Review Article

Norm-Attainable Operators and Polynomials: Theory, Characterization, and Applications in Optimization and Functional Analysis

Abstract

This research paper offers a comprehensive investigation into the concept of norm-attainability in Banach and Hilbert spaces. It establishes that norm-attainable operators exist if and only if the target space is a Banach space and that norm-attainable polynomials are inherently linear. In convex optimization scenarios, norm-attainable polynomials lead to unique global optima. The paper explores the norm of norm-attainable operators, revealing its connection to supremum norms. In Hilbert spaces, norm-attainable operators are self-adjoint. Additionally, it shows that in finite-dimensional spaces, all bounded linear operators are norm-attainable. The research also examines extremal polynomials and their relationship with derivative roots, characterizes optimal solutions in norm-attainable operator contexts, and explores equivalence between norm-attainable operators through invertible operators. In inner product spaces, norm-attainable polynomials are identified as constant. Lastly, it highlights the association between norm-attainable operators and convex optimization problems, where solutions lie on the unit ball's boundary. This paper offers a unified perspective with significant implications for functional analysis, operator theory, and optimization in various mathematical and scientific domains.

keywords{Norm-Attainable Operators, Optimization Landscape, Banach and Hilbert Spaces, Extremal Polynomials, Convex Optimization}

1 Introduction

This research paper delves into the intricate concept of norm-attainability within Banach and Hilbert spaces, investigating its implications in operator and polynomial theory as well as optimization. The paper establishes the conditions for the existence of norm-attainable operators, linking their presence to the completeness of the underlying space, and characterizes norm-attainable polynomials, emphasizing the distinction between linear and non-linear cases. It explores the optimization landscape, demonstrating the uniqueness of global optima in convex optimization problems with norm-attainable objectives. Additionally, the paper provides insights into the norm of norm-attainable operators, their self-adjointness in Hilbert spaces, and their prevalence in finite-dimensional spaces. It also uncovers connections between norm-attainable polynomials and extremal points, optimal solutions, and norm-equivalence of operators. This comprehensive study aims to deepen our understanding of the interplay between functional analysis and optimization, offering a valuable contribution to the field.

2 Preliminaries

Introduction

In mathematical analysis and functional analysis, the concept of norm-attainable operators and norm-attainable polynomials plays a fundamental role in understanding the structure and properties of Banach and Hilbert spaces. These operators and polynomials have significant implications in various mathematical disciplines and have applications in optimization, functional analysis, and convex geometry. In this section, we provide an overview of the key concepts and results that form the foundation of this research.

Banach and Hilbert Spaces

A Banach space is a complete normed vector space, equipped with a norm that allows us to measure the size or distance of vectors. On the other hand, a Hilbert space is a special type of Banach space that is equipped with an inner product, which introduces the concept of orthogonality and allows for a notion of angles and lengths of vectors. Both Banach and Hilbert spaces provide the mathematical framework for our investigation into norm-attainable operators and polynomials.

Norm-Attainable Operators

The existence and properties of norm-attainable operators are central to our research. In this context, a norm-attainable operator from a Banach space X to a Banach space Y is an operator that attains its supremum norm. We establish the necessary and sufficient conditions for the existence of such operators and explore their significance in the context of optimization and functional analysis.

Norm-Attainable Polynomials

We delve into the characterization of norm-attainable polynomials defined on normed vector spaces. Specifically, we examine the conditions under which a polynomial function is norm-attainable, shedding light on the relationship between norm-attainability and linearity. Additionally, we explore the extremal properties of norm-attainable polynomials and their optimization landscape.

Optimization and Convexity

In optimization problems over Banach spaces, we investigate the interplay between the attainability of the objective function's norm and the convexity of the feasible set. We establish the existence and uniqueness of global optima in this context, offering insights into the role of norm-attainability in convex optimization.

Norm-Equivalence and Inner Product Spaces

Norm-attainable operators in Hilbert spaces are of particular interest, and we provide a characterization of these operators in terms of self-adjointness. Moreover, we examine the behavior of norm-attainable polynomials in inner product spaces, highlighting the role of constant polynomials in this setting.

Corollaries and Implications

Finally, we discuss the corollaries and implications of our results, including their relevance to convex optimization and the boundary behavior of solutions within the unit ball of a normed vector space. This preliminary section sets the stage for the comprehensive exploration of norm-attainability and its applications in the subsequent sections of this research paper.

3 Methodology

The methodology employed to prove the results outlined below is rooted in rigorous mathematical reasoning and logical deduction. It commences with precise problem definitions and assumptions, followed by the application of mathematical techniques such as inequalities, algebraic manipulations, and analyses of special cases. These proofs often leverage the definitions of mathematical concepts and employ strategies like proof by contradiction, contrapositive reasoning, limit analysis, and proof by construction. Additionally, the structural properties of operators and polynomials are extensively analyzed. Uniqueness is demonstrated when applicable, while counterexamples may be used to illustrate cases where statements do not hold. The proofs conclude by succinctly summarizing the key findings, solidifying the validity of the theorems and propositions through a meticulous combination of mathematical rigor and logical deduction.

4 Results and Discussions

Proposition 1. *There exists a norm-attainable operator $T : X \to Y$ in a Banach space X if and only if Y is also a Banach space.*

Proof. We will prove the proposition in two parts:
Part 1: If there exists a norm-attainable operator $T : X \to Y$, then Y is a Banach space. Let $T : X \to Y$ be a norm-attainable operator. This means there exists $x_0 \in X$ such that $\|T(x_0)\| = \|T\|\|x_0\|$, where $\|T\|$ is the operator norm of T. Now, consider a Cauchy sequence $\{y_n\}$ in Y. We want to show that $\{y_n\}$ converges to some $y \in Y$, which will prove that Y is complete. Since $\{y_n\}$ is Cauchy, for any $\epsilon > 0$, there exists N such that for all $m, n \geq N$, we have $\|y_m - y_n\| < \epsilon$. Consider the sequence $\{x_n\}$ in X defined by $x_n = x_0$ for all n. Now, we have:

$$\begin{aligned} \|T(x_m) - T(x_n)\| &= \|T(x_0) - T(x_0)\| \quad (\text{since } x_n = x_0) \\ &= \|0\| \\ &= 0. \end{aligned}$$

This shows that $\{T(x_n)\}$ is a Cauchy sequence in Y. Since Y is a normed space, it is also a metric space, and in metric spaces, Cauchy sequences converge. Therefore, there exists $y \in Y$ such that $\lim_{n\to\infty} T(x_n) = y$. Now, we can use the continuity of T to show that $\lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} x_n)$. Since $\lim_{n\to\infty} x_n = x_0$, we have $T(x_0) = y$. This shows that every Cauchy sequence in Y converges to some point in Y, implying that Y is complete, which is the definition of a Banach space.
Part 2: If Y is a Banach space, then there exists a norm-attainable operator $T : X \to Y$. We will construct an example of such an operator. Let X be any normed space, and let $T : X \to Y$ be the identity operator, i.e., $T(x) = x$ for all $x \in X$. Since Y is a Banach space, every Cauchy sequence in Y converges to a point in Y. Since T is the identity operator, it preserves Cauchy sequences. Therefore, if $\{y_n\}$ is a Cauchy sequence in Y, then $\{T(y_n)\}$ is also a Cauchy sequence in Y, and it converges to $y \in Y$. This implies that $\|T(y_n) - y\| \to 0$ as $n \to \infty$. But since $T(y_n) = y_n$ (because T is the identity operator), we have $\|y_n - y\| \to 0$, which means that $\{y_n\}$ converges to y. Thus, we have shown that $T : X \to Y$ is a norm-attainable operator, as it attains the norm of each element in Y. Therefore, we have proved both directions of the proposition, and the proof is complete. □

Proposition 2. *A polynomial function $p(x)$ defined on a normed vector space X is norm-attainable if and only if it is a linear polynomial.*

Proof. Let $p(x)$ be a polynomial function defined on a normed vector space X. We will prove the proposition by considering both directions.
Direction 1: ($\Rightarrow$) If $p(x)$ is norm-attainable, then it must be a linear polynomial. Assume that $p(x)$ is norm-attainable, which means there exists $x_0 \in X$ such that $\|p(x_0)\| = \|p\|$. Since $\|p\|$ is finite, $p(x_0)$ must also be finite. Now, let's consider the degree of $p(x)$. If $p(x)$ is not a linear polynomial, then it has a degree greater than 1. Without loss of generality, we can write $p(x)$ as:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where $n > 1$ and $a_n \neq 0$. Now, let's consider the norm of $p(x_0)$:

$$\begin{aligned}
\|p(x_0)\| &= \|a_n x_0^n + a_{n-1} x_0^{n-1} + \ldots + a_1 x_0 + a_0\| \\
&\geq \|a_n x_0^n\| - \|a_{n-1} x_0^{n-1} + \ldots + a_1 x_0 + a_0\| \\
&= |a_n| \|x_0\|^n - \|a_{n-1} x_0^{n-1} + \ldots + a_1 x_0 + a_0\| \\
&> |a_n| \|x_0\|^n - (|a_{n-1}| \|x_0\|^{n-1} + \ldots + |a_1| \|x_0\| + |a_0|) \\
&= |a_n| \|x_0\|^n - |a_n| \|x_0\|^n \\
&= 0
\end{aligned}$$

The above inequality follows from the triangle inequality and the fact that $\|x_0\| > 0$ (since x_0 is a nonzero vector). However, this leads to a contradiction because we assumed that $\|p(x_0)\| = \|p\|$, but we have shown that $\|p(x_0)\| > 0$. Therefore, our assumption that $p(x)$ is not a linear polynomial must be false, and $p(x)$ must be a linear polynomial.
Direction 2: ($\Leftarrow$) If $p(x)$ is a linear polynomial, it is trivially norm-attainable. If $p(x)$ is a linear polynomial, it has the form $p(x) = ax + b$, where a and b are constants. For any $x_0 \in X$, we have:

$$\begin{aligned}
\|p(x_0)\| &= \|a(x_0) + b\| \\
&= |a| \|x_0\| + \|b\| \\
&\leq |a| \|x_0\| + |b| \\
&\leq (|a| + |b|) \|x_0\|
\end{aligned}$$

So, $\|p(x_0)\|$ is bounded by a constant multiple of $\|x_0\|$. Therefore, $p(x)$ is norm-attainable. Since we have shown both directions of the proposition, we conclude that a polynomial function $p(x)$ defined on a normed vector space X is norm-attainable if and only if it is a linear polynomial. □

Proposition 3. *In an optimization problem over a Banach space X, if the objective function is a norm-attainable polynomial and the feasible set is convex, then the global optimum exists and can be attained at a unique point.*

Proof. Let $f : X \to \mathbb{R}$ be the objective function, and let C be the convex feasible set in the Banach space X. We aim to prove that f has a unique global optimum in C.
Existence of Optimum: Since f is a norm-attainable polynomial, there exists

a point $x^* \in X$ such that $f(x^*) = \|f\|_\infty = \sup_{x \in X} |f(x)|$. By definition of the supremum, for any $\epsilon > 0$, there exists an $x_\epsilon \in X$ such that

$$f(x_\epsilon) > \|f\|_\infty - \epsilon.$$

Now, consider the sequence of values $\{f(x_\epsilon)\}_{\epsilon>0}$. Since C is a convex set, for each ϵ, there exists a corresponding point $x_\epsilon \in C$. Since f is continuous, the sequence $\{f(x_\epsilon)\}_{\epsilon>0}$ converges to $f(x^*)$. Therefore, there exists a sequence of points $\{x_\epsilon\}_{\epsilon>0}$ in C such that $\lim_{\epsilon \to 0} f(x_\epsilon) = f(x^*)$. Since C is a compact set (in a finite-dimensional Banach space), the Bolzano-Weierstrass theorem guarantees that any sequence in C has a convergent subsequence. Therefore, we can extract a subsequence $\{x_{\epsilon_k}\}_{k=1}^\infty$ that converges to some point $x_0 \in C$. By the continuity of f, we have

$$\lim_{k \to \infty} f(x_{\epsilon_k}) = f(x_0).$$

Combining the above limits, we obtain $f(x^*) = f(x_0)$, which implies that x_0 is a global optimum of the objective function f on the feasible set C.
Uniqueness of Optimum: Suppose, for the sake of contradiction, that there exist two distinct points x_1 and x_2 in C such that $f(x_1) = f(x_2) = f(x^*)$. Since C is convex, the line segment joining x_1 and x_2 lies entirely in C. By the continuity of f, the intermediate value theorem implies that for any point x on the line segment between x_1 and x_2, there exists a point c_x on the line segment such that $f(c_x) = f(x)$. Consider the set of all such points c_x for x in the line segment. Since $f(x_1) = f(x_2) = f(x^*)$, this set includes x_1, x_2, and x^*. However, this contradicts the uniqueness of x^* as the point where f attains its supremum. Therefore, there can be no two distinct points in C where f attains the same value. Thus, x_0 is the unique global optimum of the optimization problem over the convex set C. Hence, we have shown both the existence and uniqueness of the global optimum of the objective function f over the convex feasible set C. □

Theorem 1. *Let $T : X \to Y$ be a norm-attainable operator. Then,* $|T| = \sup_{|x|=1} |T(x)|$.

Proof. Let $T : X \to Y$ be a norm-attainable operator. By definition, there exists an $x_0 \in X$ such that $\|T\| = \|T(x_0)\|$. We want to show that $\|T\| = \sup_{|x|=1} |T(x)|$. First, note that for any nonzero vector $x \in X$, we can define the unit vector $\hat{x} = \frac{x}{\|x\|}$. Thus, for any $x \in X$ with $\|x\| \neq 0$, we have $|T(x)| = \|T(\hat{x})\|$, as multiplying a vector by a scalar does not change its norm. Now, consider the following inequality for any $x \in X$:

$$\begin{aligned} |T(x)| = \|T(\hat{x})\| &\leq \|T\| \cdot \|\hat{x}\| \quad \text{(by the definition of operator norm)} \\ &= \|T\| \cdot 1 = \|T\|. \end{aligned}$$

This shows that $|T(x)| \leq \|T\|$ for all x with $\|x\| \neq 0$. To complete the proof, we need to show that there exists an x with $\|x\| = 1$ such that $|T(x)| = \|T\|$.

Consider x_0, the vector for which $\|T\| = \|T(x_0)\|$. Now, let $x = \frac{x_0}{\|x_0\|}$, which is a unit vector since $\|x_0\| \neq 0$. We have:

$$|T(x)| = \|T\left(\frac{x_0}{\|x_0\|}\right)\| = \frac{1}{\|x_0\|}\|T(x_0)\| = \frac{1}{\|x_0\|}\|T\|\|x_0\| = \|T\|.$$

Thus, we've shown that $|T(x)| = \|T\|$ for some unit vector x, which implies that $\|T\| = \sup_{|x|=1} |T(x)|$. This completes the proof. □

Theorem 2. *In a Hilbert space H, an operator $T : H \to H$ is norm-attainable if and only if it is self-adjoint.*

Proof. (⇒) Suppose T is norm-attainable. This means there exists $x_0 \in H$ such that $\|Tx_0\| = \|T\|$. By definition of norm-attainability, x_0 achieves the operator norm, i.e., $\|Tx_0\| = \|T\|$. Now, let's consider the adjoint operator T^*. Using the properties of the adjoint, we have:

$$\begin{aligned}
\|Tx_0\|^2 &= \langle Tx_0, Tx_0\rangle \\
&= \langle T^*Tx_0, x_0\rangle \\
&= \langle TT^*x_0, x_0\rangle \quad \text{(since } T = T^*\text{)} \\
&= \|TT^*x_0\|\|x_0\| \quad \text{(by the Cauchy-Schwarz inequality)} \\
&= \|T\|^2\|x_0\|^2 \quad \text{(by definition of the operator norm)}
\end{aligned}$$

Since $\|Tx_0\| = \|T\|$, we have:

$$\begin{aligned}
&\|Tx_0\|^2 = \|T\|^2\|x_0\|^2 \\
&\Rightarrow \|x_0\| = \|x_0\|^2
\end{aligned}$$

This implies that $\|x_0\| = 1$, which means x_0 is a unit vector. Now, let's consider $\langle Tx_0, x_0\rangle$. Since x_0 is a unit vector, we have:

$$\begin{aligned}
\|Tx_0\|^2 &= \langle Tx_0, Tx_0\rangle \\
&= \langle TT^*x_0, x_0\rangle \\
&= \langle T^*Tx_0, x_0\rangle \quad \text{(since } T = T^*\text{)} \\
&= \langle Tx_0, T^*x_0\rangle \quad \text{(since } \langle\cdot,\cdot\rangle \text{ is conjugate linear in the second argument)} \\
&= \overline{\langle T^*x_0, Tx_0\rangle} \quad \text{(taking complex conjugate)} \\
&= \overline{\langle Tx_0, Tx_0\rangle} \quad \text{(since } T = T^*\text{)} \\
&= \langle Tx_0, Tx_0\rangle
\end{aligned}$$

Hence, we have:

$$\begin{aligned}\langle Tx_0, x_0\rangle &= \frac{1}{2}(\|Tx_0\|^2 + \|Tx_0\|^2 - \|Tx_0\|^2) \\ &= \frac{1}{2}(\|Tx_0\|^2 - \|Tx_0\|^2) \\ &= 0\end{aligned}$$

This shows that T is self-adjoint.
($\Leftarrow$) Conversely, suppose T is self-adjoint. We want to show that it is norm-attainable. Let $\lambda = \|T\|$, and let x_0 be the unit eigenvector corresponding to the eigenvalue λ. Since T is self-adjoint, all its eigenvalues are real, and we can choose a unit eigenvector x_0 such that $Tx_0 = \lambda x_0$. Then, we have:

$$\|Tx_0\| = \|\lambda x_0\| = |\lambda|\|x_0\| = \lambda \cdot 1 = \lambda = \|T\|.$$

This shows that T is norm-attainable. Therefore, we have shown both directions: if T is norm-attainable, then it is self-adjoint, and if T is self-adjoint, then it is norm-attainable. □

Theorem 3. *In a finite-dimensional normed vector space X, every bounded linear operator is norm-attainable.*

Proof. Let X be a finite-dimensional normed vector space, and let $T : X \to Y$ be a bounded linear operator, where Y is another normed vector space. We aim to show that T is norm-attainable. Since X is finite-dimensional, we can choose a basis $\{e_1, e_2, \ldots, e_n\}$ for X. Let $x \in X$ be any vector, and express x in terms of this basis as $x = \sum_{i=1}^{n} a_i e_i$, where $a_i \in \mathbb{R}$. Now, consider the operator T applied to x:

$$T(x) = T\left(\sum_{i=1}^{n} a_i e_i\right) = \sum_{i=1}^{n} a_i T(e_i).$$

Since T is a bounded linear operator, there exists a constant $M > 0$ such that $\|T(e_i)\| \leq M\|e_i\|$ for all i. Therefore, we have:

$$\|T(x)\| = \left\|\sum_{i=1}^{n} a_i T(e_i)\right\| \leq \sum_{i=1}^{n} |a_i|\|T(e_i)\| \leq M \sum_{i=1}^{n} |a_i|\|e_i\|.$$

Now, let $C = \sum_{i=1}^{n} |a_i|\|e_i\|$. Since C is a finite constant that depends only on x, we can express the above inequality as:

$$\|T(x)\| \leq MC.$$

This shows that for any $x \in X$, $\|T(x)\|$ is bounded by a constant MC, which means that T is a bounded operator. Therefore, we have shown that for any $x \in X$, there exists a constant C such that $\|T(x)\| \leq C\|x\|$. Thus, T is norm-attainable, as we can choose C as the norm-attainment constant. □

Theorem 4. *Let $p(x)$ be a polynomial of degree n defined on a normed vector space X. If $p(x)$ is norm-attainable, then its extrema are achieved at the roots of its derivative, i.e., $p'(x_0) = 0$.*

Proof. Let $p(x)$ be a norm-attainable polynomial of degree n defined on a normed vector space X. We aim to show that the extrema of $p(x)$ are achieved at the roots of its derivative, i.e., $p'(x_0) = 0$. First, suppose that $p(x)$ attains its maximum at $x = x_0$. By the definition of a maximum, we have:

$$p(x) \leq p(x_0) \quad \text{for all } x \in X.$$

Consider the function $q(t) = p(x_0+th)$ for $t \in \mathbb{R}$ and some nonzero vector $h \in X$. The function $q(t)$ is a real-valued function of a real variable, and $q(0) = p(x_0)$. Now, we differentiate $q(t)$ with respect to t and evaluate it at $t = 0$ to find the derivative of p at x_0 in the direction of h:

$$q'(0) = \lim_{t\to 0} \frac{q(t) - q(0)}{t} = \lim_{t\to 0} \frac{p(x_0 + th) - p(x_0)}{t}.$$

By the definition of the derivative, this limit exists and represents the directional derivative of p at x_0 in the direction of h. Since p attains its maximum at x_0, we have $q(t) \leq p(x_0)$ for all t, and thus:

$$q'(0) = \lim_{t\to 0} \frac{p(x_0 + th) - p(x_0)}{t} \leq 0.$$

Now, consider the case where $p(x)$ attains its minimum at $x = x_0$. Similarly, we have:

$$p(x) \geq p(x_0) \quad \text{for all } x \in X,$$

and we can repeat the above steps with $q(t) = p(x_0 + th)$ to find:

$$q'(0) \geq 0.$$

Combining both cases, we conclude that $q'(0) = 0$ for any nonzero vector $h \in X$. Therefore, the derivative of p at x_0 in any direction is zero, i.e., $p'(x_0) = 0$. Thus, we have shown that if $p(x)$ is norm-attainable, then its extrema are achieved at the roots of its derivative, as desired. □

Theorem 5. *For a norm-attainable operator $T : X \to Y$, if x_0 is the point where $|T(x_0)| = |T|$, then x_0 is the unique optimal solution to the optimization problem $\max_{|x|\leq 1} |T(x)|$.*

Proof. Let x_0 be the point where $|T(x_0)| = |T|$. We want to show that x_0 is the unique optimal solution to the optimization problem $\max_{|x|\leq 1} |T(x)|$. First, let x be any arbitrary element in X such that $|x| \leq 1$. Then, by the definition of the operator norm, we have:

$$|T(x)| \leq |T| \cdot |x_0|.$$

Since $|x| \leq 1$, we have $|x_0| \geq |x|$, and therefore:

$$|T(x)| \leq |T| \cdot |x|.$$

This shows that for any x with $|x| \leq 1$, $|T(x)|$ is bounded above by $|T| \cdot |x|$. Now, consider $x = x_0$. Since $|x_0| \leq 1$, we have:

$$|T(x_0)| \leq |T| \cdot |x_0|.$$

But by definition, $|T(x_0)| = |T|$, so we have:

$$|T| \leq |T| \cdot |x_0|.$$

Dividing both sides by $|T|$, we get:

$$1 \leq |x_0|.$$

Since $|x_0| \leq 1$ (as it is an element of the set $\{|x| \leq 1\}$), we conclude that $|x_0| = 1$. Therefore, for any x with $|x| \leq 1$, we have:

$$|T(x)| \leq |T| \cdot |x| \leq |T|.$$

This implies that x_0 maximizes $|T(x)|$ within the set $\{|x| \leq 1\}$, making it an optimal solution to the optimization problem. To prove uniqueness, suppose there exists another point x_1 in $\{|x| \leq 1\}$ such that $|T(x_1)| = |T|$. Then, we would have:

$$|T(x_1)| = |T| \leq |T| \cdot |x_1|.$$

Since $|x_1| \leq 1$, this implies $|T(x_1)| \leq |T|$. But since x_0 is also in $\{|x| \leq 1\}$ and $|T(x_0)| = |T|$, we would have $|T(x_0)| \leq |T|$, which contradicts our initial assumption that $|T(x_0)| = |T|$. Therefore, there cannot be another point in $\{|x| \leq 1\}$ with $|T(x)| = |T|$, proving the uniqueness of x_0 as the optimal solution. Hence, we have shown that x_0 is the unique optimal solution to the optimization problem $\max_{|x| \leq 1} |T(x)|$. □

Theorem 6. *Two norm-attainable operators T_1 and T_2 on the same normed vector space X are norm-equivalent if and only if there exist invertible operators A and B such that $T_1 = A^{-1}T_2B$.*

Proof. Let's prove the "if and only if" statement in two parts.
Part 1: If T_1 and T_2 are norm-equivalent, then there exist invertible operators A and B such that $T_1 = A^{-1}T_2B$. Assume that T_1 and T_2 are norm-equivalent. This means there exist constants $c_1 > 0$ and $c_2 > 0$ such that for all $x \in X$,

$$c_1\|T_1(x)\| \leq \|T_2(x)\| \leq c_2\|T_1(x)\|.$$

Now, let $A = \frac{1}{c_2}T_1$ and $B = c_1T_2^{-1}$, where T_2^{-1} is the inverse of T_2. Since $c_2 > 0$, A is invertible, and since $c_1 > 0$, B is invertible. Now, we can see that

$$A^{-1}T_2B = \frac{1}{c_2}T_1T_2c_1T_2^{-1} = \frac{c_1}{c_2}T_1 = T_1,$$

which proves the "if" part.
Part 2: If there exist invertible operators A and B such that $T_1 = A^{-1}T_2B$, then T_1 and T_2 are norm-equivalent. Assume that there exist invertible operators A and B such that $T_1 = A^{-1}T_2B$. Then, for any $x \in X$,

$$\begin{aligned}\|T_1(x)\| &= \|A^{-1}T_2Bx\| \\ &\leq \|A^{-1}\|\|T_2\|\|B\|\|x\|,\end{aligned}$$

where $\|A^{-1}\|$, $\|T_2\|$, and $\|B\|$ are the operator norms of A^{-1}, T_2, and B, respectively. Let $c_1 = \|A^{-1}\|\|B\|$ and $c_2 = \|T_2\|$. Then, we have shown that $\|T_1(x)\| \leq c_1c_2\|x\|$ for all $x \in X$. Similarly, by considering $T_2 = B^{-1}T_1A$, we can show that $\|T_2(x)\| \leq c_1c_2\|x\|$ for all $x \in X$. Therefore, T_1 and T_2 are norm-equivalent with constants c_1c_2 and c_1c_2, which completes the "only if" part. Hence, we have proven the theorem in both directions, and the proof is complete. □

Theorem 7. *In an inner product space H, a polynomial $p(x)$ is norm-attainable if and only if it is a constant polynomial.*

Proof. Let H be an inner product space, and consider a polynomial $p(x)$ defined on H. We aim to prove that $p(x)$ is norm-attainable if and only if it is a constant polynomial.
($\Rightarrow$) Suppose $p(x)$ is norm-attainable. This means there exists some $x_0 \in H$ such that $\|p(x)\| = \|p(x_0)\|$ for all $x \in H$. Now, let's consider the polynomial $q(x) = p(x) - p(x_0)$. Since both $p(x)$ and $p(x_0)$ are norm-attainable, we have:

$$\|q(x)\| = \|p(x) - p(x_0)\| = \|p(x)\| - \|p(x_0)\| = 0$$

This implies that $q(x)$ is the zero polynomial, which means $p(x) = p(x_0)$ for all $x \in H$. Thus, $p(x)$ is a constant polynomial.
($\Leftarrow$) Conversely, suppose $p(x)$ is a constant polynomial, i.e., $p(x) = c$ for some constant c. Let x_0 be any element in H. Then, for any $x \in H$, we have:

$$\|p(x) - p(x_0)\| = \|c - c\| = 0$$

This shows that $\|p(x)\| = \|p(x_0)\|$ for all $x \in H$. Therefore, $p(x)$ is norm-attainable. Hence, we have shown that $p(x)$ is norm-attainable if and only if it is a constant polynomial. □

Corollary 1. *For a norm-attainable operator $T : X \to Y$ and a convex optimization problem with the objective function $\max_{|x|\leq 1} |T(x)|$, the solution lies on the boundary of the unit ball in X.*

Proof. Assume that $T : X \to Y$ is a norm-attainable operator, and consider the convex optimization problem:

$$\max_{\|x\|\leq 1} |T(x)|.$$

Let x_0 be the point that maximizes $|T(x)|$ within the unit ball, i.e., $|T(x_0)| = \max_{\|x\|\leq 1} |T(x)|$. We will show that x_0 lies on the boundary of the unit ball in X. Suppose, for the sake of contradiction, that x_0 is an interior point of the unit ball, i.e., there exists an $\epsilon > 0$ such that $\|x_0\| + \epsilon < 1$. Consider the point $y = \frac{x_0}{\|x_0\|+\epsilon}$. Clearly, $\|y\| = 1$, and we can write x_0 as a rescaling of y: $x_0 = (\|x_0\| + \epsilon)\, y$. Now, we can compute the norm of $T(x_0)$:

$$|T(x_0)| = |T(\|x_0\|y + \epsilon y)| = |\|x_0\|T(y) + \epsilon T(y)|.$$

By the triangle inequality for norms, we have:

$$|T(x_0)| \leq \|\|x_0\|T(y)\| + \epsilon\|T(y)\|.$$

Since $\|x_0\| < 1$, and y is a unit vector, $\|x_0\|T(y)$ is strictly less than $\|T(y)\|$. Thus, we have:

$$|T(x_0)| < \|T(y)\| + \epsilon\|T(y)\| = (1 + \epsilon)\|T(y)\|.$$

However, this contradicts our assumption that $|T(x_0)|$ is the maximum value of $|T(x)|$ for all $\|x\| \leq 1$, as we have found a point y with $\|y\| = 1$ for which $|T(y)|$ is strictly greater than $|T(x_0)|$. Therefore, our assumption that x_0 is an interior point of the unit ball leads to a contradiction. Hence, the only possibility is that x_0 must lie on the boundary of the unit ball in X. This completes the proof. □

5 Conclusion

The research findings establish crucial connections between norm-attainable operators and mathematical spaces. In Banach spaces, an operator can attain its norm if and only if the target space is also a Banach space, while in Hilbert spaces, self-adjointness is the key criterion for norm-attainability. Polynomial functions in normed vector spaces can attain their norms only if they are linear, and in finite-dimensional spaces, all bounded linear operators are norm-attainable. Optimization in these contexts becomes more tractable, with unique optima guaranteed for norm-attainable polynomials. Norm-equivalent operators can be characterized by invertible operators, and in inner product spaces, only constant polynomials can achieve their norms. These conclusions significantly contribute to our understanding of norm-attainment across mathematical spaces, providing valuable insights for optimization and functional analysis.

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