Original Research Article

Improved uncertainty distribution of single expert data

Abstract

Distribution is one of the basic characteristics of statistics. There exist only two ways to obtain the distribution function for some quantity, one is frequency generated by historical data and the other is belief degree evaluated by domain experts. However, it is undoubtedly difficult for expert to give specific and accurate experimental data each time when doing the questionnaire. By improving the questionnaire, this paper proposes a new method of data collection combining uncertainty and randomness. Besides, a moment method for eatimating the distributions with known parameters is eatimated by using the collected. Several numerical examples are provided to illustrate the feasibility of the method.

Keywords: Distribution function; Randomness; Uncertainty; Chance theory; Moments

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1 Introduction

In real life, human beings may confront all kinds of things that will happen and make subjective decisions. When dealing with the the likelihood that something will happen, people have two effective tools: probability theory and uncertainty theory. With the development and improvement of the theories, probability theory and uncertainty theory have obviously become two aximatic mathematical systems. Essentially, probability theory and uncertainty theory are associated with frequencies and belief degrees respectively. As we all know, the fundamental premise of being able to apply probability theory is that the obtained distribution function is consistent with the real frequency. For the sake of obtaining the distribution function for a quantity, such as bond price, people can study historical data or consult relevant domain experts according to the specific situation. When the historical data of the quantity is sufficient, there is no doubt that it is very appropriate to use probability theory. In contrast, in the case of insufficient historical data, uncertainty theory will be a good choice. And in real life, historical data is unavailable in many cases. One example is that if we want to study the degree of marine pollution caused by the discharge of nuclear wastewater, due to the sudden impact of human activities on marine ecosystems, it is difficult to obtain a large number of historical data. In this case, owing to the lack of the historical data or the arise of some emergency, we fail to estimate the distribution via probability theory.

For purpose of dealing with this kind of problem, Liu [1] founded the uncertainty theory in 2007 and refined as a branch of axiomatic mathematics based on normality, duality, subadditivity and product axioms in 2010. Many researchers subsequently studied and made significative peogress in the area of uncertainty theory. On the basis of the uncertain measure, Liu [1] proposed the concept of uncertain variable and uncertainty distribution. Peng and Iwamura [2] proved the sufficient and necessary condition of uncertainty distribution through theoretical derivation. In the process of studying some special uncertainty distributions, Liu [3] summarized the concept of regular uncertainty distribution and naturally proposed inverse uncertainty distribution. On the basis of the independence [4], Liu [3] got some operational laws for calculating the distribution.

Uncertain statistics is an extremely crucial part of uncertainty theory. Liu [3] proposed an approach to collecting expert's experimental data through inviting expert to complete the questionaire. With the help of the questionnaire survey, Chen-Ralescu [19] got the expert's data of the travel distance between Beijing and Tianjin. For the sake of modeling and predicting the data more accurately, Liu [3] gaved the definition of the expected value. Yao [5] proposed a formula to calculate the variance by invoking inverse uncertainty distribution. Furthermore, Sheng and Kar [6] obtained some results of moments of uncertain variable. Liu [3] used the data obtained from experts to establish the empirical uncertainty distribution. Thus Liu [3] further gave the principle of least square based on the empirical uncertainty distribution. Since then, many scholars have deeply studied uncertain statistics and developed it into more research including estimation of uncertainty distribution [3], uncertain hypothesis test [7], uncertain regression analysis [8], uncertain time series [9], and uncertain differential equation [10]. Lio and Liu [11] presented the method of moments to estimate the unknown parameters in the distribution. Besides, the uncertain maximum likelihood estimation was also proposed by Lio and Liu [12] and was modified by Liu and Liu [13].

The situation we faced may be more complicated. In order to quantify an event with randomness and fuzziness, Li and Liu [14] first introduced the concepts of chance space and chance measure in 2009. Nevertheless, to describe a complex system involving both randomness and uncertainty, Liu [15] redefined the concept of chance space as the product of probability space and uncertainty space. Meanwhile, Liu [15] proposed the uncertain random variable and its chance distribution, expected value, and variance. Liu [16] provided the operational law by deriving. Yao and Gao [17] verified a law of large numbers for uncertain random variables.

In real life, it is not always so smooth for us to acquire the accurate data. On one hand, expert may have his own preferences so that the belief degrees are very subjective. On the other hand, it is undoubtedly difficult for expert to give specific and accurate experimental data each time when doing the questionnaire. By improving the questionnaire, this paper will propose a new method of data collection combining uncertainty and randomness. The main structure of this paper is organized as follows. In the next section, we will make a review of some basic concepts in uncertainty theory and chance theory. Then the improved method of data collection is introduced in Section (3). By using the collected data, the corresponding method of moments to estimate the distributions with unknown parameters is established in Section (4). After that, several numerical examples are provided to illustrate the feasibility of the method in Section (5). Finally, a conclusion is drawn in Section (6).

2 Preliminary

In this section, some basic concepts of uncertainty theory and chance theory, such as uncertain measure, uncertain variable, uncertain random variable, are given. More detailed information can refer to [1, 3, 15].

2.1 Uncertainty Theory

Let Γ_u be a nonempty set, and $\mathcal{L}_u$ be a σ-algebra over Γ_u. Each element $\Lambda \in \mathcal{L}_u$ is referred to as an event. The set function $\mathcal{M}$ satisfying the following axioms is called an uncertain measure:

Axiom 1. (Normality) $\mathcal{M}\{\Gamma_u\} = 1$ *for the universal set* Γ_u*;*

Axiom 2. (Duality) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ *for any event* $\Lambda \in \mathcal{L}_u$*;*

Axiom 3. (Subadditivity) *For every countable sequence of events* $\Lambda_i \in \mathcal{L}_u, i = 1, 2, \cdots$*, we have*

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$$

The triplet $(\Gamma_u, \mathcal{L}_u, \mathcal{M})$ is referred to as an uncertainty space [3].

Axiom 4. (Product) *Let* $(\Gamma_{uk}, \mathcal{L}_{uk}, \mathcal{M}_k)$ *be uncertainty spaces for* $k = 1, 2, \cdots$*. Denote* $\Gamma_u = \Gamma_{u1} \times \Gamma_{u2} \times \cdots$, $\Lambda = \Lambda_1 \times \Lambda_2 \times \cdots$, $\mathcal{L}_u = \mathcal{L}_{u1} \times \mathcal{L}_{u2} \times \cdots$*. Then the product uncertain maesure* $\mathcal{M}$ *is an uncertain measure which satisfies:*

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\},$$

where Λ_k *are arbitrarily chosen events from* $\mathcal{L}_{uk}$ *for* $k = 1, 2, \cdots$*, respectively.*

An uncertain variable is a measurable function ξ_u from an uncertainty space $(\Gamma_u, \mathcal{L}_u, \mathcal{M})$ to the set of real numbers.

Definition 2.1 ([1])**.** *The uncertainty distribution* $\Phi_u(x)$ *of an uncertain variable* ξ_u *is defined by*

$$\Phi_u(x) = \mathcal{M}\{\xi_u \le x\} \tag{1}$$

for any $x \in \mathbb{R}$*.*

An uncertainty distribution $\Phi_u(x)$ is called regular if it is a continous and strictly increasing function with respect to x at which $0 < \Phi_u(x) < 1$, and

$$\lim_{x \to -\infty} \Phi_u(x) = 0, \lim_{x \to +\infty} \Phi_u(x) = 1. \tag{2}$$

The inverse function $\Phi_u^{-1}(\alpha)$ is referred to as the inverse uncertainty distribution of ξ_u whose uncertainty distribution $\Phi_u(x)$ is regular.

Definition 2.2 ([1]). *Let ξ_u be an uncertain variable. Then the expected value of ξ_u is defined by*

$$E_{\mathcal{M}}\{\xi_u\} = \int_0^{+\infty} \mathcal{M}\{\xi_u \geq x\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi_u \leq x\}\mathrm{d}x \tag{3}$$

provided that at least one of the two integrals is finite.

Furthermore, according to the difinition of the uncertainty distribution and the inverse uncertainty distribution, the formula (3) can be rewritten as

$$E_{\mathcal{M}}\{\xi_u\} = \int_0^1 \Phi_u^{-1}(\alpha)\mathrm{d}\alpha, \tag{4}$$

where $\Phi_u^{-1}(\alpha)$ is the inverse uncertainty distribution of ξ_u.

2.2 Chance Theory

Let $(\Gamma_u, \mathcal{L}_u, \mathcal{M})$ be an uncertainty space and $(\Omega_r, \mathcal{A}_r, Pr)$ be a probability sapce. The product $(\Gamma_u, \mathcal{L}_u, \mathcal{M}) \times (\Omega_r, \mathcal{A}_r, Pr)$, denoted as the triplet $(\Gamma_u \times \Omega_r, \mathcal{L}_u \times \mathcal{A}_r, \mathcal{M} \times Pr)$, can be regarded as a chance space. Note that the universal set $\Gamma_u \times \Omega_r$ is clearly the set of all ordered pairs of the form (γ_u, ω_r), where $\gamma_u \in \Gamma_u$ and $\omega_r \in \Omega_r$. That is, $\Gamma_u \times \Omega_r = \{(\gamma_u, \omega_r)|\gamma_u \in \Gamma_u, \omega_r \in \Omega_r)\}$. Meanwhile, $\mathcal{L}_u \times \Omega_r$ is the product σ-algebra and $\mathcal{M} \times Pr$ is the product measure. Theoretically, $\mathcal{M} \times Pr$ is referred to as chance measure. We represent the chance measure by $Ch\{\Theta_{ur}\}$, where Θ_{ur} is an event in the chance space.

Definition 2.3 ([15]). *Let $(\Gamma_u \times \Omega_r, \mathcal{L}_u \times \mathcal{A}_r, \mathcal{M} \times Pr)$ be a chance space, and let $\Theta_{ur} \in \mathcal{L}_u \times \mathcal{A}_r$ be an event. Then the chance measure of Θ_{ur} is defined as*

$$Ch\{\Theta_{ur}\} = \int_0^1 Pr\{\omega_r \in \Omega_r|\mathcal{M}\{\gamma_u \in \Gamma_u|(\gamma_u, \omega_r) \in \Theta_{ur}\} \geq x\}\mathrm{d}x \tag{5}$$

Definition 2.4 ([15]). *Let $(\Gamma_u \times \Omega_r, \mathcal{L}_u \times \mathcal{A}_r, \mathcal{M} \times Pr)$ be a chance space. ξ_{ur} is an uncertain random variable in this space. Then its chance distribution is defined by*

$$\Upsilon_{ur}(x) = Ch\{\xi_{ur} \leq x\} \tag{6}$$

for any $x \in \mathbb{R}$.

Remark 1. *If an uncertain random variable ξ_{ur} degenerates to an uncertain variable ξ_u, its distribution also becomes the uncertainty distribution $\Phi(x) = \mathcal{M}\{\xi_u \leq x\}$, for any $x \in \mathbb{R}$. Similarly, if an uncertain random variable ξ_{ur} degenerates to a random variable ξ_r, its distribution also becomes the probability distribution $F_r(x) = Pr\{\xi_r \leq x\}$, for any $x \in \mathbb{R}$.*

According to Defininiton (2.3). and the definition of E_{Pr}, we can rewrite the chance distribution to

$$\begin{aligned} Ch\{\xi_{ur} \leq x\} &= \int_0^1 Pr\{\omega_r \in \Omega_r|\mathcal{M}\{\gamma_u \in \Gamma_u|\xi_{ur}(\gamma_u, \omega_r) \in \Theta_{ur}\} \leq x\}\mathrm{d}x \\ &= E_{Pr}[\mathcal{M}\{\gamma_u \in \Gamma_u|\xi_{ur}(\gamma_u, \omega_r) \in \Theta_{ur}\} \leq x\}] \end{aligned}$$

Definition 2.5 ([15]). *Let ξ_{ur} be an uncertain random variable. Then its expected value E_{Ch} is referred to as*

$$E_{Ch}[\xi_{ur}] = \int_0^{+\infty} Ch\{\xi_{ur} \geq x\}\mathrm{d}x - \int_{-\infty}^0 Ch\{\xi_{ur} \leq x\}\mathrm{d}x \tag{7}$$

provided that at least one of the two integrals is finite.

The formula (7) may be rewritten as follows:

$$E_{Ch}[\xi_{ur}] = \int_0^{+\infty} E_{Pr}[\mathcal{M}\{\gamma_u \in \Gamma_u | \xi_{ur}(\gamma_u, \omega_r) \in \Theta_{ur}\} \geq x\}]\mathrm{d}x - \int_{-\infty}^{0} E_{Pr}[\mathcal{M}\{\gamma_u \in \Gamma_u | \xi_{ur}(\gamma_u, \omega_r) \in \Theta_{ur}\} \leq x\}]\mathrm{d}x$$

Since $\mathcal{M}\{\gamma_u \in \Gamma_u | \xi_{ur}(\gamma_u, \omega_r) \in \Theta_{ur}\} \geq x\}$ and $\mathcal{M}\{\gamma_u \in \Gamma_u | \xi_{ur}(\gamma_u, \omega_r) \in \Theta_{ur}\} \leq x\}$ are nonnegative random variables, according to Fubini Theorem and the definition of $E_{\mathcal{M}}$, we have

$$\begin{aligned} E_{Ch}[\xi_{ur}] =& E_{Pr}\left[\int_0^{+\infty} \mathcal{M}\{\gamma_u \in \Gamma_u | \xi_{ur}(\gamma_u, \omega_r) \in \Theta_{ur}\} \geq x\}\mathrm{d}x\right] \\ &- E_{Pr}\left[\int_0^{+\infty} \mathcal{M}\{\gamma_u \in \Gamma_u | \xi_{ur}(\gamma_u, \omega_r) \in \Theta_{ur}\} \leq x\}\mathrm{d}x\right] \\ =& E_{Pr}\left[\int_0^{+\infty} \mathcal{M}\{\gamma_u \in \Gamma_u | \xi_{ur}(\gamma_u, \omega_r) \in \Theta_{ur}\} \geq x\}\mathrm{d}x\right. \\ &\left. - \int_0^{+\infty} \mathcal{M}\{\gamma_u \in \Gamma_u | \xi_{ur}(\gamma_u, \omega_r) \in \Theta_{ur}\} \leq x\}\mathrm{d}x\right] \end{aligned}$$

Furthermore, according to Definition (2.2), we can rewrite $E_{Ch}[\xi_{ur}]$ as

$$E_{Ch}[\xi_{ur}] = E_{Pr}\left[E_{\mathcal{M}}[\xi_{ur}]\right] \tag{8}$$

Theorem 2.1 ([15]). *Let ξ_{ur} be an uncertain random variable with chance distribution Υ_{ur}. Then*

$$E_{Ch}[\xi_{ur}] = \int_0^{+\infty} (1 - \Upsilon_{ur}(x))\mathrm{d}x - \int_{-\infty}^{0} \Upsilon_{ur}(x)\mathrm{d}x. \tag{9}$$

When the chance distribution Υ_{ur} of an uncertain random variable ξ_{ur} is regular, the formula (9) may be rewritten as

$$E_{Ch}[\xi_{ur}] = \int_0^1 \Upsilon_{ur}^{-1}(\alpha)\mathrm{d}\alpha. \tag{10}$$

3 Improved Experimental Data

In 2010, Liu [3] proposed a questionnaire survey for collecting expert's experimental data. We invite one domain expert who are asked to complete a questionnaire about the meaning of an uncertain variable ξ_u like 'about 10km' individually. The design of the questionnaire is roughly as follows. To begin with, we ask the expert to choose a possible value x that the uncertain variable ξ_u may take. Then, we quiz him 'how likely is ξ_u less than x?' and denote his belief degree by t. Thus, we obtain an expert's experimental data (x, t) from the domain expert. Repeating the above process, we obtain the expert's experimental data. Let $(x_1, t_1), (x_2, t_2), \cdots, (x_n, t_n)$ be the expert's experimental data that meet the following condition:

$$x_1 < x_2 < \cdots < x_n, 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq 1.$$

Based on data we collected, Liu [3] presented the empirial uncertainty distribution as follows.

$$\Phi_u(x) = \begin{cases} 0 & x < x_1 \\ t_i + \dfrac{(t_{i+1} - t_i)(x - x_i)}{x_{i+1} - x_i} & x_i \leq x \leq x_{i+1}, 1 \leq i < n \\ 1 & x > x_n \end{cases} \tag{11}$$

Since the distribution function is a monotonous increasing function, it is esay to get the corresponding inverse distribution.

$$\Phi_u^{-1}(t) = \begin{cases} x_1 & t < t_1 \\ x_i + \dfrac{(x_{i+1} - x_i)(t - t_i)}{t_{i+1} - t_i} & t_i \leq t \leq t_{i+1}, 1 \leq i < n \\ x_n & t > t_n \end{cases} \tag{12}$$

However, owing to personal preference, it is difficult to get the exact belief degree t corresponding to each possible value x. In order to be more realistic, we can make improvements in the design of the questionnaire. The specific operation is as folloes. The domain expert is firstly asked to choose a poosible value x (say 100km) that the variable ξ_{ur} may take, and is then quizzed on the question,

"How likely is ξ_{ur} less than or equal to x? Give an interval."

Denote the expert's belif degree interval by (α, β) (say(0.65,0.7)). An expert's experimental data

$$(100, (0.65, 0.7))$$

is thus acquired from the domain expert. In this way, we replace the belief degree t with the belif degree interval (α, β). In fact, the exact value of t is just a number in the interval (α, β). Generally, the probability of t appearing in each value in the interval (α, β) is equal. So we can recognize that $t_i \sim \mathcal{U}(\alpha, \beta)$. In this case, the variable ξ_{ur} should be an uncertain random variable instead of an uncertain variable. Repeating the above process, the questionnaire may yield the following expert's experimental data,

$$(x_1, (\alpha_1, \beta_1)), (x_2, (\alpha_2, \beta_2)), \cdots, (x_n, (\alpha_n, \beta_n)).$$

Denote t_i as a random variable which is subject to the uniform distribution on the interval $[\alpha_i, \beta i]$. Let $(x_1, (\alpha_1, \beta_1)), (x_2, (\alpha_2, \beta_2)), \cdots, (x_n, (\alpha_n, \beta_n))$ meet the following condition:

$$x_1 < x_2 < \cdots < x_n; t_i \sim \mathcal{U}(\alpha_i, \beta_i), 0 \leq \alpha_1 \cdots \leq \alpha_i \leq \beta_i \leq \alpha_{i+1} \leq \cdots \leq \beta_n \leq 1, i = 1, 2, \cdots, n$$

On the basis of the data above, we can get the empirical chance distribution as follows:

$$\Upsilon_{ur}(x) = \begin{cases} 0 & x < x_1 \\ \dfrac{\alpha_i + \beta_i}{2} + \dfrac{(\alpha_{i+1} + \beta_{i+1} - \alpha_i - \beta_i)(x - x_i)}{2(x_{i+1} - x_i)} & x_i \leq x \leq x_{i+1}, 1 \leq i < n \\ 1 & x > x_n \end{cases} \tag{13}$$

Since the belief degrees are given in the form of intervals, we can't acquire the inverse distribution.

4 Method of Moments

In this section, a method of moments based on expert's experimental data is proposed to estimate the unknown parameters. The k-th moment of the empirical chance distribution is presented as well.

Definition 4.1. *Let ξ_{ur} be an uncertain random variable and let k be a positive integer. Then $E_{Ch}[\xi_{ur}^k]$ is called the k-th moment of ξ_{ur}.*

Theorem 4.1. *Let ξ_{ur} be an uncertain random variable with regular chance distribution Υ_{ur} and let k be a positive integer. Then*

$$E_{Ch}[\xi_{ur}^k] = \int_0^1 (\Upsilon_{ur}^{-1}(\alpha))^k \mathrm{d}\alpha.$$

Proof. Since $\alpha = \Upsilon_{ur}(\sqrt[k]{x})$ and $x = (\Upsilon_{ur}^{-1}(\alpha))^k$ represent the same curve in the rectangular coordinate system (x, α), we have

$$\int_0^{+\infty}(1-\Upsilon_{ur}(\sqrt[k]{x}))\mathrm{d}x = \int_{\Upsilon_{ur}(0)}^{1}(\Upsilon_{ur}^{-1}(\alpha))^k\mathrm{d}\alpha$$

because the two integrals make an identical acreage. Similarly, we also have

$$\int_{-\infty}^{0}\Upsilon_{ur}(\sqrt[k]{x})\mathrm{d}x = -\int_0^{\Upsilon_{ur}(0)}(\Upsilon_{ur}^{-1}(\alpha))^k\mathrm{d}\alpha.$$

Then we can rewrite the k-th moment as

$$\begin{aligned}
E_{Ch}[\xi_{ur}^k] &= \int_0^{+\infty} Ch\{\xi_{ur}^k \geq x\}\mathrm{d}x - \int_{-\infty}^{0} Ch\{\xi_{ur}^k \leq x\}\mathrm{d}x \\
&= \int_0^{+\infty}(1-\Upsilon_{ur}(\sqrt[k]{x}))\mathrm{d}x - \int_{-\infty}^{0}\Upsilon_{ur}(\sqrt[k]{x})\mathrm{d}x \\
&= \int_{\Upsilon_{ur}(0)}^{1}(\Upsilon_{ur}^{-1}(\alpha))^k\mathrm{d}\alpha + \int_0^{\Upsilon_{ur}(0)}(\Upsilon_{ur}^{-1}(\alpha))^k\mathrm{d}\alpha \\
&= \int_0^1(\Upsilon_{ur}^{-1}(\alpha))^k\mathrm{d}\alpha.
\end{aligned}$$

Theorem 4.2. *Let $(x_i, t_i), i = 1, 2, \cdots, n$ be the expert's experimental data that meet the following condition:*

$$0 \leq x_1 < x_2 < \cdots < x_n, t_i \sim \mathcal{U}(\alpha_i, \beta_i), 0 \leq \alpha_1 \cdots \leq \alpha_i \leq \beta_i \leq \alpha_{i+1} \leq \cdots \leq \beta_n \leq 1, i = 1, 2, \cdots, n. \tag{14}$$

Then for any positive integer k, the uncertain random variable ξ_{ur} with the empirical chance distribution has the k-th empirical moment

$$E_{Ch}[\xi_{ur}^k] = \frac{\alpha_1+\beta_1}{2}x_1^k + \frac{1}{k+1}\sum_{i=1}^{n-1}\sum_{j=0}^{k}(\frac{\alpha_{i+1}+\beta_{i+1}}{2} - \frac{\alpha_i+\beta_i}{2})x_i^j x_{i+1}^{k-j} + (1-\frac{\alpha_n+\beta_n}{2})x_n^k. \tag{15}$$

Proof. Since $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$, according to the formula (8) and Definition (2.2), we have

$$\begin{aligned}
E_{Ch}[\xi_{ur}^k] =& E_{Pr}\left[E_{\mathcal{M}}[\xi_{ur}^k]\right] \\
=& E_{Pr}\left[\int_0^{+\infty}\mathcal{M}\{\xi_{ur}^k \geq x\}\mathrm{d}x\right].
\end{aligned}$$

Then by using integral substitution method and duality axiom, we can rewrite $E_{Ch}[\xi_{ur}^k]$ as

$$\begin{aligned}
E_{Ch}[\xi_{ur}^k] =& E_{Pr}\left[\int_0^{+\infty} kx^{k-1}\mathcal{M}\{\xi_{ur} \geq x\}\mathrm{d}x\right] \\
=& E_{Pr}\left[\int_0^{+\infty} kx^{k-1}(1-\mathcal{M}\{\xi_{ur} \leq x\})\mathrm{d}x\right] \\
=& E_{Pr}\left[k\int_0^{x_1} x^{k-1}(1-\mathcal{M}\{\xi_{ur} \leq x\})\mathrm{d}x + k\sum_{i=1}^{n-1}\int_{x_i}^{x_{i+1}} x^{k-1}(1-\mathcal{M}\{\xi_{ur} \leq x\})\mathrm{d}x\right. \\
&\left. + k\int_{x_1}^{+\infty} x^{k-1}(1-\mathcal{M}\{\xi_{ur} \leq x\})\mathrm{d}x\right] \\
=& E_{Pr}\left[k\int_0^{x_1} x^{k-1}\mathrm{d}x + k\sum_{i=1}^{n-1}\int_{x_i}^{x_{i+1}} x^{k-1}(1-\mathcal{M}\{\xi_{ur} \leq x\})\mathrm{d}x\right] \\
=& E_{Pr}\left[x_1^k + k\sum_{i=1}^{n-1}\int_{x_i}^{x_{i+1}} x^{k-1}(1-t_i-\frac{(t_{i+1}-t_i)(x-x_i)}{x_{i+1}-x_i}\mathrm{d}x\right] \\
=& E_{Pr}\left[\frac{kt_1+t_2}{k+1}x_1^k + \frac{1}{k+1}\sum_{i=1}^{n-1}\sum_{j=1}^{k-1}(t_{i+1}-t_i)x_i^j x_{i+1}^{k-j}\right. \\
&\left. + \frac{1}{k+1}\sum_{i=2}^{n-1}(t_{i+1}-t_{i-1})x_i^k + (1-\frac{1}{k+1}t_{n-1}-\frac{k}{k+1}t_n)x_n^k\right] \\
=& E_{Pr}\left[t_1x_1^k + \frac{1}{k+1}\sum_{i=1}^{n-1}\sum_{j=0}^{k}(t_{i+1}-t_i)x_i^j x_{i+1}^{k-j} + (1-t_n)x_n^k\right]
\end{aligned}$$

As is known to all, random variables have some good properties. Thus $E_{Ch}[\xi_{ur}^k]$ can be further rewritten as

$$\begin{aligned}
E_{Ch}[\xi_{ur}^k] =& E_{Pr}\left[t_1x_1^k\right] + E_{Pr}\left[\frac{1}{k+1}\sum_{i=1}^{n-1}\sum_{j=0}^{k}(t_{i+1}-t_i)x_i^j x_{i+1}^{k-j}\right] + E_{Pr}\left[(1-t_n)x_n^k\right] \\
=& E_{Pr}[t_1]x_1^k + \frac{1}{k+1}\sum_{i=1}^{n-1}\sum_{j=0}^{k}(E_{Pr}[t_{i+1}]-E_{Pr}[t_i])x_i^j x_{i+1}^{k-j} + (1-E_{Pr}[t_n])x_n^k \\
=& \frac{\alpha_1+\beta_1}{2}x_1^k + \frac{1}{k+1}\sum_{i=1}^{n-1}\sum_{j=0}^{k}(\frac{\alpha_{i+1}+\beta_{i+1}}{2}-\frac{\alpha_i+\beta_i}{2})x_i^j x_{i+1}^{k-j} + (1-\frac{\alpha_n+\beta_n}{2})x_n^k.
\end{aligned}$$

The theorem is proved.

Definition 4.2. *Let $(x_i, t_i), i = 1, 2, \cdots, n$ be the expert's experimental data, $0 \leq x_1 < x_2 < \cdots < x_n, t_i \sim \mathcal{U}(\alpha_i, \beta_i), 0 \leq \alpha_1 \cdots \leq \alpha_i \leq \beta_i \leq \alpha_{i+1} \leq \cdots \leq \beta_n \leq 1, i = 1, 2, \cdots, n$. Then for any positive integer k,*

$$\bar{\xi_{ur}^k} = \frac{\alpha_1+\beta_1}{2}x_1^k + \frac{1}{k+1}\sum_{i=1}^{n-1}\sum_{j=0}^{k}(\frac{\alpha_{i+1}+\beta_{i+1}}{2}-\frac{\alpha_i+\beta_i}{2})x_i^j x_{i+1}^{k-j} + (1-\frac{\alpha_n+\beta_n}{2})x_n^k \quad (16)$$

is referred to as the k-th empirical moment.

As we all know, method of moments is usually used for estimating the unknown parameter. Let ξ_{ur} be an uncertain random variable with regular chance disetribution $\Upsilon_{ur}(x; \theta_1, \theta_2, \cdots, \theta_p)$, where

$\theta_1, \theta_2, \cdots, \theta_p$ are unknown parameters. Let $(x_i, t_i), i = 1, 2, \cdots, n$ be the expert's experimental data, $0 \leq x_1 < x_2 < \cdots < x_n, t_i \sim \mathcal{U}(\alpha_i, \beta_i), 0 \leq \alpha_1 \cdots \leq \alpha_i \leq \beta_i \leq \alpha_{i+1} \leq \cdots \leq \beta_n \leq 1, i = 1, 2, \cdots, n.$ Then we can use the k-th empirical moment $\bar{\xi^k_{ur}}$ to replace he k-th moment $E_{Ch}[\xi^k_{ur}]$. Now that we have p unknown parameters, we need to construct p equations:

$$E_{Ch}[\xi^k_{ur}] = \bar{\xi^k_{ur}}, k = 1, 2, \cdots, p. \tag{17}$$

If the equation group has a solution, then the solution is referred to as the moment estimators. We denote them as $\hat{\theta_1}, \hat{\theta_2}, \cdots, \hat{\theta_p}$, respectively. Thus we obtain the estimated distribution function $\Upsilon_{ur}(x; \hat{\theta_1}, \hat{\theta_2}, \cdots, \hat{\theta_p})$. If the equation group has no solution, we can use the principle of least squares to get estimators. Let $\gamma_i = \Upsilon_{ur}(x_i)$. In fact, the unknown parameters $\theta_i, i = 1, 2, \cdots, p$ are the solution of the following minimization problem:

$$\min_{\theta_1, \theta_2, \cdots, \theta_p} \sum_{i=1}^{n} (\Upsilon(x_i; \theta_1, \theta_2, \cdots, \theta_p) - \gamma_i)^2.$$

5 Numerical examples

In order to demonstrate the effectiveness of the improved moments method presented in the previous section, two numerical examples is given. Besides, when the the improved moments method fails to acquire the results, another example is provided.

Example 5.1. Suppose that the uncertainty distribution of uncertain variable ξ_u has a functional form with one unknown parameter θ as follows:

$$\Upsilon_u(x; \theta) = \theta x^{\frac{1}{2}}, \theta > 0, \Upsilon_u(x; \theta) \leq 1$$

By consulting a domain expert, we got the values of ξ_u and its corresponding belief degree intervals, which are shown in Table 1 and Figure 1.

Table 1: the data given by the domain expert

x	0.002	0.011	0.032	0.049	0.072	0.110
t	(0.02, 0.07)	(0.09, 0.13)	(0.18, 0.22)	(0.23, 0.25)	(0.27, 0.31)	(0.32, 0.39)
x	0.213	0.302	0.395	0.480	0.574	0.723
t	(0.47, 0.53)	(0.56, 0.63)	(0.66, 0.70)	(0.74, 0.76)	(0.80, 0.84)	(0.89, 0.94)

Considering that the belief degree t is presented in the form of an interval (α, β), we regard the uncertain variable ξ_u as an uncertain random variable ξ_{ur}, while $t \sim \mathcal{U}(\alpha, \beta)$. By using the moment method, we have

$$E_{Ch}[\xi_{ur}] = \bar{\xi_{ur}}$$

Since $E_{Ch}[\xi_{ur}] = \int_0^1 \Upsilon^{-1}_{ur}(\alpha) \mathrm{d}\alpha = \dfrac{1}{3\theta^2}$, we have

$$\begin{aligned}\frac{1}{3\theta^2} =& \frac{\alpha_1 + \beta_1}{2} x_1 + \frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=0}^{1} (\frac{\alpha_{i+1} + \beta_{i+1}}{2} - \frac{\alpha_i + \beta_i}{2}) x_i^j x_{i+1}^{k-j} + (1 - \frac{\alpha_n + \beta_n}{2}) x_n \\ =& \frac{\alpha_1 + \beta_1 + \alpha_2 + \beta_2}{4} x_1 + \sum_{i=2}^{n-1} \frac{\alpha_{i+1} + \beta_{i+1} - \alpha_{i-1} - \beta_{i-1}}{4} x_i + (1 - \frac{\alpha_{n-1} + \beta_{n-1} + \alpha_n + \beta_n}{4}) x_n.\end{aligned}$$

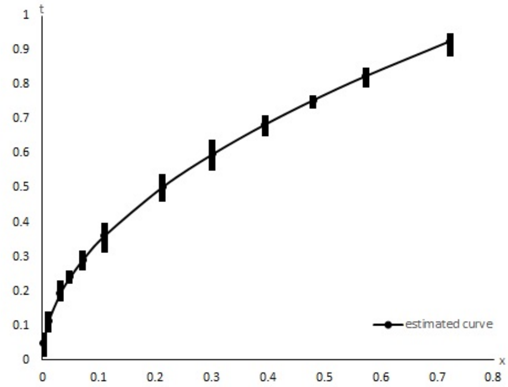

Figure 1: the data given by the domain expert

Hence, we obtain the estimate value of unknown parameter θ,

$$\hat{\theta} = \left\{3\left[\frac{\alpha_1+\beta_1+\alpha_2+\beta_2}{4}x_1 + \sum_{i=2}^{n-1}\frac{\alpha_{i+1}+\beta_{i+1}-\alpha_{i-1}-\beta_{i-1}}{4}x_i + (1-\frac{\alpha_{n-1}+\beta_{n-1}+\alpha_n+\beta_n}{4})x_n\right]\right\}^{-\frac{1}{2}}$$

By calculating, we have the moment estimator

$$\hat{\theta} = 1.089,$$

and the moment estimate distribution is

$$\Upsilon_{ur}(x) = 1.089x^{\frac{1}{2}}, \Upsilon_{ur}(x) \leq 1.$$

Example 5.2. Suppose that the uncertainty distribution of uncertain variable ξ_u has a functional form with two unknown parameters a, b as follows:

$$\Upsilon_u(x; a, b) = ax + b, (a > 0, 0 \leq \Upsilon_{ur}(x; a, b) \leq 1)$$

By consulting a domain expert, we got the values of ξ_u and its corresponding belief degree intervals, which are shown in Table 2 and Figure 2.

Table 2: the data given by the domain expert

x	0.4	1.0	1.5	2.0	3.0	4.0
t	$(0.08, 0.12)$	$(0.17, 0.23)$	$(0.28, 0.32)$	$(0.39, 0.41)$	$(0.67, 0.73)$	$(0.88, 0.92)$

Considering that the belief degree t is presented in the form of an interval (α, β), we regard the uncertain variable ξ_u as an uncertain random variable ξ_{ur}, while $t \sim \mathcal{U}(\alpha, \beta)$.In according to the moment method, we will solve the system of equations as follows:

$$\begin{cases} E[\xi_{ur}] = \bar{\xi_{ur}} \\ E[\xi_{ur}^2] = \bar{\xi_{ur}^2} \end{cases} \tag{18}$$

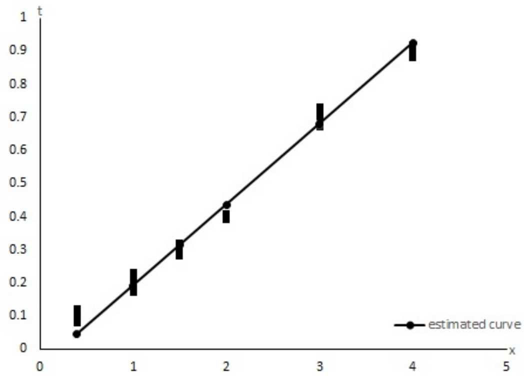

Figure 2: the data given by the domain expert

Additionally, the inverse chance distribution is $\Upsilon_{ur}^{-1}(\alpha; a, b) = \dfrac{\alpha - b}{a}$. We have

$$E[\xi_{ur}] = \int_0^1 \Upsilon_{ur}^{-1}(\alpha)\mathrm{d}\alpha = \frac{1-2b}{2a},$$

and

$$E[\xi_{ur}^2] = \int_0^1 (\Upsilon_{ur}^{-1}(\alpha))^2\mathrm{d}\alpha = \frac{1+3b^2-3b}{3a^2}.$$

Thus, we have the following system of equations:

$$\begin{cases} \dfrac{1-2b}{2a} = \bar{\xi_{ur}} \\ \dfrac{1+3b^2-3b}{3a^2} = \bar{\xi_{ur}^2} \end{cases} \tag{19}$$

Then the unique solution of the above equations

$$\begin{cases} \hat{a} = \dfrac{1}{2\sqrt{3}}(\bar{\xi_{ur}^2} - (\bar{\xi_{ur}})^2)^{-\frac{1}{2}} \\ \hat{b} = \dfrac{1}{2}(1 - 2\hat{a}\bar{\xi_{ur}}) \end{cases} \tag{20}$$

is the moment estimate value of unknown parameters a and b, and the moment estimate distribution is

$$\Upsilon_{ur}(x) = \hat{a}x + \hat{b}$$

By calculating, we have the moment estimate values $\hat{a} = 0.2445, \hat{b} = -0.0526$ and the estimate distribution is $\Upsilon_{ur}(x) = 0.2445x - 0.0526, 0 \leq \Upsilon_{ur}(x) \leq 1.$

Example 5.3. Suppose that the uncertainty distribution of uncertain variable ξ_u has a functional form with three unknown parameters a, b, θ as follows:

$$\Upsilon_u(x; a, b, \theta) = \theta^x + ax + b, (a > 0, \theta > 1, 0 \leq \Upsilon_{ur}(x; a, b) \leq 1)$$

By consulting a domain expert, we got the values of ξ_u and its corresponding belief degree intervals, which are shown in Table 3 and Figure 3.

Table 3: the data given by the domain expert

x	0.1	0.2	0.3	0.4	0.5
t	$(0.17, 0.21)$	$(0.24, 0.29)$	$(0.32, 0.36)$	$(0.41, 0.43)$	$(0.47, 0.53)$
x	0.6	0.7	0.8	0.9	1.0
t	$(0.57, 0.62)$	$(0.66, 0.72)$	$(0.77, 0.80)$	$(0.86, 0.92)$	$(0.99, 1.00)$

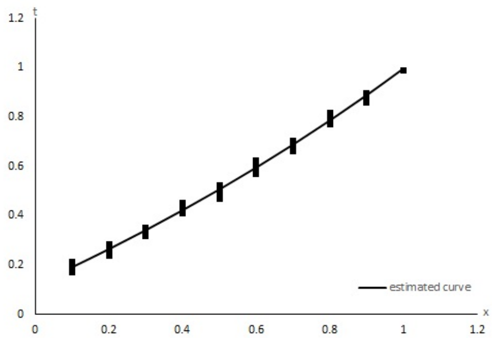

Figure 3: the data given by the domain expert

Considering that the belief degree t is presented in the form of an interval (α, β), we regard the uncertain variable ξ_u as an uncertain random variable ξ_{ur}, while $t \sim \mathcal{U}(\alpha, \beta)$. In this example, we can't get the moment estimate values easily. As the number of unknown parameters increases, the calculation of the estimators will become larger and harder. So we use the least squares estimation. The unknown parameters a, b, θ are the solution of the following minimization problem:

$$\min_{a,b,\theta} \sum_{i=1}^{10} (\theta^{x_i} + ax_i + b - \gamma_i)^2.$$

By calculating, we have the estimate values $\hat{\theta} = 1.750, \hat{a} = 0.326, \hat{b} = -0.879$ and the estimate distribution is $\Upsilon_{ur}(x) = 1.75^x + 0.326x - 0.879, 0 \leq \Upsilon_{ur}(x) \leq 1$.

6 Conclusion

In this paper, we mainly provide a new method of data collection based on chance theory. By improving the method collecting the data from the domain expert, we make the data more realistic and make the fault tolerance of the estimated value stronger. The method of moments in this paper is used to estimate the unknown parameters in the distribution. By using this method, we can easily and conveniently calculate the unknown parameters of the distribution in the real experiment.

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