\mathcal{G}$-Metric space and f,g be two weakly compatible self mappings on $\mathbb{X}$ satisfying

1. $CLR_{(\mathcal{K},\mathcal{L})}$ property.
2. $\mathcal{G}(ga, gb, gc, kt) < \max\{\mathcal{G}(ha, hb, hc), \frac{\mathcal{G}(ga,ha,ha,t)+\mathcal{G}(gb,hb,hb,t)+\mathcal{G}(gc,hc,hc,t)}{3}, \frac{\mathcal{G}(gb,ha,ha,t)+\mathcal{G}(gc,hb,hb,t)+\mathcal{G}(ga,hc,hc,t)}{3}\}$ For all $a, b, c \in \mathbb{X}$ with $a \neq b$.

Then g and h have a unique common fuzzy fixed point.

Proof:

Since g and h satisfies $\mathrm{CLR}_{(\mathcal{K},\mathcal{L})}$ property, there exists a sequence $\{x_n\}$ in $\mathbb{X}$ such that $lim_n g x_n = lim_n h x_n = h x$ for some $x \in \mathbb{X}$.

Consider

$\mathcal{G}(g x_n, ga, ga, kt) <$

$\max\{\mathcal{G}(h x_n, ha, ha), \frac{\mathcal{G}(g x_n, h x_n, h x_n, t)+\mathcal{G}(ga,ha,ha,t)+\mathcal{G}(ga,ha,ha,t)}{3}, \frac{\mathcal{G}(ga, h x_n, h x_n, t)+\mathcal{G}(ga,ha,ha,t)+\mathcal{G}(ga,hc,hc,t)}{3}\}$

Letting $n \to \infty$, we obtain $\mathcal{G}(ha, ga, ga) \leq \frac{2}{3}\mathcal{G}(ha, ga, ga)$ which implies

$ha = ga$.

Thus x is the coincidence point of g and h. Let $c = ga = ha$.

Since (g, h) are weakly compatible, we have $gc = ghc = hgc = hc$.

Now we will prove that $gc = c$.

Suppose $gc \neq c$, then

$$\mathcal{G}(gc, c, c, kt) = \mathcal{G}(gc, ga, ga, kt)$$
$$< max\{\mathcal{G}(gc, ga, ga, t), \frac{\mathcal{G}(ga, ha, ha, t) + \mathcal{G}(gb, hb, hb, t) + \mathcal{G}(gc, hc, hc, t)}{3}, \frac{\mathcal{G}(gb, ha, ha, t) + \mathcal{G}(gc, hb, hb, t) + \mathcal{G}(}{3}$$

$< \mathcal{G}(gc, c, c, t)$,whichisacontradiction.

Hence$gc = c = hc$.
Thus$c$isthecommonfixedpointof$g$and$h$.
Theuniquenessofthefixedpointcanbeprovedeasily.
We now illustrate this theorem by giving an example.

Example 2.1:

Let $\mathbb{X} = [2,20]$ and $\mathcal{G}: \mathbb{X} \times \mathbb{X} \times \mathbb{X} \times (0,\infty) \longrightarrow [0,\infty)$defined by$\mathcal{G}(a,b,c,t) = 0$ if $a = b = c$and $\mathcal{G}(a,b,c,t) = max\,\{a,b,c\}$ in all other cases.

Then $(\mathbb{X}, \mathcal{G}, *)$ is a symmetric $\mathcal{G}$-Metric space.

Let g, hbe two self maps on $\mathbb{X}$definedby,

$gx = 5 if x \leq 5, gx = 3$ if $x > 5$and

$hx = \frac{x+5}{2} if x \leq 5, hx = 10ifx > 5$.

Here$g$and$h$satisfiestheCLR_hproperty.
Toseethis,considerasequence

$$\{x_n\} = \left\{5 - \frac{1}{n}\right\} for\ all\ n.$$

Then $gx_n = g\left(5 - \frac{1}{n}\right) \longrightarrow 5$ and $hx_n = h\left(5 - \frac{1}{n}\right) = \frac{5-\frac{1}{n}+5}{2} \longrightarrow 5$
Therefore$lim_n gx_n = lim_n hx_n = 5 = h5$.

Further,(g, h)areweaklycompatibleand
$\mathcal{G}(ga, gb, gc, kt)$
$< max\{\mathcal{G}(ha, hb, hc, t), \frac{\mathcal{G}(ga, ha, ha, t) + \mathcal{G}(gb, hb, hb, t) + \mathcal{G}(gc, hc, hc, t)}{3}, \frac{\mathcal{G}(gb, ha, ha, t) + \mathcal{G}(gc, hb, hb, t) + \mathcal{G}}{3}$
Thus$g$ and$h$satisfyalltheconditionsofTheorem2.1andhaveaunique
Common fuzzyfixedpointat$x$=5.
In1977,Mathkowski[4]introducedtheΦ-mapasthefollowing:
Let Φ be the set of auxiliary functions φ such that $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is anon-decreasing function satisfying $lim_n \varphi^n(t) = 0$ for all $t \in (0,\infty)$. If $\varphi \in \Phi$, then φ is called a $\Phi -$map. further $\varphi(t) < t$ for all $t \in (0,\infty)$.
If $\varphi(0) = 0$.

Inthenextresult,weextractedauniquecommonfixedpointfortwopairsofselfmappingswhichinvolvesaφ-mapunderthelipschitztypeofcontractivecondition.This result extend and generalize Theorem 2 of M.Aamri and ElMoutawakil[12].

Theorem 2.2:

Let $(\mathbb{X}, \mathcal{G}, *)$be a symmetric fuzzy$\mathcal{G}$-Metric space and $g, h, \mathcal{K}, \mathcal{L}$ befourselfmappingsonXsuchthat

1. $(g, \mathcal{K})$and$(h, \mathcal{L})$satisfies$CLR_{(\mathcal{K},\mathcal{L})}$property.
2. $\mathcal{G}(ga, hb, hc, kt) \leq \varphi(max\{\mathcal{G}(\mathcal{K}a, \mathcal{L}b, \mathcal{L}c, t), \mathcal{G}(\mathcal{K}a, \mathcal{L}b, hc, t), \mathcal{G}(\mathcal{L}b, hb, hc, t), \mathcal{G}(hb, \mathcal{L}b, \mathcal{L}c, t)\}) \forall\ a, b, c \in \mathbb{X}$.
3. $(g, \mathcal{K})$and$(h, \mathcal{L})$areweaklycompatible.

Then$g, h, \mathcal{K} and\ \mathcal{L}$haveaunique fuzzycommonfixedpoint.

Proof:

Since $(g, \mathcal{K})$and$(h, \mathcal{L})$ satiesfies $CLR_{(\mathcal{K},\mathcal{L})}$ property, there exists two sequences

$\{x_n\}$ and $\{y_n\}$ such that $lim_n g x_n = lim_n \mathcal{L} x_n = lim_n h x_n = lim_n \mathcal{L} x_n = t$ with $t = \mathcal{K}x = \mathcal{L}y$ for some $t, x, y \in \mathbb{X}$.

Consider,

$\mathcal{G}(ga, hy_n, hy_n, kt) < \varphi(\max\{\mathcal{G}(\mathcal{K}a, \mathcal{L}y_n, \mathcal{L}y_n, t), \mathcal{G}(\mathcal{K}a, hy_n, hy_n, t), \mathcal{G}(\mathcal{L}y_n, hy_n, hy_n, t), \mathcal{G}(hy_n, \mathcal{L}y_n, \mathcal{L}y_n, t)\}\}$

On letting $n \to \infty$, we obtain $\mathcal{G}(gx, u, u, t) \leq \varphi(0) = 0$ which implies $gx = u = \mathcal{K}x$.

Hence x is the coincidence point of g and $\mathcal{K}$. Since $(g, \mathcal{K})$ are weakly compatible, we have $ggx = g\mathcal{K}x = \mathcal{K}gx = \mathcal{K}\mathcal{K}x$.

Now we prove that $\mathcal{L}b = hb$.

Consider

$$\mathcal{G}(gx, hy, hy, kt) < \varphi(\max\{\mathcal{G}(\mathcal{K}x_n, \mathcal{L}b, \mathcal{L}b, t), \mathcal{G}(\mathcal{K}a, hy_n, hy_n, t), \mathcal{G}(\mathcal{L}y_n, hy_n, hy_n, t), \mathcal{G}(hy_n, \mathcal{L}y_n, \mathcal{L}y_n, t)\})$$

On letting $n \to \infty$, we have

$\mathcal{G}(u, hb, hb, t) \leq \varphi(\mathcal{G}(u, hb, hb, t))$ which implies $\mathcal{G}(u, hb, hb, t) = 0$.

Therefore $hb = u = \mathcal{L}b$.that is b is the coincidence point of h and $\mathcal{L}$.

Since $(h, \mathcal{L})$ are weakly compatible, we prove that $hhb = h\mathcal{L}b = \mathcal{L}hb = \mathcal{L}\mathcal{L}b$.

Also note that $ga = \mathcal{K}a = hb = \mathcal{L}b = u$.

Now we prove that $gga = ga$.suppose $ga \neq gga$, then

$$\mathcal{G}(gga, ga, ga, kt) = \mathcal{G}(gga, hb, hb, t)$$

$$\leq \varphi(\max\{\mathcal{G}(\mathcal{K}ga, \mathcal{L}b, \mathcal{L}b, t), \mathcal{G}(\mathcal{K}ga, hb, hb, t), \mathcal{G}(\mathcal{K}b, hb, hb, t), \mathcal{G}(hb, \mathcal{L}b, \mathcal{L}b, t)\})$$

$< G(gga, ga, ga, t)$ a contradiction.

Hence $gga = ga = \mathcal{K}ga$, which implies ga is the common fuzzy fixed point of g and $\mathcal{K}$.

Similarly one can prove hb is the common fuzzy fixed point of g and $\mathcal{L}$.

Since $ga = hb, c = ga$ is the common fuzzy fixed point of $g, h, \mathcal{K}, and\ \mathcal{L}$.

The uniqueness of the fuzzy fixed point follows easily.

Corollary 2.1:

Let $(\mathbb{X}, \mathcal{G}, *)$be a symmetric fuzzy $\mathcal{G}$-Metric space and $g, h, \mathcal{K}, \mathcal{L}$ befourselfmappingsonXsuchthat

1. $(g, \mathcal{K})$and$(h, \mathcal{L})$satisfies$\mathrm{CLR}_{(\mathcal{K},\mathcal{L})}$property.
2. $\mathcal{G}(ga, hb, hc, kt) \leq \varphi(max\{\mathcal{G}(\mathcal{K}a, \mathcal{L}b, \mathcal{L}b, t), \mathcal{G}(\mathcal{K}a, hb, hb, t), \mathcal{G}(\mathcal{L}b, hb, hb, t), \mathcal{G}(hb, \mathcal{L}b, \mathcal{L}b, t)\}) \forall\ a, b, c \in \mathbb{X}$.
3. $(g, \mathcal{K})$and$(h, \mathcal{L})$areweaklycompatible.

Then$g, h, \mathcal{K} and\ \mathcal{L}$haveaunique fuzzycommonfixedpoint.

Proof:

By restricting $g, h, , \mathcal{K}, \mathcal{L}$ suitly, one can derive the corollaries involving two as well as three self mappings as follows:

Corollary 2.2:

Let $(\mathbb{X},\mathcal{G},*)$ be a symmetric $\mathcal{G}-$metric space and $\mathscr{g},\hbar,\mathcal{K}$ be three self mappings on $\mathbb{X}$ such that

1. $(\mathscr{g},\mathcal{K})and(\hbar,\mathcal{L})$satisfies$\mathrm{CLR}_{\mathcal{K}}$property.
2. $\mathcal{G}(\mathscr{g}a,\hbar b,\hbar c,kt) \le \varphi(max\{\mathcal{G}(\mathcal{K}a,\mathcal{K}b,\mathcal{K}c,t),\mathcal{G}(\mathcal{K}a,\hbar b,\hbar c,t),\mathcal{G}(\mathcal{L}b,\hbar b,\hbar c,t),$
$\mathcal{G}(\hbar b,\mathcal{L}b,\mathcal{L}c,t)\})\forall\ a,b,c \in \mathbb{X}.$
3. $(\mathscr{g},\mathcal{K})and(\hbar,\mathcal{K})$areweaklycompatible.

Then$\mathscr{g},\hbar$ *and* $\mathcal{K}$haveaunique fuzzycommonfixedpoint.

Proof:

Follows from theorem 2.2 by setting $\mathcal{K} = \mathcal{L}$.

We now exhibit this theorem through the subsequent example.

Example 2.2:

Let $\mathbb{X} = [0,6]$ and $\mathcal{G}:\mathbb{X}\times\mathbb{X}\times\mathbb{X}\times(0,\infty) \longrightarrow [0,\infty)$defined by$\mathcal{G}(a,b,c,t) = 0$ if $a = b = c$and $\mathcal{G}(a,b,c,t) = max\,\{a,b,c\}$ in all other cases.

Then clearly $(\mathbb{X},\mathcal{G},*)$ is a symmetric $\mathcal{G}$- Metric space.

Let $\mathscr{g},\hbar,\mathcal{K},\mathcal{L}$be fourself maps on $\mathbb{X}$definedby,

$\mathscr{g}a = 3 if a \le 3, \mathscr{g}a = 4$ if $a > 3$and

$\mathcal{K}a = 6 - a\ if\ a \le 3, \mathcal{K}a = 5\ if\ a > 3$,

$$\hbar a = 4\ if\ a < 3,$$

$$\mathcal{L}a = \frac{2a+3}{3} if x \ge 3$$

Now$(\mathscr{g},\mathcal{K})$*and* $(\hbar,\mathcal{L})$ satisfy the $\mathrm{CLR}_{(\mathcal{K},\mathcal{L})}$ property.

Toseethis,choose twosequence

$$\{x_n\} = \left\{3 - \frac{1}{n}\right\} for\ all\ n.$$

Then $\mathscr{g}x_n = \mathscr{g}\left(3 + \frac{1}{n}\right)$for all n.
Then $\mathscr{g}x_n = \mathscr{g}\left(3 - \frac{1}{n}\right) \longrightarrow 3$

$$\mathcal{K}x_n = \mathcal{K}\left(3 - \frac{1}{n}\right) = 6 - \left(3 - \frac{1}{n}\right) \longrightarrow 3$$

$\hbar x_n = \hbar\left(3 + \frac{1}{n}\right) = \frac{3+\frac{1}{n}+3}{2} \longrightarrow 3$ and $\mathcal{L}x_n = \mathcal{L}\left(3 + \frac{1}{n}\right) = \frac{2(3+\frac{1}{n})+3}{2} \longrightarrow 3.$

Therefore$lim_n\mathscr{g}x_n = lim_n\mathcal{K}x_n = lim_n\hbar x_n = lim_n\mathcal{L}x_n = t = 3$ with $3 = \mathcal{K}3 = \mathcal{L}3$.

Also $(\mathscr{g},)\mathcal{K}$ and $(\hbar,\mathcal{L})$areweaklycompatible.
Let $\varphi:[0,\infty) \to [0,\infty)$ be a function defined by $\varphi(u) = \frac{u}{2}, for\ all\ u \in [0,\infty)$.
Then $\varphi(0) = 0$ and $0 < \varphi(u) < u, for\ all\ u \in (0,\infty)$.
Further,
$\mathcal{G}(\mathscr{g}a,\hbar b,\hbar c,kt)$
$< \varphi(max\{\mathcal{G}(\mathcal{K}a,\mathcal{L}b,\mathcal{L}c,t), \frac{\mathcal{G}(\mathcal{K}a,\hbar b,\hbar c,t)+\mathcal{G}(\mathcal{L}b,\hbar b,\hbar c,t)+\mathcal{G}(\hbar c,\mathcal{L}b,\mathcal{L}c,t)}{3}, \frac{\mathcal{G}(\mathcal{L}b,\hbar b,\hbar c,t)+\mathcal{G}(\mathscr{g}c,\hbar b,\hbar b,t)+}{3}$

Thus all the conditions of theorem 2.2 are satisfied and $a = 3$ is the unique common fuzzy fixed point of $g, \hbar, \mathcal{K}\ and \mathcal{L}$.

Next two theorems involved with Hardy Roger's type of contractive condition for two pairs of self mappings, which extend the results contained in theorem 2.8[6], and theorem 2.2[6], theorem 3.11[9].

Theorem 2.3:

Let $(\mathbb{X}, \mathcal{G}, *)$be a fuzzy $\mathcal{G}$-Metric space and $g, \hbar, \mathcal{K}, \mathcal{L}$ befourselfmappingson$\mathbb{X}$suchthat

4. $(g, \mathcal{K})$and$(\hbar, \mathcal{L})$satisfies$\mathrm{CLR}_{(\mathcal{K},\mathcal{L})}$property.
5. $\mathcal{G}(ga, \hbar b, \hbar c, kt) \leq$ $p\mathcal{G}(ga, \mathcal{K}a, \mathcal{K}a, t) + q\mathcal{G}(\mathcal{K}a, \mathcal{L}b, \mathcal{L}b, t) + r\mathcal{G}(\mathcal{L}b, \hbar c, \hbar c, t) + w[\mathcal{G}(ga, \mathcal{L}b, \mathcal{L}c, t) + \mathcal{G}(\mathcal{K}a, \hbar b, \hbar c, t)] \forall\ a, b, c \in \mathbb{X}$

Where $p, q, r, w \in [0,1)$ satisfying $p + q + r + 2w < 1$.

Then $(g, \mathcal{K})$and$(\hbar, \mathcal{L})$ have unique point of coincidence in $\mathbb{X}$.

Moreover, $(g, \mathcal{K})$and$(\hbar, \mathcal{L})$ are areweaklycompatible, then

$g, \hbar, \mathcal{K} and\ \mathcal{L}$haveaunique fuzzycommonfixedpoint.

Proof:

Since $(g, \mathcal{K})$and$(\hbar, \mathcal{L})$ satiesfies $\mathrm{CLR}_{(\mathcal{K},\mathcal{L})}$ property, there exists two sequences

$\{x_n\}$ and $\{y_n\}$ such that $lim_n g x_n = lim_n \mathcal{L}x_n = lim_n \hbar x_n = lim_n \mathcal{L}x_n = t$ with $t = \mathcal{K}a = \mathcal{L}b$ for some $t, a, \in \mathbb{X}$.

Consider,

$$\mathcal{G}(ga, \hbar y_n, \hbar y_n, kt) < p\mathcal{G}(ga, \mathcal{K}a, \mathcal{K}a, t) + q\mathcal{G}(\mathcal{K}a, \mathcal{L}y_n, \mathcal{L}y_n, t) + r\mathcal{G}(\mathcal{L}y_n, \hbar y_n, \hbar y_n, t) + w[\mathcal{G}(ga, \mathcal{L}y_n, \mathcal{L}y_n, t) + \mathcal{G}(\mathcal{K}a, \hbar y_n, \hbar y_n, t)]$$

On letting $n \to \infty$,

we obtain $[1 - (p + w)]\mathcal{G}(ga, \ell, \ell, t) \leq 0$ which gives $ga = \ell = \mathcal{K}a$, since

$$p + q + r + 2w < 1.$$

Hence a is the coincidence point of g and $\mathcal{K}$. Similarly$b$ is the coincidence point of $\hbar$ and $\mathcal{L}$.

Thus $l = ga = \mathcal{K}a = \hbar b = \mathcal{L}b$.

Uniqueness of coincidence point:

Let l_1 and l_2 be two points of coincidence of $(g, \mathcal{K})$ and $(\hbar, \mathcal{L})$.

$l_1 = ga_1 = \mathcal{K}a_1 = \hbar b_1 = \mathcal{L}b_1$ and $l_2 = ga_2 = \mathcal{K}b_2 = \mathcal{L}b_2$.

Consider

$$\mathcal{G}(l_1, l_1, l_2, t) = \mathcal{G}(ga_1, \hbar b_2, \hbar b_2, t) \leq p\mathcal{G}(ga_1, \mathcal{K}a_1, \mathcal{K}a_1, t) + q\mathcal{G}(\mathcal{K}a_1, \mathcal{L}b_2, \mathcal{L}b_2, t) + r\mathcal{G}(\mathcal{L}b_2, \hbar b_2, \hbar b_2, t) + w[\mathcal{G}(ga_1, \mathcal{L}b_2, \hbar b_2, t) + \mathcal{G}(\mathcal{K}a_1, \hbar b_2, \hbar b_2, t)]$$

which implies$[1 - (q + 2w)]\mathcal{G}(l_1, l_1, l_2, t) \leq 0$

that is $l_1 = l_2$.

Since $(g, \mathcal{K})$ and $(h, \mathcal{L})$ are weakly compatible, we have

$gga = g\mathcal{K}a = \mathcal{K}ga = \mathcal{K}\mathcal{K}\mathcal{K}a$ and $hhb = h\mathcal{L}b = \mathcal{L}hb = \mathcal{L}\mathcal{L}b$.

Now we prove that $gga = ga$.

Consider,

$$\begin{aligned}\mathcal{G}(gga = ga = ga, t) = \mathcal{G}(gga, hb, hb, t) \\ &\leq p\mathcal{G}(gga, \mathcal{K}ga, \mathcal{K}ga, t) + q\mathcal{G}(\mathcal{K}ga, \mathcal{L}b, \mathcal{L}b, t) + r\mathcal{G}(\mathcal{L}b, hb, hb, t) + w[\mathcal{G}(gga, \mathcal{L}b, \mathcal{L}b, t) \\ &+ \mathcal{G}(\mathcal{K}ga, hb, hb, t)]\end{aligned}$$

Which gives $[1 - (q + 2w)]\mathcal{G}(gga, ga, ga, t) \leq 0$.

Hence $ga = gga = \mathcal{K}ga$.

That is ga is the common fuzzy fixed point of g and $\mathcal{K}$.

Similarly we can prove hb is the common fuzzy fixed point of h and $\mathcal{L}$.

Hence $c = gc$ is the common fuzzy fixed point of $g, h, \mathcal{K}$ and $\mathcal{L}$.

The uniqueness of the fuzzy fixed can be proved easily.

Example 2.3:

Let $\mathbb{X} = [0,4]$ and$\mathcal{G}: \mathbb{X} \times \mathbb{X} \times \mathbb{X} \times (0, \infty) \longrightarrow [0, \infty)$ defined by $\mathcal{G}(a, b, c, t) = \max\ \{d(a, b, t), d(b, c, t), d(c, a, t)\}$, where d is the usual metric on $\mathbb{X}$. Then clearly $(\mathbb{X}, \mathcal{G}, *)$ is a fuzzy $\mathcal{G}$ –fuzzy metric space.

Let $g, h, \mathcal{K}$,$\mathcal{L}$ be fourself maps on $\mathbb{X}$ defined by

$ga = 2, \mathcal{K}a = 4 - a, ha = \frac{5a+12}{11}$ and $\mathcal{L}a = \frac{a+2}{2}$ for all $a \in \mathbb{X}$.

Now $(g, \mathcal{K})$ *and* $(h, \mathcal{L})$ satisfy the $\mathrm{CLR}_{(\mathcal{K},\mathcal{L})}$ property.

Toseethis, choose twosequence

$\{x_n\} = \left\{2 + \frac{1}{n}\right\}$and $\{y_n\} = \left\{2 + \frac{1}{n}\right\}$ for all n.

Then $gx_n = g\left(2 + \frac{1}{n}\right) \longrightarrow 2, \mathcal{K}x_n = \mathcal{K}\left(2 + \frac{1}{n}\right) = 4 - \left(2 + \frac{1}{n}\right) \longrightarrow 2,$

$hy_n = h\left(2 - \frac{1}{n}\right) = \frac{5(2-\frac{1}{n})+12}{11} \longrightarrow 2.$

$\mathcal{L}y_n = \mathcal{L}\left(2 - \frac{1}{n}\right) = \frac{2-\frac{1}{n}+2}{2} \longrightarrow 2.$

Therefore$lim_n gx_n = lim_n \mathcal{K}x_n = lim_n hx_n = lim_n \mathcal{L}x_n = t = 2$ with $2 = \mathcal{K}2 = \mathcal{L}2$.

Further, $(g, \mathcal{K})$ and $(h, \mathcal{L})$ are weakly compatible and $g, h, \mathcal{K}$ and $\mathcal{L}$ satisfy the contractive condition equation (2) for $p = \frac{1}{8}, q = \frac{1}{3}, r = \frac{1}{8}, w = \frac{1}{7}$.

Thus all the conditions of theorem 2.5 are satisfied and $a = 2$ is the unique common fuzzy fixed point of $g, h, \mathcal{K}$ and $\mathcal{L}$.

Theorem 2.4:

Let $(\mathbb{X},\mathcal{G},*)$be a fuzzy $\mathcal{G}$-Metric space and $\mathcal{g},\hbar,\mathcal{K},\mathcal{L}$ befourselfmappingson$\mathbb{X}$suchthat

1. $(\mathcal{g},\mathcal{K})and(\hbar,\mathcal{L})$satisfies$\mathrm{CLR}_{(\mathcal{K},\mathcal{L})}$property.
2. $\mathcal{G}(\mathcal{g}a,\hbar b,\hbar c,kt) \leq hw(a,b,c,t)$ where $h \in (0,1)$ and for all $a,b,c \in \mathbb{X} u(a,b,c) \in \left\{\mathcal{G}(\mathcal{g}a,\mathcal{K}a,\mathcal{K}a,t),\mathcal{G}(\mathcal{K}a,\mathcal{L}b,\mathcal{L}b,t),\mathcal{G}(\mathcal{L}b,\hbar c,\hbar c,t),\frac{\mathcal{G}(\mathcal{g}a,\mathcal{L}a,\mathcal{L}a,t)+\mathcal{G}(\mathcal{K}a,\mathcal{L}b,\mathcal{L}c,t)}{2}\right\}$
3. $(\mathcal{g},\mathcal{K})and(\hbar,\mathcal{L})$ are areweaklycompatible.

Then $\mathcal{g},\hbar,\mathcal{K} and\ \mathcal{L}$haveaunique fuzzycommonfixedpoint.

Proof:

Since $(\mathcal{g},\mathcal{K})and(\hbar,\mathcal{L})$ satiesfies $\mathrm{CLR}_{(\mathcal{K},\mathcal{L})}$ property, there exists two sequences

$\{x_n\}$ and $\{y_n\}$ such that $lim_n \mathcal{g}x_n = lim_n \mathcal{K}x_n = lim_n \hbar y_n = lim_n \mathcal{L}y_n = t$ with $t = \mathcal{K}a = \mathcal{L}b$ for some $t,a,b \in \mathbb{X}$.

Consider,

$$\mathcal{G}(\mathcal{g}a,\hbar y_n,\hbar y_n,kt) \leq hw(a,\hbar y_n,\hbar y_n,t)$$

Where

$u(a,y_n,y_n,kt) \in$
$\left\{\mathcal{G}(\mathcal{g}a,\mathcal{K}a,\mathcal{K}a,t),\mathcal{G}(\mathcal{K}a,\mathcal{L}y_n,\mathcal{L}y_n,t),\mathcal{G}(\mathcal{L}y_n,\hbar y_n,\hbar y_n,t),\frac{\mathcal{G}(\mathcal{g}a,\mathcal{L}y_n,\mathcal{L}y_n,t)+\mathcal{G}(\mathcal{K}a,\hbar y_n,\hbar y_n,t)}{2}\right\}$

On letting $n \to \infty$,

$$\mathcal{G}(\mathcal{g}a,w,w,t) \leq hw(a,\hbar y_n,\hbar y_n,t)$$

Where $u(a,y_n,y_n,kt) \in \left\{\mathcal{G}(\mathcal{g}a,w,w,t),\frac{\mathcal{G}(\mathcal{g}a,w,w,t)}{2}\right\}$,

If $u(a,y_n,y_n,kt) = \mathcal{G}(\mathcal{g}a,w,w,t)$ then $\mathcal{G}(\mathcal{g}a,w,w,t) \leq h\mathcal{G}(\mathcal{g}a,w,w,t)$

Which also gives $\mathcal{g}a = w = \mathcal{K}a$.

If $u(a,y_n,y_n,kt) = \frac{\mathcal{G}(\mathcal{g}a,w,w,t)}{2}$ then $\mathcal{G}(\mathcal{g}a,w,w,t) \leq \frac{\mathcal{G}(\mathcal{g}a,w,w,t)}{2}$ which also gives $\mathcal{g}a = w = \mathcal{K}a$.

Therefore in both the cases $\mathcal{g}a = \mathcal{K}a = t$. hence a is the coincidence point of $\mathcal{g}$ and $\mathcal{K}$. similarly b is the coincidence point of $\hbar$ and $\mathcal{L}$. thus $w = \mathcal{g}a = \mathcal{K}a = \hbar b = \mathcal{L}b$.

Since $(\mathcal{g},\mathcal{K})and(\hbar,\mathcal{L})$ are areweaklycompatible,

We have$\mathcal{g}\mathcal{g}a = \mathcal{g}\mathcal{K}a = \mathcal{K}\mathcal{g}a = \mathcal{K}\mathcal{K}a$ and $\hbar\hbar b = \hbar\mathcal{L}b = \mathcal{L}\hbar b = \mathcal{L}\mathcal{L}b$.

Now prove that, $\mathcal{g}\mathcal{g}a = \mathcal{g}a$.

Consider $\mathcal{G}(\mathcal{g}\mathcal{g}a,\mathcal{g}a,\mathcal{g}a,t) = \mathcal{G}(\mathcal{g}\mathcal{g}a,\hbar b,\hbar b,t) \leq hu(\mathcal{g}a,b,b,t)$,

Where $u((\mathcal{g}a,b,b,kt) \in \{\mathcal{G}(\mathcal{g}\mathcal{g}a,\mathcal{K}\mathcal{g}a,\mathcal{K}\mathcal{g}a,t),\mathcal{G}(\mathcal{K}\mathcal{g}a,\mathcal{L}b,\mathcal{L}b,t),\mathcal{G}(\mathcal{L}b,\hbar b,\hbar b,t),\frac{\mathcal{G}(\mathcal{g}\mathcal{g}a,\mathcal{L}b,\mathcal{L}b,t)+\mathcal{G}(\mathcal{K}\mathcal{g}a,\hbar b,\hbar b,t)}{2}$

That is $u((\mathcal{g}a,b,b,kt) = \mathcal{G}(\mathcal{g}\mathcal{g}a,\mathcal{g}a,\mathcal{g}a,t)$.

Therefore $\mathcal{G}(\mathcal{g}\mathcal{g}a,\mathcal{g}a,\mathcal{g}a,t) \leq h\mathcal{G}(\mathcal{g}\mathcal{g}a,\mathcal{g}a,\mathcal{g}a,t)$this implies$\mathcal{g}a = \mathcal{g}\mathcal{g}a = \mathcal{K}\mathcal{g}a$.

Hence $\mathcal{g}a$ is the common fuzzy fixed point of $\mathcal{g}$ and $\mathcal{K}$. Similarly we can prove $\hbar b$ is the common fuzzy fixed point of $\hbar$ and $\mathcal{L}$.

Since $\mathcal{g}a = \hbar b, c = \mathcal{g}a$ is the common fuzzy fixed point of $\mathcal{g},\hbar,\mathcal{K}\ and\ \mathcal{L}$.

The uniqueness of the fuzzy fixed point can be proved easily.

Example 2.4:

Let $\mathbb{X} = [0,4]$ and $\mathcal{G}: \mathbb{X} \times \mathbb{X} \times \mathbb{X} \times (0,\infty) \longrightarrow [0,\infty)$ defined by $\mathcal{G}(a,b,c,t) = \max\{d(a,b,t), d(b,c,t), d(c,a,t)\}$, where d is the usual metric on $\mathbb{X}$. Then clearly $(\mathbb{X},\mathcal{G},*)$ is a fuzzy $\mathcal{G}$ –fuzzy metric space.

Let $g, h, \mathcal{K}, \mathcal{L}$ be fourself maps on $\mathbb{X}$ defined by

$$ga = 3, \mathcal{K}a = a, ha = \frac{a+24}{9} \text{ and } \mathcal{L}a = \frac{a+3}{2} \text{ for all } a \in \mathbb{X}.$$

Now $(g,\mathcal{K})$ *and* $(h,\mathcal{L})$ satisfy the $\mathrm{CLR}_{(\mathcal{K},\mathcal{L})}$ property.

Toseethis, choose twosequence

$\{x_n\} = \left\{3 + \frac{1}{n}\right\}$ and $\{y_n\} = \left\{3 - \frac{1}{n}\right\}$ for all n.

Then $gx_n = g\left(3 + \frac{1}{n}\right) \longrightarrow 3, \mathcal{K}x_n = \mathcal{K}\left(3 + \frac{1}{n}\right) = 3 + \frac{1}{n} \longrightarrow 3$

$hy_n = h\left(3 - \frac{1}{n}\right) = \frac{3 - \frac{1}{n} + 24}{9} \longrightarrow 3.$

$\mathcal{L}y_n = \mathcal{L}\left(3 - \frac{1}{n}\right) = \frac{3 - \frac{1}{n} + 3}{2} \longrightarrow 3.$

Therefore $lim_n gx_n = lim_n \mathcal{K}x_n = lim_n hx_n = lim_n \mathcal{L}x_n = t = 3$ with $3 = \mathcal{K}3 = \mathcal{L}3$.

Further, $(g,\mathcal{K})$ and $(h,\mathcal{L})$ are weakly compatible and $g, h, \mathcal{K}$ and $\mathcal{L}$ satisfy the contractive condition equation (2) for $h = \frac{1}{2}$.

Thus all the conditions of theorem 2.4 are satisfied and $a = 3$ is the unique common fuzzy fixed point of $g, h, \mathcal{K}$ and $\mathcal{L}$.

Our final end result is likewise a common fuzzy fixed factor theorem however expansive sort of contractive condition, which increase and enhance the effects theorem 3.1of [9].

Theorem 2.5:

Let g, h be two self maps of a fuzzy $\mathcal{G}$-Metric space $(\mathbb{X},\mathcal{G},*)$ satisfying CLR_h property and

$$\begin{aligned}\mathcal{G}(ha, hb, hc, kt) &\geq \mathcal{A}\mathcal{G}(ga, gb, gc, t), +\mathcal{B}\mathcal{G}(ga, ha, ha, t) + \mathcal{C}\mathcal{G}(gb, hb, hb, t) \\ &+ \mathcal{D}\mathcal{G}(gc, hc, hc, t)\})\forall\ a, b, c \in \mathbb{X}.\end{aligned}$$

Where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} > 0$. then g *and* h have a coincidence point. If $\mathcal{A} > 1$, then the coincidence point is unique. Moreover, if g *and* h are weakly compatible, then they have a unique common fuzzy fixed point.

Proof:

Since g and h satisfies CLR_h property, there exist a sequence $\{x_n\}$ in $\mathbb{X}$ such that $lim_n gx_n = lim_n hx_n = ha$ for some $a \in \mathbb{X}$.

Consider,

$\mathcal{G}(gx_n, ha, ha, kt) \geq \mathcal{A}\mathcal{G}(gx_n, ga, ga, t) + \mathcal{B}\mathcal{G}(gx_n, hx_n, hx_n, t) + \mathcal{C}\mathcal{G}(ga, ha, ha, t) + \mathcal{D}\mathcal{G}(ga, ha, ha, t)$

On letting $n \to \infty$,

We obtain $\left(\frac{\mathcal{A}}{2} + \mathcal{C} + \mathcal{D}\right)\mathcal{G}(ga, ha, haa, t) \leq 0$ which implies $ga = ha$.

Thus a is coincidence point of g *and* h.

Uniqueness of coincidence point:

Let $c = ga = ha$ and $w = gb = hb$ be the two points of coincidence of g and h.

Then, $\mathcal{G}(c, w, w, t) = \mathcal{G}(ha, hb, hb, t)$

$$\geq \mathcal{A}\mathcal{G}(ga, gb, gb, t), +\mathcal{B}\mathcal{G}(ga, ha, ha, t) + \mathcal{C}\mathcal{G}(gb, hb, hb, t) + \mathcal{D}\mathcal{G}(gb, hb, hb, t)$$

That is $(\mathcal{A} - 1)\mathcal{G}(c, w, w, t) \leq 0$.

Since $\mathcal{A} > 1$, we get $c = w$.

Now (g, h) are weakly compatible implies $gga = gha = hga = hha$.

To prove that $ga = gga$.

Consider $\mathcal{G}(gga, ga, ga, t) = \mathcal{G}(hga, ha, ha, t)$

$$\geq \mathcal{A}\mathcal{G}(gga, ga, ga, t), +\mathcal{B}\mathcal{G}(gga, hga, hga, t) + \mathcal{C}\mathcal{G}(ga, ha, ha, t) + \mathcal{D}\mathcal{G}(ga, ha, ha, t)$$

Which implies $gga = ga = hga$.

Hence ga is the common fuzzy fixed point of g and h.

The forte of the fuzzy fixed point may be proved easily.

References:

1) Krishnakumar.R , Dhamodharan. D; Fixed point theorems in normal cone metric space, International J. of Math. Sci. &Engg. Appls.(IJMSEA), ISSN 0973-9424, Vol. 10 No. III, 2016, pp. 213-224.

2) Giniswamy and Maheshwari P.G, Some Common Fixed Point Theorems on G-Metric Space, Gen. Math. Notes, Vol. 21, No 2, (2014) pp. 114-124.

3) Mustafa. Z and Sims. B, A New Approach to generalized metric Spaces; J. Nonlinear Convex Anal., 7(2006), pp. 289-297.

4) Mathkowski. J, Fixed Point Theorems for Mappings with a Contractive Iterate at a Point, Proc. Amer. Math. Soc., 62(1977), 344-348.

5) Senthil Kumar. P, Thiruveni. P; Estimation of fuzzy metric spaces from metric spaces, Ratio Mathematica, Volume 41, 2021, pp. 64-70.

6) Abbas. M, Rhoades. B.E and Nazir. T, Common fixed points for fourmapsinconemetricspaces,*Appl.Math.Comput.*,216(1)(2010),80-86.

7) Senthil kumar. P, Thiruveni. P , Dhamodharan. D; Some results on common fixed point in fuzzy 2-normed linear space, Advances And Applications In Mathematical Sciences, Volume 21, Issue 11, September 2022, Pages 6425-6435.

8) Senthil kumar. P, Thiruveni. P , Rostam. K Saeed; Fuzzy Fixed PointTheorems for $\boldsymbol{\gamma}$ – WeakContractive Mappings in 3 Metric Spaces, Journal of algebraic statistics, Volume 13, No. 2, 2022, p. 723-729SN: 1309-3452 .

9) Z. Kadelburg, P.P. Murthy and S. Radenovic, Common fixed points forexpansive mappings in cone metric spaces,*Int. J. Math. Anal.*,5(27)(2011),1309-1319.

10) J.G. Mehta and M.L. Joshi, On complete cone metric space and fixedpointtheorems,*J.Sci.Res.*,3(2)(2011),303-309.

11) Mohamed Ali. A and RanjithKanna. G, Intuitionistic Fuzzy Cone Metric Spaces and Fixed Point Theorems, International Journal of Mathematics And its Applications, Volume 5, Issue 1–A (2017), 25–36.

12) Aamri. M and Moutawakil. D.El., Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl., 270 (2002), 181-188.

13) Senthil Kumar. P and Thiruveni. P, Fuzzy Fixed Point Theorems in Normal Cone Metric Space, Asian Research journal of Mathematics, Vol. 19, Issue 10(2023), 57-66.

14) Senthil Kumar. P and Thiruveni. P, Novel Way of C^* – Algebra Valued 2- metric Space, Proceedings of the International Conference on Recent Innovations in Applications of Mathematics, 2023, 194 – 201.